The First Triangular Representation of The Symmetric Groups when n=6

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Abstract

In this paper we study a special case of the first triangular representation of the symmetric groups when n=6 over a field K of characteristic p .

Keywords: *symmetric group,group algebraKS_n ,KS_n −module, Specht module ,exact sequence*.

الخلاصة

في هذا البحث تم دراسة حالة خاصة من مقاسات التمثيل الاول المثلثي . p ذو مميز K ضمن حقل n=6 عندما $M(n-3,2,1)$

الكلمات المفتاحية:الزمرة المتناظرة , الجبر الزمري KS_n , KS_n − موديول , سبخت موديل ، المتتالية التامه .

I. Introduction

In 1962 H.K. Farahat studied the representation which deals with the partition $\lambda = (n-1,1)$ of the positive integer n and called it the natural representation of the symmetric groups [1].

In 1969 M. H. Peel renamed the natural representation of the symmetric groups by the first natural representation of the symmetric groups and studied the second representation of the symmetric group which deal with the partition $\lambda = (n-2,2)$ of the positive integer n [٢].

In 1971 Peel introduced the r^{th} Hook representations which deals with the partitions $\lambda = (n-r,1^r)$; r≥ 1. [٣]

In 2016 we introduced the r^{th} triangular representations which deals with the partition $\lambda = \left(n-\frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right)$; r≥ 1 ,and study the first of them which call it the first triangular representation of the symmetric groups when p divides (n-1) [٤].

Through this paper let **K** be the field of characteristic p and n=6.

II. Preminaries

Definition 1: [Peel:1969] Let S_n be the set of all permutations τ on the set $\{x_1, x_2, \dots, x_n\}$ and $K[x_1, x_2, \dots, x_n]$ be the ring of polynomials in $x_1, x_2, \dots, x_n$ with coefficients in K. Then each permutation $\tau \in S_n$ can be regarded as a bijective function from $K[x_1, x_2, \dots, x_n]$ onto $K[x_1, x_2, \dots, x_n]$

defined by $(f(x_1,x_2,\dots,x_n)) = f(\tau(x_1),\tau(x_2),\dots,\tau(x_n)) \forall f(x_1,x_2,\dots,x_n) \in K[x_1,x_2,\dots,x_n]$. Then KS_n forms a group algebra with respect to addition of functions, product of functions by scalars and composition of functions which is called the group algebra of the symmetric group S_n.

Definition2: [٥] Let n be a positive integer then the sequence $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_l)$ is called a partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Then the set $D_\lambda = \{(i,j) | i = 1,2,\dots,l; 1 \leq j \leq \lambda_i\}$ is called $\lambda -$diagram .And any bijective function $t : D_\lambda \to \{x_1,x_2,\dots,x_n\}$ is called a λ–tableau. A λ–tableau may be thought as an array consisting of l rows and λ_1 columns of distinct variables $t((i,j))$ where the variables occur in the first λ_i positions of the i^{th} row and each variable $t((i,j))$ occurs in the i^{th} row and the j^{th} column ((i,j)-position)of the array.$t((i,j))$ will be denoted by $t(i,j)$for each $(i,j) \in D_\lambda$. The set of all λ-tableaux will be denoted by T_λ. i.e $T_\lambda = \{t | t\ is\ a\ \lambda - tableau\}$. Then the function $g: T_\lambda \to K[x_1,x_2,\dots,x_n]$ which is defined by $g(t) = \prod_{i=1}^{l} \prod_{j=1}^{\lambda_i} (t(i,j))^{i-1}$, $\forall\ t \in T_\lambda$ .is called the row position monomial function of T_λ,and for each λ–tableau t, $g(t)$ is called the row position monomial of t .So $M(\lambda)$ is the cyclic $KS_n -$module generated by $g(t)$ over KS_n. [٥]

III. The First Triangular Representation of S_n

In the beginning ,we define some denotations and state some theorems which we need them in this paper.

1)Let $\sigma_1(n) = \sum_{i=1}^{n} x_i$.

2)Let $\sigma_2(n) = \sum_{1 \leq i < j \leq n} x_i x_j$.

3)Let $C_l(n) = x_l^2 \, (\sigma_2(n) - \sum_{\substack{j=1 \\ j \neq l}}^{n} x_l\, x_j); l = 1,2,\dots,n.$

We denote $\overline{N}$ to be the KS_nmodule generated by $C_1(n)$ over KS_n. The set $B = \{C_i(n) |\ i = 1,2,\dots,n\}$ is a K-basis for $\overline{N} = KS_n C_1(n)$ and $dim_K \overline{N} = n$.

4)Let $u_{ij}(n) = C_i(n) - C_j(n)$; $i,j = 1,2,\dots,n$.

we denote $\overline{N}_0$ the KS_n submodule of $\overline{N}$ generated by $u_{12}(n)$.

5)Let $\sigma_3(n) = \sum_{1 \leq i < j \leq n} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} x_i\, x_j x_k^2$. Then $\sum_{l=1}^{n} C_l(n) = \sigma_3(n)$and $dim_K(K\sigma_1(n))$=$dim_K(K\sigma_2(n))$=$dim_K(K\sigma_3(n)) = 1$. $K\sigma_1(n), K\sigma_2(n)$ and $K\sigma_3(n)$ are all KS_n-modules, since $\tau\sigma_k(n) = \sigma_k(n)\ \forall\ k = 1,2,3$.

Definition 3:The KS_n-module $M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right)$ defined by

$$M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right) = KS_n x_1 x_2 \dots x_{r+1} x_{r+2}^2 \dots x_{2r+1}^2 x_{2r+2}^3 \dots x_m^{r+1}$$

is called the r^{th} triangular representation module of S_n over K ,where $n \geq \frac{(r+3)(r+2)}{2}$ and m=$\frac{(r+3)(r+2)}{2}$.

Remark: The first triangular representation module of S_n over K is the $KS_n -$module $M(n-3,2,1)$, the second triangular representation module of S_n over K is the $KS_n -$module $M(n-6,3,2,1)$,the third triangular representation module of S_n over K is the $KS_n -$module $M(n-10,4,3,2,1)$,and so on.

Theorem1: The set $B_0(n-3,2,1) = \{ x_i x_j x_l^2 - x_1 x_2 x_3^2 |\ 1 \leq i < j \leq n, 1 \leq l \leq n, l \neq i,j,(i,j,l) \neq (1,2,3)\}$ is a K-basis of $M_0(n-3,2,1)$, and $dim_K M_0(n-3,2,1) = \binom{n}{2}(n-2) - 1$; $n \geq 6$ [٤]

Theorem2: $\overline{N} = KS_n\, C_1(n)$ and $M(n-1,1)$ are isomorphic over KS_n. [٤]

Proposition1: If p does not divide n then $\overline{N} = \overline{N}_0 \oplus K\sigma_3(n)$. [٤]

Proposition 2: If p does not divides n,then $\overline{N}$ has the following two composition series

$0 \subset \overline{N}_0 \subset \overline{N}$ and $0 \subset K\sigma_3(n) \subset \overline{N}$. [٤]

Definitions 4:

1) The KS_n–homomorphism $d : M\ (n-3,2,1) \rightarrow M\ (n-2,2)$ is defined in terms of the partial operators by

$$d\left(x_i x_j x_l^2 \right) = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \left(x_i x_j x_l^2 \right),$$

2) The KS_n–homomorphism $\bar{d}$ which is the restriction of d to $M_0\ (n-3,2,1)$. i.e.

$$\bar{d}:\ M_0\ (n-3,2,1) . \rightarrow\ M_0\ (n-3,2).$$

Theorem 3: The following sequence of $KS_n -$ modules is exact

$$0 \rightarrow Ker\ d \overset{i}{\rightarrow} M(n-3,2,1) \overset{d}{\rightarrow} M(n-2,2) \rightarrow 0 \ldots\ldots\ldots (1)\ [٤]$$

Corollary 2: The dimension of ker$d$ over K of the $KS_n -$ homomorphism

$$d: M(n-3.2.1) \rightarrow M(n-2,2) \text{ is } \frac{n(n-1)(n-3)}{2}. \quad [٤]$$

Corollary 3: The following sequence of $KS_n -$ modules is exact

$$0 \rightarrow Kerd \overset{i}{\rightarrow} M_0(n-3,2,1) \overset{\bar{d}}{\rightarrow} M_O(n-2,2) \rightarrow 0 \ \ldots\ldots (2)\ [٤]$$

Lemma 2: $dim_K\ S(n-3,2,1) = \frac{n(n-2)(n-4)}{3}$.

Proposition 3: $S(n-3,2,1)$ is a proper submodule of $kerd$. [٤]

Theorem4 : If p=5 and n=6,then we get the following series:

$$0 \subset \overline{N_0} \subset \overline{N} \subset KS_6 y\ , \qquad 0 \subset K\sigma_3 \subset \overline{N} \subset KS_6 y$$

Where $KS_6 y = KS_6(x_1x_4x_3^2 - x_1x_2x_3^2)$

Proof:

Since $\overline{N} = KS_n(C_1(n))$,where $C_1(n) = \sum_{1<i<j\leq n} x_i x_j x_1^2$, then when n=6 we get

$$C_1(6) = x_2x_3x_1^2 + x_2x_4x_1^2 + x_2x_5x_1^2 + x_2x_6x_1^2 + x_3x_4x_1^2 + x_3x_5x_1^2 + x_3x_6x_1^2 + x_4x_5x_1^2 + x_4x_6x_1^2 + x_5x_6x_1^2$$

Thus if p=5 and $y = x_1x_4x_3^2 + x_1x_2x_3^2$ we get that

$\big((x_1x_2x_3) + (x_1x_3) + (x_1x_5x_3) + (x_1x_6x_3) + (x_1x_3)(x_4x_5) + (x_1x_3)(x_4x_6) + (x_1x_5x_3)(x_4x_6) - (x_1x_2x_3)(x_4x_5) + (x_1x_2x_3)(x_4x_6) + 4(x_1x_2x_6x_3)(x_4x_5)\big)y = (x_1x_2x_3)y + (x_1x_3)y + (x_1x_5x_3)y + (x_1x_6x_3)y + (x_1x_3)(x_4x_5)y + (x_1x_3)(x_4x_6)y + (x_1x_5x_3)(x_4x_6)y - (x_1x_2x_3)(x_4x_5)y + (x_1x_2x_3)(x_4x_6)y + 4(x_1x_2x_6x_3)(x_4x_5)y = x_2x_4x_1^2 - x_2x_3x_1^2 + x_3x_4x_1^2 - x_2x_3x_1^2 + x_4x_5x_1^2 -$

$x_2x_5x_1^2+x_4x_6x_1^2-x_2x_6x_1^2+x_3x_5x_1^2-x_2x_3x_1^2+x_3x_6x_1^2-x_2x_3x_1^2+x_5x_6x_1^2-x_2x_5x_1^2+x_2x_3x_1^2-x_2x_5x_1^2+x_2x_6x_1^2-x_2x_3x_1^2+4x_2x_5x_1^2-4x_2x_6x_1^2=$
$x_2x_3x_1^2+x_2x_4x_1^2+x_2x_5x_1^2+x_2x_6x_1^2+x_3x_4x_1^2+x_3x_5x_1^2+x_3x_6x_1^2+x_4x_5x_1^2+x_4x_6x_1^2+x_5x_6x_1^2=C_1(6)$

Hence $C_1(6)\in KS_6y$ which implies that $KS_6C_1(6)\subset KS_6y.$ *i.e*

$\bar{N}\subset KS_6y$. Moreover since p=5 and n=6 then p dose not divide n and by [Al-Aamily:2016] we get the following two composite series:

$0\subset \bar{N}_0\subset \bar{N}$ and $0\subset K\sigma_3\subset \bar{N}$

Therefore if p=5 and n=6 we get the following two series:

$0\subset \bar{N}_0\subset \bar{N}\subset KS_6y$ and $0\subset K\sigma_3\subset \bar{N}\subset KS_6y$.

Theorem5: We have the following series:

$$0\subset S(n-3,2,1)\subset KS_nw\subset kerd\subset M_0(n-3,2,1)\subset M(n-3,2,1).$$

Where $w=x_2x_5x_3^2-x_2x_4x_3^2-x_2x_5x_1^2+x_2x_4x_1^2$

Proof: we have $S(n-3,2,1)=\Delta(x_1,x_2,x_3)\Delta(x_4,x_5)$.

Let $y=\Delta(x_1,x_2,x_3)\Delta(x_4,x_5)=(x_3-x_2)(x_3-x_1)(x_2-x_1)(x_5-x_4)$

$=x_2x_5x_3^2-x_2x_4x_3^2-x_3x_5x_2^2+x_3x_4x_2^2+x_1x_5x_2^2-x_1x_4x_2^2-x_1x_5x_3^2+x_1x_4x_3^2+x_3x_5x_1^2-x_3x_4x_1^2-x_2x_5x_1^2+x_2x_4x_1^2$

$=(x_2x_5x_3^2-x_2x_4x_3^2-x_2x_5x_1^2+x_2x_4x_1^2)+(x_3x_4x_2^2-x_3x_5x_2^2-x_3x_4x_1^2+x_3x_5x_1^2)+(x_1x_5x_2^2-x_1x_4x_2^2-x_1x_5x_3^2+x_1x_4x_3^2)\in KS_nw.$

Hence $S(n-3,2,1)\subset KS_nw$.Moreover by definition of *w* we get that $dw=0$,thus $KS_nw\subset kerd$ **.**Hence we get the following

$$0\subset S(n-3,2,1)\subset KS_nw\subset kerd\subset M_0(n-3,2,1)\subset M(n-3,2,1).$$

Theorem6: If n=6 then we get the following series:

$0\subset S(3,2,1)\subset T\subset KS_6y\subset M_0(3,2,1)\subset M(3,2,1)$

Where $T=KS_6(x_1x_3x_5^2-x_1x_4x_5^2+x_2x_4x_5^2-x_2x_3x_5^2)$ and

$KS_6y=KS_6(x_1x_4x_3^2-x_1x_2x_3^2)$.

Proof: Let $y=x_1x_4x_3^2-x_1x_2x_3^2$ .If $\tau_1=(x_3x_5),\tau_2=(x_1x_3x_5)\in S_6$.

Then we get that

$(\tau_2-\tau_1)y=x_3x_4x_5^2-x_2x_3x_5^2+x_1x_2x_5^2-x_1x_4x_5^2$ which implies that

$x_3x_4x_5^2-x_2x_3x_5^2+x_1x_2x_5^2-x_1x_4x_5^2\in KS_6y$.Thus$T\subset KS_6y$.

Moreover we have if $\sigma_1=(x_3x_4)(x_2x_5),\sigma_2=(x_3x_4)(x_1x_5)\in S_6$ .Then

$(i+\sigma_1+\sigma_2)(x_1x_3x_5^2-x_1x_4x_5^2+x_2x_4x_5^2-x_2x_3x_5^2)=$

$x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2 + x_1x_4x_2^2 - x_1x_3x_2^2 + x_3x_5x_2^2 - x_4x_5x_2^2 + x_4x_5x_1^2 - x_3x_5x_1^2 + x_2x_3x_1^2 - x_2x_4x_1^2 =$

$(x_2 - x_1)(x_5 - x_1)(x_5 - x_2)(x_4 - x_3) = \Delta(x_1, x_2, x_5)\Delta(x_3, x_4)$

Which implies that $\Delta(x_1, x_2, x_5)\Delta(x_3, x_4) \in T$.Thus$S(3,2,1) \subset T$. By definition of y we have $KS_6y \subset M_0(3,2,1)$ .Hence we get the following series $0 \subset S(3,2,1) \subset T \subset KS_6y \subset M_0(3,2,1) \subset M(3,2,1)$.

Theorem7: If n=6 then we have the following series:

1) $0 \subset KS_6y_3 \subset KS_6y_6 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$
2) $0 \subset KS_6y \subset KS_6y_2 \subset KS_6y_4 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$
3) $0 \subset KS_6y_1 \subset KS_6y_5 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$.
4) $0 \subset KS_6y \subset KS_6y_2 \subset KS_6y_5 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$

   Where
   $y = x_1x_4x_3^2 - x_1x_2x_3^2$, $y_1 = x_1x_2x_4^2 - x_1x_2x_3^2$,
   $y_2 = x_2x_4x_6^2 - x_1x_2x_3^2$, $y_3 = x_2x_3x_1^2 - x_1x_2x_3^2$,
   $y_4 = x_2x_4x_1^2 - x_1x_2x_3^2$, $y_5 = x_4x_5x_1^2 - x_1x_2x_3^2$,
   $y_6 = x_5x_6x_4^2 - x_1x_2x_3^2$, $y_7 = x_3x_5x_4^2 - x_1x_2x_3^2$.

Proof: Let $y_7 = x_3x_5x_4^2 - x_1x_2x_3^2$. Then $y_7 \in M_0(3,2,1)$ which implies that $KS_6y_7 \subset M_0(3,2,1)$. So if $\tau_1 = (x_3x_5x_6), \tau_2 = (x_1x_6x_3x_2x_4x_5), \tau_3 = (x_1x_6x_5)(x_2x_4x_3) \in S_6$, then we get $(\tau_1 + \tau_2 - \tau_3)y_7 = x_5x_6x_4^2 - x_1x_2x_3^2 = y_6$ which implies that $y_6 \in KS_6y_7$ .Thus $KS_6y_6 \subset KS_6y_7$ .Moreover if $\tau = (x_1x_3) \in S_6$ then $(i - \tau)y_6 = x_5x_6x_4^2 - x_1x_2x_3^2 - x_5x_6x_4^2 + x_2x_3x_1^2 = x_2x_3x_1^2 - x_1x_2x_3^2 = y_3$ .So $y_3 \in KS_6y_6$, thus $KS_6y_3 \subset KS_6y_6$ and we get the following series $0 \subset KS_6y_3 \subset KS_6y_6 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$

Now let $y_4 = x_2x_4x_1^2 - x_1x_2x_3^2$ .Then if $\tau = (x_2x_4) \in S_6$ we get that $(i - \tau)y_4 = x_1x_4x_3^2 - x_1x_2x_3^2 = y$.Hence $y \in KS_6y_4$ which implies that $KS_6y \subset KS_6y_4$ .Moreover when $\tau = (x_1x_5x_2x_3x_4) \in S_6$ we get that $(i + \tau)y_7 = x_2x_4x_1^2 - x_1x_2x_3^2 = y_4$ .Thus $y_4 \in KS_6y_7$ which implies that $KS_6y_4 \subset KS_6y_7$.Then we get $KS_6y \subset KS_6y_4 \subset KS_6y_7$ .From other side if $\tau_1 = (x_4x_6), \tau_2 = (x_1x_6x_5x_3) \in S_6$ we get $(\tau_1 + \tau_2)y_4 = x_2x_4x_6^2 - x_1x_2x_3^2 = y_2$ .Thus $KS_6y_2 \subset KS_6y_4$.While when $\rho_1 = (x_1x_2x_4)(x_3x_6), \rho_2 = (x_1x_4)(x_3x_6) \in S_6$ we get $(\rho_1 - \rho_2)y_2 = x_1x_4x_3^2 - x_1x_2x_3^2 = y$.Thus $y \in KS_6y_2$ which implies that $KS_6y \subset KS_6y_2$ .Therefore we get the following series
$0 \subset KS_6y \subset KS_6y_2 \subset KS_6y_4 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$.

If $\tau_1 = (x_1x_3x_4)(x_2x_6), \tau_2 = (x_1x_2)(x_5x_6), \tau_3 = (x_1x_2) \in S_6$ .Then we get that $(\tau_1 + \tau_2)y_7 = x_4x_5x_1^2 - x_1x_2x_3^2 = y_5$ and $(i - \tau_3)y_5 = x_4x_5x_1^2 - x_4x_5x_2^2 = (x_1x_4)(x_2x_5x_3)y_1$ .Hence $y_5 \in KS_6y_7$ and $y_1 \in KS_6y_5$ which implies that $KS_6y_1 \subset KS_6y_5 \subset KS_6y_7$ .Thus we get the following series
$0 \subset KS_6y_1 \subset KS_6y_5 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$.
Also we have $y_2 \in KS_6y_5$ since $\big(i - (x_2x_6) + (x_1x_6x_5x_2)\big)y_5 = y_2$ then
$KS_6y_2 \subset KS_6y_5$ .Thus we get the following series
$0 \subset KS_6y \subset KS_6y_2 \subset KS_6y_5 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$.
Hence when n=6 we get the following series

1. $0 \subset KS_6y_3 \subset KS_6y_6 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$
2. $0 \subset KS_6y \subset KS_6y_2 \subset KS_6y_4 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$
3. $0 \subset KS_6y_1 \subset KS_6y_5 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$.
4. $0 \subset KS_6y \subset KS_6y_2 \subset KS_6y_5 \subset KS_6y_7 \subset M_0(3,2,1) \subset M(3,2,1)$

Theorem8: If n=6 and p=5 then we get the following series:

1. $0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

2. $0 \subset S_6(3,2,1) \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.
3. $0 \subset K\sigma_3 \subset K\sigma_3 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.
4. $0 \subset S_6(3,2,1) \subset K\sigma_3 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.
5. $0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N}_0 \oplus KS_6w \subset \bar{N}_0 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.
6. $0 \subset S_6(3,2,1) \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N}_0 \oplus KS_6w \subset \bar{N}_0 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.
7. $0 \subset K\sigma_3 \subset K\sigma_3 \oplus S_6(3,2,1) \subset K\sigma_3 \oplus KS_6w \subset K\sigma_3 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.
8. $0 \subset S_6(3,2,1) \subset K\sigma_3 \oplus S_6(3,2,1) \subset K\sigma_3 \oplus KS_6w \subset K\sigma_3 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

Proof: Since $\bar{N} = KS_n(C_1(n))$,where $C_1(n) = \sum_{1<i<j\leq n} x_i x_j x_1^2$ .Then the sum of coefficients is $\frac{n(n-1)(n-2)}{2}$ which implies that $\bar{N} \subset M_0(3,2,1)$.Moreover we have $d\big(C_1(n)\big) = 2\sum_{1<i<j\leq n} x_i x_j \neq 0$.Thus we get that $\bar{N} \cap kerd = 0$.By [Al-Aamily:2016] we have if $p \neq 2$ and p divides $(n-1)$ then we have the following series:

1) $0 \subset \bar{N}_0 \subset \bar{N} \subset \bar{N}\oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.
2) $0 \subset \bar{N}_0 \subset \bar{N}_0\oplus ker\, d \subset \bar{N}\oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$
3) $0 \subset K\sigma_3 \subset K\sigma_3\oplus ker\, d \subset \bar{N}\oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.
4) $0 \subset K\sigma_3 \subset \bar{N} \subset \bar{N}\oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

5) $0 \subset ker\, d \subset \bar{N}_0\oplus \ker d \subset \bar{N}\oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

6) $0 \subset ker\, d \subset K\sigma_3\oplus ker\, d \subset \bar{N}\oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

Therefore by Theorem 1 and Theorem 2 we get the following series:

1 .$0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

2. $0 \subset S_6(3,2,1) \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

3. $0 \subset K\sigma_3 \subset K\sigma_3 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

4. $0 \subset S_6(3,2,1) \subset K\sigma_3 \oplus S_6(3,2,1) \subset \bar{N} \oplus S_6(3,2,1) \subset \bar{N} \oplus KS_6w \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

5. $0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N}_0 \oplus KS_6w \subset \bar{N}_0 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

6. $0 \subset S_6(3,2,1) \subset \bar{N}_0 \oplus S_6(3,2,1) \subset \bar{N}_0 \oplus KS_6w \subset \bar{N}_0 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

7. $0 \subset K\sigma_3 \subset K\sigma_3 \oplus S_6(3,2,1) \subset K\sigma_3 \oplus KS_6w \subset K\sigma_3 \oplus kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

8. $0 \subset S_6(3,2,1) \subset K\sigma_3 \oplus S_6(3,2,1) \subset K\sigma_3 \oplus KS_6w \subset K\sigma_3 \oplus \quad kerd \subset \bar{N} \oplus kerd \subset M_0(3,2,1) \subset M(3,2,1)$.

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