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Whitney Multiapproximation

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Abstract

In this article we prove that Whitney theorem for the value of the best multiapproximation of a function $f \in L_p([a,b]^d)$, $0 < p < \infty$ by algebraic multipolynomial p_{m-1} of degree $\leq m-1$.

1. Introduction, definitions and main result

Whitney theorem has applications in many areas and has been further generalized to various classes of function and other approximating spaces .

Whitney theorem was proved by Burkill [1] when $(k = 2,\ p = \infty)$ and Storozhenko [2] when $(0 < p < 1)$.

In **[3], [4]** Whitney proved that if $f \in C([a,b])$ then $E_{k-1}(f)_{[a,b]} \leq W_k \omega_k \left(f, \frac{b-a}{k}, [a,b]\right)$ where $W_k =$ const depends only on k.

In 2003 E.S. Bhaya [5] proved the following theorem by using Whitney theorem of interpolatory type for k-monotone functions for K. A. Kopotun.

Theorem A: *Let* $m, k \in N$, $m < k$ *and* $f \in \Delta^k \cap W_p^m(I)$. Then for any, $n \geq k-1$, there exists *apolynomial* $p_n \in \Pi_n$ such that for any $p < 1$

$$\| f^{(j)} - p_n^{(j)} \|_p \leq c(p,k)\omega_{k-j}^{\varphi}(f^{(j)}, n^{-1}, I)_p \ \ for\ j = 1, \dots, m .$$

In 2004 S.Dekel and D.Leviatan [6] proved the following Whitney estimate.

Theorem B: Given $0 < p \le \infty$, $r \in N$, and $d \ge 1$, there exists a constant $C(d,r,p)$, depending only on the three parameters, such that for every bounded convex domain $\Omega \subset R^d$, and each function $f \in L_p(\Omega)$,

$$E_{r-1}(f,\Omega)_p \le C(d,r,p)\omega_r(f, diam(\Omega),\Omega)_p,$$

where $E_{r-1}(f,\Omega)_p$ is the degree of approximation by polynomials of total degree $r-1$, and $\omega_r(f,\cdot)_p$ is the modulus of smoothness of order r.

In 2011 Dinh Dung and Tino Ullrich [7] proved the following Whitney type inequalities

Theorem C: Let $1 \le p \le \infty, r \in \mathrm{N}^d$. then there is a constant C depending only on r,d such that for every $f \in L_p(Q)$

$$\left(\sum_{e\subset[d]}\prod_{i\in e}2^{r_i}\right)^{-1}\Omega(f,\delta,Q)_{p,Q} \le E_r(f)_{p,Q} \le C\Omega(f,\delta,Q)_{p,Q},$$

Where $Q := [a_1,b_1]\times\ldots\times[a_d,b_d]$ *and* $\delta = \delta(Q) := (b_1 - a_1,\ldots,b_d - a_d)$ *is the size of* Q.

For the proof our main result we need the following definitions **:**

Let us introduce a new version of Lagrange polynomial on R^d, and call it a Lagrange multipolynomial .

Definition 1.1.

A Lagrange multipolynomial $L(x,f) = L\big((x_1,x_2,\ldots,x_d);f\big)$

$$L\big((x_1,x_2,\ldots,x_d);f\big) = L\big((x_1,\ldots,x_d);f;(x_{01},\ldots,x_{0d}),(x_{11},\ldots,x_{1d}),\ldots,(x_{m1},\ldots,x_{md})\big) \quad (1)$$

that interpolates a function f at points $x_0 = (x_{01},\ldots,x_{0d})$, $x_{1=}(x_{11},\ldots,x_{1d})$ $,\ldots,x_m = (x_{m1},\ldots,x_{md})$ (interpolation nodes) is defined as an algebraic multipolynomial of at most mth order that takes the same values at these points as the function f , that is

$$L(x_i;f) = L\big((x_{i1},\ldots,x_{id});f\big) = f\big((x_{i1},\ldots,x_{id})\big) \quad (2)$$

where $i = 0,\ldots,m$.

Example , for $m = 1$ we have

$L(x; f; x_0, x_1) = L((x_1, \ldots, x_d); f; (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}))$

$$= \frac{(x_1 - x_{11}) \ldots (x_d - x_{1d})}{(x_{01} - x_{11}) \ldots (x_{0d} - x_{1d})} f((x_{01}, \ldots, x_{0d})) + \frac{(x_1 - x_{01}) \ldots (x_d - x_{0d})}{(x_{11} - x_{01}) \ldots (x_{1d} - x_{0d})} f((x_{11}, \ldots, x_{1d}))$$

$$= f(x_{01}, \ldots, x_{0d}) + \frac{f((x_{11}, \ldots, x_{1d})) - f((x_{01}, \ldots, x_{0d}))}{(x_{11} - x_{01}) \ldots (x_{1d} - x_{0d})} ((x_1 - x_{01}) \ldots (x_d - x_{0d})) \qquad (3)$$

where $x_{0j} \neq x_{1j}$, $j = 1, \ldots, d$

Definition 1.2.

Let $I_k(x) = I_k((x_1, \ldots, x_d)) = I_k((x_1, \ldots, x_d); (x_{01}, \ldots x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}))$

$$= \prod_{\substack{i=0 \\ k \neq i}}^{m} \frac{(x_1 - x_{i1}) \ldots (x_d - x_{id})}{(x_{k1} - x_{i1}) \ldots (x_{kd} - x_{id})} \quad , \qquad k = 0, \ldots, m , \qquad (4)$$

a new version of fundamental Lagrange multi polynomials.

We set

$p(x) = p((x_1, \ldots, x_d)), x \in R^d$

$$= ((x_1 - x_{01}) \ldots (x_d - x_{0d}))((x_1 - x_{11}) \ldots (x_d - x_{1d})) \ldots ((x_1 - x_{m1}) \ldots (x_d - x_{md})).$$

And note that

$$\acute{p}((x_{k1}, \ldots, x_{kd})) = \lim_{\substack{x_j \to x_{kj} \\ j=1,\ldots,d}} \frac{p((x_1, \ldots, x_d))}{((x_1 - x_{k1}) \ldots (x_d - x_{kd}))}$$

$$= \lim_{\substack{x_j \to x_{kj} \\ j=1,\ldots,d}} \prod_{i=0}^{m} ((x_1 - x_{i1}) \ldots (x_d - x_{id}))$$

$$= \prod_{i=0}^{m} ((x_{k1} - x_{i1}) \ldots (x_{kd} - x_{id})) .$$

Therefore , for any $k = 0, \ldots, m$, the new version of the fundamental Lagrange multipolynomials are represented in the form

$$I_k\big((x_1,\dots,x_d)\big) = I_k\big((x_1,\dots,x_d);(x_{01},\dots,x_{0d}),(x_{11},\dots,x_{1d}),\dots,(x_{m1},\dots,x_{md})\big)$$

$$= \frac{p\big((x_1,\dots,x_d)\big)}{\big((x_1-x_{k1})\dots(x_d-x_{kd})\big)\,\acute{p}\,\big((x_{k1},\dots,x_{kd})\big)} \quad ,$$

where $x_j \neq x_{kj}$, $j = 1,\dots,d$, $k = 0,\dots,m.$

Let $\delta_{i,k}$ denote the Kronecker symbol , which is equal to 1 for $i = k$ and to 0 otherwise .

It follows from the obvious equality $I_k(x_{i1},\dots,x_{id}) = \delta_{i,k}$, $i,k = 0,\dots,m$, that the Lagrange multipolynomial exists and is represented by the relation

$$L\big((x_1,\dots,x_d);f;(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md})\big)$$

$$= \sum_{k=0}^{m} f\big((x_{k1},\dots,x_{kd})\big)\; I_k\,\big((x_1,\dots,x_d);(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md})\big) \qquad (5)$$

Definition 1.3.

The expression $[(x_{01},\dots,x_{0d}),(x_{11},\dots,x_{1d}),\dots,(x_{m1},\dots,x_{md});f]$

is called the divided difference of order m for the function f at the points $x_0 = (x_{01},\dots,x_{0d}), x_1 = (x_{11},\dots,x_{1d}),\dots,x_m = (x_{m1},\dots,x_{md})$

For example

$$[x_0,x_1;f] = \frac{f\big((x_{01},\dots,x_{0d})\big)}{(x_{01}-x_{11})\dots(x_{od}-x_{1d})} + \frac{f\big((x_{11},\dots,x_{1d})\big)}{(x_{11}-x_{01})\dots(x_{1d}-x_{0d})}$$

$$= \frac{f\big((x_{01},\dots,x_{0d})\big) - f\big((x_{11},\dots,x_{1d})\big)}{(x_{01}-x_{11})\dots(x_{od}-x_{1d})} \qquad (6)$$

Let $[x_0;f] = [(x_{01},\dots,x_{0d});f] = f\big((x_{01},\dots,x_{0d})\big)$. (7)

Definition 1.4.

The expression

$$\Delta_h^m\big(f;(x_{01},\dots,x_{0d})\big) := \sum_{k=0}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^d f\big((x_{01}+kh_1,\dots,x_{0d}+kh_d)\big) \quad (8)$$

where $d \in N$ chosen so that $(-1)^{m-k} = (-1)^d$

is called the multi mth difference of the function $f \in L_p([a,b]^d), 0<p<\infty$ at the point $x_0 = (x_{01},\dots,x_{0d})$ with step $h=(h_1,\dots,h_d)$.

Denote $\Delta_h^0\big(f;(x_{01},\dots,x_{0d})\big) = f\big((x_{01},\dots,x_{0d})\big)$ and $\Delta_0^m\big(f;(x_{01},\dots,x_{0d})\big) = 0$.

Our main result is:

Theorem 1.1.

If $f \in L_p([a,b]^d)$ $,0<p<\infty, then$

$$E_{m-1}(f)_{L_p[a,b]^d} \le C(p,m,d)\,\omega_m(f;h;[a,b]^d)_p$$

where $h=(h_1,\dots,h_d)$.

Now to prove our theorem we need the lemmas and theorems which will be stated and proved in the following sections :

2. Divided differences

Let us define the difference

$f\big((x_1,\dots,x_d)\big) - L\big((x_1,\dots,x_d);f;(x_{01},\dots,x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big),$

by the product $\big((x_1-x_{01})\dots(x_d-x_{0d})\big)\dots\big((x_1-x_{m1-1})\dots(x_d-x_{md-1})\big)$

Using (4) and (5) , we represent the quotient at the points $x_1 = x_{m1}$,…, $x_d = x_{md}$ as follows :

$$\frac{f\big((x_{m1},\dots,x_{md})\big)-L\big((x_{m1},\dots,x_{md});f;(x_{01},\dots,x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big)}{\prod_{k=0}^{m-1}\big((x_{m1}-x_{k1})\dots(x_{md}-x_{kd})\big)}$$

$$=\sum_{k=0}^{m}\frac{f\big((x_{k1},\dots,x_{kd})\big)}{\prod_{\substack{i=0\\ i\neq k}}^{m}\big((x_{k1}-x_{i1})\dots(x_{kd}-x_{id})\big)}$$

$$=[(x_{01},\dots,x_{0d}),(x_{11},\dots,x_{1d}),\dots,(x_{m1},\dots,x_{md});f] \qquad (9)$$

Theorem 2.1.

The Lagrange multipolynomial $L(x;f;x_0,\dots,x_m)$ is represented by the following Newton formula:

$$L(x;f;x_0,\dots,x_m)=L\big((x_1,\dots,x_d);f;(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md})\big)$$

$$=[(x_{01},\dots x_{0d});f]+[(x_{01},\dots x_{0d}),(x_{11},\dots x_{1d});f]\big((x_1-x_{01})\dots(x_d-x_{0d})\big)+\dots+ [(x_{01},\dots x_{0d}),(x_{11},\dots x_{1d}),\dots,(x_{m1},\dots x_{md});f]\big((x_1-x_{01})\dots(x_d-x_{0d})\big)\big((x_1-x_{11})\dots(x_d-x_{1d})\big)\dots\big((x_1-x_{m1-1})\dots(x_d-x_{md-1})\big) \qquad (10)$$

Proof:

For $m=1$, formula (10) follows from (3), (6) and (7) .

Assume that (10) is true for a number $m-1$.

By induction , let us prove that this formula is true for the number m , that is

$$L\big((x_1,\dots,x_d);f;(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md})\big)$$

$$=L\big((x_1,\dots,x_d);f;(x_{01},\dots x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big)+[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md});f]\big((x_1-x_{01})\dots(x_d-x_{0d})\big)\dots\big((x_1-x_{m1-1})\dots(x_d-x_{md-1})\big).$$

Since both parts of this equality are multipolynomials of degree $\leq m$, it suffices to prove that this equality holds at all points x_i , $i=0,\dots,m$.

By the definition of Lagrange multipolynomial (Definition1.1) , for all $i=0,\dots,m-1$, we have

$$L\big((x_{i1},\dots,x_{id});f;(x_{01},\dots x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big)$$

$+[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md});f]\left((x_{i1}-x_{01})\dots(x_{id}-x_{0d})\right)...\left((x_{i1}-x_{m1-1})\dots(x_{id}-x_{md-1})\right)=f\left((x_{i1},\dots,x_{id})\right)+0$

$=L\left((x_{i1},\dots,x_{id});f;(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md})\right)$,

for $i=m$ according to (9) we obtain

$L\left((x_{m1},\dots,x_{md});f;(x_{01},\dots x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\right)+$

$[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md});f]\left((x_{m1}-x_{01})\dots(x_{md}-x_{0d})\right)...\left((x_{m1}-x_{m1-1})\dots(x_{md}-x_{md-1})\right)$

$=f\left((x_{m1},\dots,x_{md})\right)=L\left((x_{m1},\dots,x_{md});f;(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md})\right)$ □

Lemma 2.1.

$$L_{x_j}^{(m)}(f)=m!\,\psi\quad[(x_{01},\dots x_{0d}),(x_{11},\dots,x_{1d}),\dots,(x_{m1},\dots,x_{md});f] \tag{11}$$

where ψ is a constant and $j=1,\dots,d$.

proof:

We have

$$\left(\left((x_1-x_{01})(x_1-x_{11})\dots(x_1-x_{m1-1})\right)\left((x_2-x_{02})(x_2-x_{12})\dots(x_2-x_{m2-1})\right)\left((x_3-x_{03})(x_3-x_{13})\dots(x_3-x_{m3-1})\right)\dots\left((x_d-x_{0d})(x_d-x_{1d})\dots(x_d-x_{md-1})\right)\right)^{(m)}$$

$$=\left(x_j^m\prod_{i=0}^{m-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{d}(x_\ell-x_{i\ell})\right)^{(m)}+\left(c_1x_j^{m-1}\prod_{i=0}^{m-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{d}(x_\ell-x_{i\ell})\right)^{(m)}$$

$$+\cdots+\left(c_m\prod_{i=0}^{m-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{d}(x_\ell-x_{i\ell})\right)^{(m)},$$

where c_1 , c_2 , … , c_m are constant and $j = 1,\dots,d$

$$= m!\left(\prod_{i=0}^{m-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{d}(x_\ell - x_{i\ell})\right) \text{ , so}$$

$$L_{x_j}^{(m)}(f) = m!\left(\prod_{i=0}^{m-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{d}(x_\ell - x_{i\ell})\right) \quad [(x_{01}, \dots x_{0d}), (x_{11}, \dots, x_{1d}), \dots, (x_{m1}, \dots, x_{md}); f]$$

$$= m!\,\psi\,[(x_{01}, \dots x_{0d}), (x_{11}, \dots, x_{1d}), \dots, (x_{m1}, \dots, x_{md}); f],$$

where $\psi = \prod_{i=0}^{m-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{d}(x_\ell - x_{i\ell})$ is a constant □

Lemma 2.2.

The following identity is true

$$(x_{0j} - x_{mj})[(x_{01}, \dots x_{0d}), \dots, (x_{m1}, \dots, x_{md}); f] = [(x_{01}, \dots x_{0d}), \dots, (x_{m1-1}, \dots, x_{md-1}); f] - [(x_{11}, \dots x_{1d}), \dots, (x_{m1}, \dots, x_{md}); f] \quad (12)$$

where $j = 1,\dots,d$.

Proof :

Let $L((x_1, \dots, x_d)) = L((x_1, \dots, x_d); f; (x_{01}, \dots x_{0d}), \dots, (x_{m1}, \dots, x_{md}))$.

It follows from (10) and (11) that

$$L_{x_j}^{(m-1)}(f) = [(x_{01}, \dots x_{0d}), \dots, (x_{m1-1}, \dots, x_{md-1}); f](m-1)!\,\psi + [(x_{01}, \dots x_{0d}), \dots, (x_{m1}, \dots, x_{md}); f]\,\psi\left(m!\ x_j - (m-1)!\left(x_{0j} + \cdots + x_{mj-1}\right)\right).$$

Interchanging the points $x_0 = (x_{01}, \dots x_{0d})$ and $x_m = (x_{m1}, \dots, x_{md})$ in (10) we get

$$L_{x_j}^{(m-1)}(f) = [(x_{m1}, \dots x_{md}), (x_{11}, \dots, x_{1d}), \dots, (x_{m1-1}, \dots, x_{md-1}); f](m-1)!\,\psi$$

$$+[(x_{m1}, \dots, x_{md}), (x_{11}, \dots, x_{1d}), \dots, (x_{m1-1}, \dots, x_{md-1}), (x_{01}, \dots, x_{0d}); f]$$

$$\psi\left(m!\ x_j - (m-1)!\left(x_{mj} + x_{1j} + \cdots + x_{mj-1}\right)\right)$$

$$= [(x_{11}, \dots x_{1d}), \dots, (x_{m1}, \dots, x_{md}); f](m-1)!\,\psi$$

$+[(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md});f]\,\psi\,\left(m!\ x_j-(m-1)!\left(x_{1j}+\cdots+x_{mj}\right)\right)$.
Subtracting equalities

$[(x_{01},\dots x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1});f](m-1)!\,\psi\ +[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md});f]\,\psi$ $\left(m!\ x_j-(m-1)!\left(x_{0j}+\cdots+x_{mj-1}\right)\right)$, and
$[(x_{11},\dots x_{1d}),\dots,(x_{m1},\dots,x_{md});f](m-1)!\,\psi+[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots x_{md});f]\,\psi\,\left(m!\ x_j-(m-1)!\left(x_{1j}+\cdots+x_{mj}\right)\right)$, we get

$$([(x_{01},\dots x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1});f]-[(x_{11},\dots x_{1d}),\dots,(x_{m1},\dots,x_{md});f])(m-1)!\,\psi$$

$$-\left((m-1)!\,\psi[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md});f]\left(x_{0j}-x_{mj}\right)\right)=0.$$

By dividing on $(m-1)!\,\psi$, we get

$$\left(x_{0j}-x_{mj}\right)[(x_{01},\dots x_{0d}),\dots,(x_{m1},\dots,x_{md});f]$$

$$=[(x_{01},\dots x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1});f]-[(x_{11},\dots x_{1d}),\dots,(x_{m1},\dots,x_{md});f]\quad\square$$

Now let $x_0,x_1\in[a,b]^d$ and let a function f be absolutely continuous on $[a,b]^d$. Then according to the Lebesgue theorem we have

$$f\left((x_{11},\dots x_{1d})\right)-f\left((x_{01},\dots x_{0d})\right)$$

$$=\int_{x_{01}}^{x_{11}}\dots\int_{x_{0d}}^{x_{1d}}f_{t_1,t_2,\dots,t_d}\left((t_1,,\dots,t_d)\right)dt_1\dots dt_d\quad .$$

Performing the change of variables $t_1=x_{01}+(x_{11}-x_{01})\tilde{t}_1$, . . . ,$t_d=x_{0d}+(x_{1d}-x_{0d})\tilde{t}_1$ we obtain

$$[(x_{01},\dots x_{0d}),(x_{11},\dots x_{1d});f]$$

$$=\frac{1}{(x_{11}-x_{01})\dots(x_{1d}-x_{0d})}\int_{x_{01}}^{x_{11}}\dots\int_{x_{0d}}^{x_{1d}}f_{t_1,\dots,t_d}\ \left((t_1,\dots,t_d)\right)\ dt_1\dots\ dt_d$$

$$=\int_0^1\dots\int_0^1 f'\left((x_{01}+(x_{11}-x_{01})\tilde{t}_1\ ,\dots,x_{0d}+(x_{1d}-x_{0d})\tilde{t}_1\ )\right)d\tilde{t}_1^d\ , \tag{13}$$

where $dt_1=(x_{11}-x_{01})\,d\tilde{t}_1$, . . . , $dt_d=(x_{1d}-x_{0d})\,d\tilde{t}_1$, $f'=f_{t_1,\dots,t_d}$ and $d\tilde{t}_1^d=d\ \tilde{t}_1\ d\ \tilde{t}_1\ \dots\ d\ \tilde{t}_1$, d times .

A similar representation is true for any m by virtue of the following theorem:

Theorem 2.2.

Let $x_i \in [a,b]^d$ where $x_i = (x_{i1}, \dots, x_{id})$ for $i = 0, \dots, m$.

If the function f has the absolute continuous $(m-1)\ th$ derivative on $[a,b]^d$, then

$$[(x_{01}, \dots x_{0d}), (x_{11}, \dots x_{1d}), \dots, (x_{m1}, \dots, x_{md}); f]$$

$$= \int_0^1 \dots \int_0^1 \int_0^{\tilde{t}_1} \dots \int_0^{\tilde{t}_1} \dots \int_0^{\tilde{t}_{m-1}} \dots \int_0^{\tilde{t}_{m-1}} f^{(m)}\Big((x_{01}, \dots x_{0d}) + \big((x_{11} - x_{01})\tilde{t}_1, \dots, (x_{1d} - x_{0d})\tilde{t}_1\big) + \cdots + \big((x_{m1} - x_{m1-1})\tilde{t}_m, \dots, (x_{md} - x_{md-1})\tilde{t}_m\big)\Big)\, d\tilde{t}_m^d \dots d\tilde{t}_1^d \qquad (14)$$

Proof:

Assume that representation (14) is true for a number $m-1$. By induction , let us prove that (14) is also true for the number m . Denote $\tilde{t}_0$:=1. According to relation (12) and the induction hypothesis , we have

$$\left(x_{mj} - x_{mj-1}\right)[(x_{01}, \dots x_{0d}), \dots, (x_{m1}, \dots, x_{md}); f]$$

$$= [(x_{01}, \dots x_{0d}), \dots, (x_{m1-2}, \dots, x_{md-2}), (x_{m1}, \dots x_{md}); f] - [(x_{01}, \dots, x_{0d}), \dots, (x_{m1-1}, \dots, x_{md-1}); f]$$

$$= \int_0^{\tilde{t}_0} \dots \int_0^{\tilde{t}_0} \dots \int_0^{\tilde{t}_{m-2}} \dots \int_0^{\tilde{t}_{m-2}} \left(\int_{u_1}^{v_1} \dots \int_{u_d}^{v_d} f^{(m)}(t_1, \dots, t_d)\, dt_1 \dots dt_d\right) d\tilde{t}_{m-1}^d \dots d\tilde{t}_1^d,$$

where

$$f^{(m)} = f_{\underbrace{t_1 \dots t_1}_{m\ times}, t_2 \dots t_2, \dots, \underbrace{t_d \dots t_d}_{m\ times}} ,$$

$$v_1 = x_{01} + \cdots + (x_{m1-2} - x_{m1-3})\tilde{t}_{m-2} + (x_{m1} - x_{m1-2})\tilde{t}_{m-1},$$

$$\vdots$$

$$v_d = x_{0d} + \cdots + (x_{md-2} - x_{md-3})\tilde{t}_{m-2} + (x_{md} - x_{md-2})\tilde{t}_{m-1},$$

$u_1 = \ x_{01} + \cdots + (x_{m1-1} - x_{m1-2})\tilde{t}_{m-1}$,

$\vdots$

$u_d = \ x_{0d} + \cdots + (x_{md-1} - x_{md-2})\tilde{t}_{m-1}$.

It remains to introduce a new integration variable $\tilde{t}_m$ instead of $t = (t_1, \dots, t_d)$ in the last integral by using the change of variables

$t_1 = x_{01} + (x_{11} - x_{01})\tilde{t}_1 + \cdots + (x_{m1-1} - x_{m1-2})\tilde{t}_{m-1} + (x_{m1} - x_{m1-1})\tilde{t}_m$,

$\vdots$

$t_d = x_{0d} + (x_{1d} - x_{0d})\tilde{t}_1 + \cdots + (x_{md-1} - x_{md-2})\tilde{t}_{m-1} + (x_{md} - x_{md-1})\tilde{t}_m$.

And then note that this change of variables transforms the segment $[0, \tilde{t}_m]^d$ into the segment that connects the points $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ □

Lemma 2.3.

Let $i \in N$, $i \le m$ and let $x_i \in [a, b]^d$ then

$$[(x_{01}, \dots, x_{0d}), \dots, (x_{m1}, \dots, x_{md}); f] = [(x_{i1}, \dots, x_{id}), \dots, (x_{m1}, \dots, x_{md}); f_i] \qquad (15)$$

where $f_i\big((x_1, \dots, x_d)\big) = [(x_{01}, \dots, x_{0d}), \dots, (x_{i1-1}, \dots, x_{id-1}); (x_1, \dots, x_d); f]$

proof :

Can easily be proved by induction with the use of (12)

Lemma 2.4.

Let $k \in N$, $k \le m$, and let $x_i \in [a, b]^d$ for all $i = 0, \dots, m$. If a function f is k times continuously differentiable on $[a, b]^d$ or f has the $(k-1)$ th absolutely continuous derivative on $[a, b]^d$, then

$$[(x_{01}, \dots, x_{0d}), \dots, (x_{m1}, \dots, x_{md}); f] = \int_0^1 \dots \int_0^1 \left[(x_{11}, \dots, x_{1d}), \dots, (x_{m1}, \dots, x_{md}); f_{\tilde{t}_1^d}\right] d\tilde{t}_1^d \qquad (16)$$

Proof:

From (13) , (14) and (15) we get

$[(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md});f]$

$$= \int_0^1 \dots \int_0^1 \dots \int_0^{\tilde{t}_{k-1}} \dots \int_0^{\tilde{t}_{k-1}} \left[(x_{k1},\dots x_{kd}),\dots,(x_{m1},\dots,x_{md}); f_{\tilde{t}_1^d,\dots,\tilde{t}_k^d}\right] d\tilde{t}_k^d \dots d\tilde{t}_1^d,$$

where

$f_{\tilde{t}_1^d,\dots,\tilde{t}_k^d}\big((x_1,\dots x_d)\big)$

$$= f^{(k)}\Big((x_{01},\dots,x_{0d}) + \big((x_{11},\dots,x_{1d}) - (x_{01},\dots,x_{0d})\big)\tilde{t}_1 + \cdots + \big((x_{k1-1},\dots,x_{kd-1}) - (x_{k1-2},\dots,x_{kd-2})\big)\tilde{t}_{k-1} + \big((x_{k1},\dots,x_{kd}) - (x_{k1-1},\dots,x_{kd-1})\big)\tilde{t}_k\Big).$$

In particular , if $k = 1$, then

$[(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md});f]$

$$= \int_0^1 \dots \int_0^1 \left[(x_{11},\dots,x_{1d}),\dots,(x_{m1},\dots,x_{md}); f_{\tilde{t}_1^d}\right] d\tilde{t}_1^d \quad \square$$

3. Finite differences

In this section , we assume that the points $x_i = (x_{i1},\dots,x_{id})$ are equidistant , that is , for all $i = 0,\dots,m$ we have

$x_{i1} = x_{01} + ih_1 ,\dots, x_{id} = x_{0d} + ih_d , h \in R^d, h_j \neq 0 , j = 1,\dots,d$.

For the Lagrange interpolation multipolynomial

$L\big((x_1,\dots,x_d)\big) = L\big((x_1,\dots,x_d); f; (x_{01},\dots,x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big)$

$$= \sum_{k=0}^{m-1} f\big((x_{k1},\dots,x_{kd})\big)\, I_k\big((x_1,\dots,x_d),(x_{01},\dots,x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big).$$

We determine the values of the new version of the fundamental Lagrange multipolynomials I_k at the point $x_1 = x_{m1} ,\dots, x_d = x_{md}$.

According to (4) , we have

$I_k\big((x_{m1},\dots,x_{md}),(x_{01},\dots,x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big)$

$$=\prod_{\substack{i=0\\k\neq i}}^{m-1}\frac{(x_{m1}-x_{i1})\dots(x_{md}-x_{id})}{(x_{k1}-x_{i1})\dots(x_{kd}-x_{id})}$$

$$=\prod_{\substack{i=0\\i\neq k}}^{m-1}\frac{(x_{01}+mh_1-x_{01}-ih_1)\dots(x_{0d}+mh_d-x_{0d}-ih_d)}{(x_{01}+kh_1-x_{01}-ih_1)\dots(x_{0d}+kh_d-x_{0d}-ih_d)}$$

$$=\prod_{\substack{i=0\\i\neq k}}^{m-1}\frac{(m-i)h_1\dots(m-i)h_d}{(k-i)h_1\dots(k-i)h_d}$$

$$=\prod_{\substack{i=0\\i\neq k}}^{m-1}\frac{(m-i)\dots(m-i)}{(k-i)\dots(k-i)}$$

$$=\left(-(-1)^{m-k}\binom{m}{k}\right)\dots\left(-(-1)^{m-k}\binom{m}{k}\right)$$

$$=\left(-(-1)^{m-k}\binom{m}{k}\right)^d.$$

We represent the difference $f\big((x_{m1},\dots,x_{md})\big)-L\big((x_{m1},\dots,x_{md})\big)$ in the form

$f\big((x_{m1},\dots,x_{md})\big)-L\big((x_{m1},\dots,x_{md})\big)$

$$=\sum_{k=0}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^d f\big((x_{01}+kh_1,\dots,x_{0d}+kh_d)\big). \tag{17}$$

Lemma 3.1.

$\Delta_h^m\big(f;(x_{01},\dots,x_{0d})\big)$

$$=(m!)^d\,(h_1\dots h_d)^m[(x_{01},\dots,x_{0d}),(x_{01}+h_1,\dots,x_{0d}+h_d),\dots,(x_{01}+mh_1,\dots,x_{0d}+mh_d);f]\,, \tag{18}$$

Proof:

Since $f\big((x_{m1},\dots,x_{md})\big)-L\big((x_{m1},\dots,x_{md})\big)$

$$=\sum_{k=0}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^{d} f\big((x_{01}+kh_1,\dots,x_{0d}+kh_d)\big)$$

and

$$f\big((x_{m1},\dots,x_{md})\big)-L\big((x_{m1},\dots,x_{md});f;(x_{01},\dots,x_{0d}),\dots,(x_{m1-1},\dots,x_{md-1})\big)$$

$$=\prod_{k=0}^{m-1}\big((x_{m1-}x_{k1})\dots(x_{md}-x_{kd})\big)\,[(x_{01},\dots,x_{0d}),(x_{11},\dots,x_{1d}),\dots,(x_{m1},\dots,x_{md});f],$$

then

$$\Delta_h^m\big(f;(x_{01},\dots,x_{0d})\big)$$

$$=\prod_{k=0}^{m-1}\big((x_{01}+mh_1-x_{01}-kh_1)\dots(x_{0d}+mh_d-x_{0d}-kh_d)\big)\,[(x_{01},\dots x_{0d}),(x_{01}+h_1,\dots,x_{0d}+h_d),\dots,(x_{01}+mh_1,\dots,x_{0d}+mh_d);f]$$

$$=\prod_{k=0}^{m-1}\Big(\big((m-k)h_1\big)\dots\big((m-k)h_d\big)\Big)\,[(x_{01},\dots,x_{0d}),(x_{01}+h_1,\dots,x_{0d}+h_d),\dots,(x_{01}+mh_1,\dots,x_{0d}+mh_d);f]$$

$$=\Big((mh_1)\big((m-1)h_1\big)\big((m-2)h_1\big)\dots\big((m-m+1)h_1\big)\Big)\dots\Big((mh_d)\big((m-1)h_d\big)\big((m-2)h_d\big)\dots\big((m-m+1)h_d\big)\Big)\,[(x_{01},\dots,x_{0d}),(x_{01}+h_1,\dots,x_{0d}+h_d),\dots,(x_{01}+mh_1,\dots,x_{0d}+mh_d);f]$$

$$=(m!\,{h_1}^{m})\dots(m!\,{h_d}^{m})[(x_{01},\dots,x_{0d}),(x_{01}+h_1,\dots,x_{0d}+h_d),\dots,(x_{01}+mh_1,\dots,x_{0d}+mh_d);f]$$

$$=(m!)^d\,(h_1\dots h_d)^m\,[(x_{01},\dots,x_{0d}),(x_{01}+h_1,\dots,x_{0d}+h_d),\dots,(x_{01}+mh_1,\dots,x_{0d}+mh_d);f]\ \square$$

Lemma 3.2.

Let $x_0\in[a,b]^d$, $h_j>0$, $j=1,\dots,d$, $x_k=(x_{k1},\dots,x_{kd})$ such that

$x_{k1}=x_{01}+kh_1,\dots,\ x_{kd}=x_{0d}+kh_d$ and $x_m\in[a,b]^d, x_m=(x_{m1},\dots,x_{md})$.

If $F \in L_p^1([a,b]^d)$, then for every $x \in [a,b]^d$ the following inequality is true:

$$\left\|F\big((x_1,\dots,x_d)\big) - L\big((x_1,\dots,x_d);F;(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md})\big)\right\|_{L_p[a,b]^d}$$

$$\le C(p)\frac{1}{(m!)^d(h_1\dots h_d)^m}\left\|\big((x_1-x_{01})\dots(x_d-x_{0d})\big)\dots\big((x_1-x_{m1})\dots(x_d - x_{md})\big)\right\|_{L_p[a,b]^d}\ \omega_m(F',h,[a,b]^d)_p \tag{19}$$

Proof:

For every $\tilde{t}_1 \in [0,1]$, we set

$$F_{\tilde{t}_1^d}\big((u_1,\dots,u_d)\big) = F'\Big((x_1,\dots,x_d) + \big((u_1,\dots,u_d)-(x_1,\dots,x_d)\big)\tilde{t}_1\Big)$$

$$= F'\big((x_1+(u_1-x_1)\tilde{t}_1,\dots,x_d+(u_d-x_d)\tilde{t}_1)\big),$$

where $u \in [a,b]^d$.
Then relations (16) and (18) yield

$$[(x_1,\dots,x_d),(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md});F]$$

$$= \int_0^1 \dots \int_0^1 \Big[(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md});F_{\tilde{t}_1^d}\Big]\,d\tilde{t}_1{}^d$$

$$= \frac{1}{(m!)^d(h_1\dots h_d)^m}\int_0^1 \dots \int_0^1 \Delta_h^m\Big(F_{\tilde{t}_1^d};(x_{01},\dots,x_{0d})\Big)\,d\tilde{t}_1{}^d$$

Since

$$\left\|\Delta_h^m\Big(F_{\tilde{t}_1^d};(u_1,\dots,u_d)\Big)\right\|_{L_p[a,b]^d}$$

$$= \left\|\Delta_{h\tilde{t}_1}^m\big(F';(x_1+(u_1-x_1)\tilde{t}_1,\dots,x_d+(u_d-x_d)\tilde{t}_1)\big)\right\|_{L_p[a,b]^d}$$

$$\le \omega_m(F',h\tilde{t}_1,[a,b]^d)_p \le \omega_m(F',h,[a,b]^d)_p \quad ,$$
then

$$\|[(x_1,\dots,x_d),(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md});F]\|_{L_p[a,b]^d}$$

$$= \left\| \frac{1}{(m!)^d (h_1 \dots h_d)^m} \int_0^1 \dots \int_0^1 \Delta_h^m \left(F_{\tilde{t}_1^d}; (x_{01}, \dots, x_{0d}) \right) d\tilde{t}_1{}^d \right\|_{L_p[a,b]^d}$$

$$\leq C(p) \frac{1}{(m!)^d (h_1 \dots h_d)^m} \left\| \Delta_h^m \left(F_{\tilde{t}_1^d}; (x_{01}, \dots, x_{0d}) \right) \right\|_{L_p[a,b]^d}$$

$$= C(p) \frac{1}{(m!)^d (h_1 \dots h_d)^m} \left\| \Delta_{h\tilde{t}_1}^m \left(F'; (x_1 + (x_{01} - x_1)\tilde{t}_1, \dots, x_d \quad + (x_{0d} - x_d)\tilde{t}_1) \right) \right\|_{L_p[a,b]^d}$$

$$\leq C(p) \frac{1}{(m!)^d (h_1 \dots h_d)^m} \omega_m(F', h, [a,b]^d)_p \ ,$$

relation (19) follows from (9) such that

$$F\big((x_1, \dots, x_d)\big) - \ L\big((x_1, \dots, x_d); F; (x_{01}, \dots, x_{0d}), \dots, (x_{m1}, \dots, x_{md})\big)$$

$$= \big((x_1 - x_{01}) \dots (x_d - x_{0d})\big) \dots \big((x_1 - x_{m1}) \dots (x_d - x_{md})\big) [(x_1, \dots, x_d), (x_{01}, \dots, x_{0d}), \dots, (x_{m1}, \dots, x_{md}); F],$$

since
$\|[(x_1, \dots, x_d), (x_{01}, \dots, x_{0d}), \dots, (x_{m1}, \dots, x_{md}); F]\|_{L_p[a,b]^d}$

$$\leq C(p) \frac{1}{(m!)^d (h_1 \dots h_d)^m} \omega_m(F', h, [a,b]^d)_p \ ,$$

then

$$\left\| F\big((x_1, \dots, x_d)\big) - \ L\big((x_1, \dots, x_d); F; (x_{01}, \dots, x_{0d}), \dots, (x_{m1}, \dots, x_{md})\big) \right\|_{L_p[a,b]^d}$$

$$\leq C(p) \frac{1}{(m!)^d (h_1 \dots h_d)^m} \left\| \big((x_1 - x_{01}) \dots (x_d - x_{0d})\big) \dots \big((x_1 - x_{m1}) \dots (x_d - x_{md})\big) \right\|_{L_p[a,b]^d} \omega_m(F', h, [a,b]^d)_p \quad \square$$

4. Proof of theorem 1.1.

Let $x_k = (x_{k1}, \dots, x_{kd})$ and $x_0 = (x_{01}, \dots, x_{0d})$ such that $x_{0j} := a$,

$h_j := \dfrac{b-a}{m}$, $j = 1, \dots, d$ and $x_{k1} = x_{01} + kh_1, \dots, x_{kd} = x_{0d} + kh_d$,

$$F\big((x_1,\dots,x_d)\big):=\int_a^{x_1}\dots\int_a^{x_d} f\big((u_1,\dots,u_d)\big)\,d\,u_1\dots du_d\ ,$$

$$G\big((x_1,\dots,x_d)\big):=\ F\big((x_1,\dots,x_d)\big)-\ L\big((x_1,\dots,x_d);F;(x_{01},\dots,x_{0d}),\dots,(x_{m1},\dots,x_{md})\big),$$

$$g\big((x_1,\dots,x_d)\big)=G'\big((x_1,\dots,x_d)\big)\ ,$$

$$\omega_m\big((t_1,\dots,t_d)\big)=\omega_m\Big((t_1,\dots,t_d),f,[a,b]^d\Big)_p\equiv\omega_m\Big((t_1,\dots,t_d),g,[a,b]^d\Big)_p.$$

We fix $x\in[a,b]^d$, choose δ for which $(x+m\delta)\in[a,b]^d$ and let $\delta'=(t_1\delta_1,\dots,t_d\delta_d)$.

As a result , we get

$$\int_0^1\dots\int_0^1\Delta_{\delta'}^m\big(g;(x_1,\dots,x_d)\big)\,dt_1\dots dt_d$$

$$=(-1)^{md}\,g\big((x_1,\dots,x_d)\big)+\sum_{k=1}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^d\int_0^1\dots\int_0^1 g\big((x_1+kt_1\delta_1,\dots,x_d+kt_d\delta_d)\big)\,dt_1\dots dt_d$$

$$=(-1)^{md}\,g\big((x_1,\dots,x_d)\big)+\sum_{k=1}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^d\int_0^1\dots\int_0^1 G'\big((x_1+kt_1\delta_1,\dots,x_d+kt_d\delta_d)\big)\,dt_1\dots dt_d$$

$$=(-1)^{md}\,g\big((x_1,\dots,x_d)\big)+\sum_{k=1}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^d\frac{1}{k^d(\delta_1\dots\delta_d)}\Big(G\big((x_1+k\delta_1,\dots,x_d+k\delta_d)\big)-G\big((x_1,\dots,x_d)\big)\Big),$$

whence

$$\big|\,g\big((x_1,\dots,x_d)\big)\big|\le\int_0^1\dots\int_0^1\big|\Delta_{\delta'}^m\big(g;(x_1,\dots,x_d)\big)\big|\,dt_1\dots dt_d+\left|\sum_{k=1}^{m}\left((-1)^{m-k}\binom{m}{k}\right)^d\frac{1}{k^d(\delta_1\dots\delta_d)}\Big(G\big((x_1+k\delta_1,\dots,x_d+k\delta_d)\big)-G\big((x_1,\dots,x_d)\big)\Big)\right|$$

Then

$$\|g\|_{L_p[a,b]^d} \le \Bigg\| \int_0^1 \dots \int_0^1 \left|\Delta_{\delta'}^m\big(g;(x_1,\dots,x_d)\big)\right| dt_1 \dots dt_d + \sum_{k=1}^{m} \left((-1)^{m-k}\binom{m}{k}\right)^d \frac{1}{k^d(\delta_1 \dots \delta_d)} \Big(G\big((x_1+k\delta_1,\dots,x_d+k\delta_d)\big) - G\big((x_1,\dots,x_d)\big)\Big) \Bigg\|_{L_p[a,b]^d}$$

$$\le C(p) \left\| \int_0^1 \dots \int_0^1 \Delta_{\delta'}^m\big(g;(x_1,\dots,x_d)\big)\, dt_1 \dots dt_d \right\|_{L_p[a,b]^d} + C(p) \left\| \sum_{k=1}^{m} \left((-1)^{m-k}\binom{m}{k}\right)^d \frac{1}{k^d(\delta_1 \dots \delta_d)} \Big(G\big((x_1+k\delta_1,\dots,x_d+k\delta_d)\big) - G\big((x_1,\dots,x_d)\big)\Big) \right\|_{L_p[a,b]^d}$$

$$\le C(p)\omega_m(g,|\delta|,[a,b]^d)_p + C(p)\frac{2}{|\delta_1| \dots |\delta_d|}\|G\|_{L_p[a,b]^d} \sum_{k=1}^{m}\left(\binom{m}{k}\frac{1}{k}\right)^d .$$

By virtue of Lemma(3.2) we have $\|G\|_{L_p[a,b]^d} \le C(p)(h_1 \dots h_d)\omega_m(F',h,[a,b]^d)_p$

$= C(p)(h_1 \dots h_d)\omega_m(f,h,[a,b]^d)_p$.

Therefore

$E_{m-1}(f)_{L_p[a,b]^d} \le \|g\|_{L_p[a,b]^d}$

$$\le C(p)\,\omega_m(g,|\delta|,[a,b]^d)_p + C(p)\frac{2}{|\delta_1| \dots |\delta_d|}(h_1 \dots h_d)\sum_{k=1}^{m}\left(\binom{m}{k}\frac{1}{k}\right)^d \omega_m(f,h,[a,b]^d)_p,$$

note that δ_j can always be chosen so that $h_j \ge |\delta_j| \ge h_j/2$, then $E_{m-1}(f)_{L_p[a,b]^d} \le C(p)\omega_m(f,h,[a,b]^d)_p$

$$+C(p)\frac{2}{(h_1 \dots h_d)}(h_1 \dots h_d)\sum_{k=1}^{m}\left(\binom{m}{k}\frac{1}{k}\right)^d \omega_m(f,h,[a,b]^d)_p$$

$$= \left(C(p) + 2C(p) \sum_{k=1}^{m} \left(\binom{m}{k} \frac{1}{k} \right)^{d} \right) \omega_m(f, h, [a,b]^d)_p$$

$$= C(p,m,d)\omega_m(f, h, [a,b]^d)_p ,$$

where $C(p,m,d)$ is Whitney constant

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الخلاصة

برهنا في هذا البحث نظرية وتني لأفضل تقريب متعدد للدالة f التي تنتمي إلى الفضاء L_p , $0 < p < \infty$ بواسطة متعددة الحدود الجبرية متعددة المتغيرات p_{m-1} من الدرجة اقل أو تساوي $m-1$.

الكلمات المفتاحية: نظرية وتني, التقريب المتعدد, متعددة حدود لاكرانج .