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Fuzzy Continuous Mappings in Fuzzy Normed Linear Spaces

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Abstract: In this paper we continue the study of fuzzy continuous mappings in fuzzy normed linear spaces initiated by T. Bag and S.K. Samanta, as well as by I. Sadeqi and F.S. Kia, in a more general settings. Firstly, we introduce the notion of uniformly fuzzy continuous mapping and we establish the uniform continuity theorem in fuzzy settings. Furthermore, the concept of fuzzy Lipschitzian mapping is introduced and a fuzzy version for Banach's contraction principle is obtained. Finally, a special attention is given to various characterizations of fuzzy continuous linear operators. Based on our results, classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem) can be extended in a more general fuzzy context.
Keywords: Fuzzy normed linear spaces; fuzzy continuous mapping; fuzzy bounded linear operators.

1 Introduction and preliminaries

The concept of fuzzy set was introduced by L. Zadeh [14] in 1965. If X is a nonempty set, a fuzzy set in X is a function μ from X into the unit interval $[0,1]$. The classical union and intersection of ordinary subsets of X can be extended by the following formulas, proposed by L. Zadeh

$$\left(\bigvee_{i\in I}\mu_i\right)(x)=\sup\{\mu_i(x)\ :\ i\in I\}\ ,\quad \left(\bigwedge_{i\in I}\mu_i\right)(x)=\inf\{\mu_i(x)\ :\ i\in I\}\ .$$

From here to the notion of fuzzy topological space, there was one more step to be taken. Thus, in 1968, C.L. Chang [4] introduced the notion of fuzzy topological space. The definition is a natural translation to fuzzy sets of the ordinary definition of topological space. Indeed, a fuzzy topology is a family $\mathcal{T}$, of fuzzy sets in X, such that $\mathcal{T}$ is closed with respect to arbitrary union and finite intersection and every constant function belong to $\mathcal{T}$.

One of the important problems concerning the fuzzy topological spaces is to obtain an adequate notion of fuzzy metric space. Many authors have investigated this question and several notions of fuzzy metric space have been defined and studied. We just mention the definition given by I. Kramosil and J. Michálek [9] in 1975.

Definition 1. The pair (X,M) is said to be a fuzzy metric space if X is an arbitrary set and M is a fuzzy set in $X\times X\times[0,\infty)$ satisfying the following conditions:

(M1) $M(x,y,0)=0,(\forall)x,y\in X;$

(M2) $(\forall)x,y\in X, x=y$ if and only if $M(x,y,t)=1$ for all $t>0;$

(M3) $M(x,y,t)=M(y,x,t),(\forall)x,y\in X,(\forall)t>0;$

(M4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s),\ (\forall)x,y,z\in X,(\forall)t,s>0;$

(M5) $(\forall)x, y \in X, M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

We note that, in previous definition, $*$ denotes a continuous t-norm (see [13]). The basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0, 1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

In studying fuzzy topological linear spaces, A.K. Katsaras [8], in 1984, first introduced the notion of fuzzy norm on a linear space. Since then many mathematicians have introduced several notions of fuzzy norm from different points of view. Thus, C. Felbin [6] in 1992 introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of linear space. In 1994, S.C. Cheng and J.N. Mordeson [5] introduced a concept of fuzzy norm on a linear space whose associated metric is Kramosil and Michálek type. Following S.C. Cheng and J.N. Mordeson, in 2003, T. Bag and S.K. Samanta [2] proposed another concept of fuzzy norm.

In this paper we continue the study of fuzzy continuous mappings in fuzzy normed linear spaces initiated by T. Bag and S.K. Samanta [3], as well as by I. Sadeqi and F.S. Kia [12], in a more general settings:

Definition 2. [10] Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, t) = 1, (\forall)t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s), (\forall)x, y \in X, (\forall)t, s \geq 0$;

(N5) $(\forall)x \in X$, $N(x, \cdot)$ is left continuous and $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Remark 3. a) T. Bag and S.K. Samanta [2], [3] gave a similar definition for $* = \wedge$, but in order to obtain some important results they assumed that the fuzzy norm also satisfied the following conditions:

(N6) $N(x, t) > 0, (\forall)t > 0 \Rightarrow x = 0$;

(N7) $(\forall)x \neq 0, N(x, \cdot)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x, t) < 1\}$ of $\mathbb{R}$.

The results obtained by T. Bag and S.K. Samanta [3], as well as by I. Sadeqi and F.S. Kia [12], can be found in this more general setting.
b) I. Goleţ [7], C. Alegre and S. Romaguera [1] also gave this definition in the context of real vector spaces.
c) $N(x, \cdot)$ is nondecreasing, $(\forall)x \in X$.

Example 4. *[2] Let X be a linear space and $||\cdot||$ be a norm on X. Let*

$$N(x, t) := \begin{cases} 1 & if \quad |x| < t \\ 0 & if \quad |x| \geq t \end{cases}$$

Then $(X, N, \wedge)$ is a FNLS. In particular, $(\mathbb{C}, N, \wedge)$ is a FNLS.

Theorem 5. *[10] Let $(X, N, *)$ be a FNLS. For $x \in X, r \in (0,1), t > 0$ we define the open ball*

$$B(x,r,t) := \{y \in X \; : \; N(x-y,t) > r\} \; .$$

Then

$$\mathcal{T}_N := \{T \subset X \; : \; x \in T \text{ iff } (\exists) t > 0, r \in (0,1) \; : \; B(x,r,t) \subseteq T\}$$

is a topology on X.

Moreover, if the t-norm $*$ satisfies $\sup_{x \in (0,1)} x * x = 1$, then $(X, \mathcal{T}_N)$ is Hausdorff.

Theorem 6. *[10] Let $(X, N, *)$ be a FNLS. Then $(X, \mathcal{T}_N)$ is a metrizable topological vector space.*

Definition 7. [2] Let $(X, N, *)$ be a FNLS and (x_n) be a sequence in X.

1. The sequence (x_n) is said to be convergent if $(\exists) x \in X$ such that

$$\lim_{n \to \infty} N(x_n - x, t) = 1 \; , \; (\forall) t > 0 \; .$$

   In this case, x is called the limit of the sequence (x_n) and we denote $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

2. The sequence (x_n) is called Cauchy sequence if

$$\lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1 \; , \; (\forall) t > 0, (\forall) p \in \mathbb{N}^* \; .$$

3. $(X, N, *)$ is said to be complete if any Cauchy sequence in X is convergent to a point in X. A complete FNLS will be called a fuzzy Banach space.

Theorem 8. *Let $(X, N, *)$ be a FNLS and*

$$p_\alpha(x) := \inf\{t > 0 \; : \; N(x,t) > \alpha\}, \alpha \in (0,1) \; .$$

Then, for $x \in X, s > 0, \alpha \in (0,1)$, we have:

$$p_\alpha(x) < s \text{ if and only if } N(x,s) > \alpha \; .$$

Proof: The proof is entirely the same as in [10], where there are considered FNLSs of type $(X, N, \wedge)$. □

The structure of the paper is as follows: in Section 2, we introduce the notion of uniformly fuzzy continuous mapping and we establish the uniform continuity theorem in fuzzy settings. The concept of fuzzy Lipschitzian mapping is introduced and a fuzzy version for Banach's contraction principle is obtained. In Section 3, special attention is given to various characterizations of fuzzy continuous linear operators. Based on our results, classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem) can be extended in a more general fuzzy context.

Even if the structure of fuzzy F-spaces, recently introduced in [11], is much more complicated than that of fuzzy Banach spaces, we intent to study, in a further paper, fuzzy continuous linear operators on fuzzy F-spaces and to prove that the well-known principles of functional analysis are valid in this context too.

In the following sections $(X, N_1, *_1)$, $(Y, N_2, *_2)$ will be FNLSs with the t-norms $*_1, *_2$ which satisfy $\sup_{x \in (0,1)} x *_i x = 1, (\forall) i = 1, 2$.

2 Fuzzy continuous mappings

Definition 9. [3] A mapping $T : X \to Y$ is said to be fuzzy continuous at $x_0 \in X$, if

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\varepsilon, \alpha, x_0) > 0, (\exists)\beta = \beta(\varepsilon, \alpha, x_0) \in (0,1)$$

$$\text{such that } (\forall)x \in X : N_1(x - x_0, \delta) > \beta \text{ we have that } N_2(T(x) - T(x_0), \varepsilon) > \alpha\ .$$

If T is fuzzy continuous at each point of X, then T is called fuzzy continuous on X.

Theorem 10. *[3] A mapping $T : X \to Y$ is fuzzy continuous at $x_0 \in X$, if and only if $(\forall)(x_n) \subseteq X$, $x_n \to x_0$, we have that $T(x_n) \to T(x_0)$.*

Definition 11. A mapping $T : X \to Y$ is said to be uniformly fuzzy continuous on X, if

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\varepsilon, \alpha) > 0, (\exists)\beta = \beta(\varepsilon, \alpha) \in (0,1)$$

$$\text{such that } (\forall)x, y \in X : N_1(x - y, \delta) > \beta \text{ we have that } N_2(T(x) - T(y), \varepsilon) > \alpha\ .$$

Remark 12. If T is uniformly fuzzy continuous, then T is fuzzy continuous.

Theorem 13. (Uniform continuity theorem). *Let $(X, N_1, *_1)$ be a compact FNLS and $(Y, N_2, *_2)$ be a FNLS. If $T : X \to Y$ is a fuzzy continuous mapping, then T is uniformly fuzzy continuous.*

Proof: Let $\varepsilon > 0$ and $\alpha \in (0,1)$.

As $\sup\limits_{x\in(0,1)} x *_2 x = 1$, then there exists $\alpha_0 \in (0,1)$ such that $\alpha_0 *_2 \alpha_0 > \alpha$.

As $T : X \to Y$ is a fuzzy continuous on X, for all $x \in X$, there exist $\delta_x = \delta\left(\frac{\varepsilon}{2}, \alpha_0, x\right) > 0$, $\beta_x = \beta\left(\frac{\varepsilon}{2}, \alpha_0, x\right) \in (0,1)$ such that

$$(\forall)y \in X \ :\ N_1(x - y, \delta_x) > \beta_x \Rightarrow N_2\left(T(x) - T(y), \frac{\varepsilon}{2}\right) > \alpha_0\ .$$

As $\sup\limits_{x\in(0,1)} x *_1 x = 1$, we can take $\gamma_x > \beta_x$ such that $\gamma_x *_1 \gamma_x > \beta_x$.

Since X is compact and $\left\{B\left(x, \gamma_x, \frac{\delta_x}{2}\right)\right\}_{x\in X}$ is an open covering of X, there exist $x_1, x_2, \cdots, x_n$ in X such that $X = \bigcup\limits_{i=1}^{n} B\left(x_i, \gamma_{x_i}, \frac{\delta_{x_i}}{2}\right)$. Let $\beta = \max\{\gamma_{x_i}\}$ and $\delta = \min\left\{\frac{\delta_{x_i}}{2}\right\}$, for $i = 1, 2, \cdots, n$.

Let $x, y \in X$ arbitrary, such that $N_1(x - y, \delta) > \beta$. As $x \in X$, there exists $i \in \{1, 2, \cdots, n\}$ such that $x \in B\left(x_i, \gamma_{x_i}, \frac{\delta_{x_i}}{2}\right)$, namely $N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) > \gamma_{x_i}$. Hence

$$N_1(x - x_i, \delta_{x_i}) \geq N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) > \gamma_{x_i} > \beta_{x_i}\ .$$

Thus

$$N_2\left(T(x) - T(x_i), \frac{\varepsilon}{2}\right) > \alpha_0\ .$$

We remark that

$$N_1(y - x_i, \delta_{x_i}) \geq N_1\left(y - x, \frac{\delta_{x_i}}{2}\right) *_1 N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) \geq$$

$$\geq N_1(y - x, \delta) *_1 N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) > \beta *_1 \gamma_{x_i} \geq \gamma_{x_i} *_1 \gamma_{x_i} > \beta_{x_i}\ .$$

Thus $N_2\left(T(y)-T(x_i),\frac{\varepsilon}{2}\right)>\alpha_0$.

In conclusion

$$N_2(T(x)-T(y),\varepsilon)\geq N_2\left(T(x)-T(x_i),\frac{\varepsilon}{2}\right)*_2 N_2\left(T(x_i)-T(y),\frac{\varepsilon}{2}\right)>$$

$$>\alpha_0*_2\alpha_0>\alpha\ .$$

□

Definition 14. A mapping $T:X\to Y$ is said to be fuzzy Lipschitzian on X if $(\exists)L>0$ such that

$$N_2(T(x)-T(y),t)\geq N_1\left(x-y,\frac{t}{L}\right)\ ,\ (\forall)t>0,(\forall)x,y\in X\ .$$

If $L<1$ we say that T is a fuzzy contraction.

Remark 15. It is clear that a fuzzy Lipschitzian mapping is necessarily fuzzy continuous.

Theorem 16. (Banach's contraction principle). *Let $(X,N,*)$ be a fuzzy Banach space and $T:X\to X$ be a fuzzy contraction. Then T has a unique fixed point $z\in X$ and*

$$\lim_{n\to\infty}T^n(x)=z,(\forall)x\in X\ .$$

Proof: Let $x\in X$ be arbitrary. Then $\{T^n(x)\}$ is a Cauchy sequence. Indeed, for $t>0$ and $p\in\mathbb{N}^*$, we have

$$N(T^{n+p}(x)-T^n(x),t)\geq N\left(T^{n+p-1}(x)-T^{n-1}(x),\frac{t}{L}\right)\geq$$

$$\geq\cdots\geq N\left(T^p(x)-x,\frac{t}{L^n}\right).$$

As $L\in(0,1)$, we have that $\lim\limits_{n\to\infty}\frac{t}{L^n}=\infty$. Thus

$$\lim_{n\to\infty}N\left(T^p(x)-x,\frac{t}{L^n}\right)=1\ .$$

Hence $\lim\limits_{n\to\infty}N(T^{n+p}(x)-T^n(x),t)=1$, namely $\{T^n(x)\}$ is a Cauchy sequence.

Since X is complete, we have that $\{T^n(x)\}$ is a convergent sequence. Thus $(\exists)z\in X$ such that $\lim\limits_{n\to\infty}T^n(x)=z$. We note that

$$z=\lim_{n\to\infty}T^{n+1}(x)=\lim_{n\to\infty}T(T^n(x))=T(z)\ .$$

Now we show the uniqueness. Suppose that there exist $z,y\in X,z\neq y$ with the property $z=T(z),y=T(y)$. As $z\neq y$, there exists $s>0$ such that $N(z-y,s)=a<1$. Then, for all $n\in\mathbb{N}$, we have

$$a=N(y-z,s)=N(T^n(y)-T^n(z),s)\geq N\left(y-z,\frac{s}{L^n}\right)\to 1\ .$$

Thus $a=1$, which contradicts our assumption. □

3 Fuzzy continuous linear operators

Theorem 17. *Let $T : X \to Y$ be a linear operator. Then T is fuzzy continuous on X, if and only if T is fuzzy continuous at a point $x_0 \in X$.*

Proof: " $\Rightarrow$ " It is obvious.
" $\Leftarrow$ " Let $y \in Y$ be arbitrary. We will show that T is fuzzy continuous at y. Let $\varepsilon > 0, \alpha \in (0,1)$. Since T is fuzzy continuous at $x_0 \in X$, there exist $\delta > 0$, $\beta \in (0,1)$ such that

$$(\forall)x \in X : N_1(x - x_0, \delta) > \beta \Rightarrow N_2(T(x) - T(x_0), \epsilon) > \alpha \, .$$

Replacing x by $x + x_0 - y$, we obtain that

$$(\forall)x \in X : N_1(x + x_0 - y - x_0, \delta) > \beta \Rightarrow N_2(T(x + x_0 - y) - T(x_0), \epsilon) > \alpha \, ,$$

namely

$$(\forall)x \in X : N_1(x - y, \delta) > \beta \Rightarrow N_2(T(x) - T(y), \epsilon) > \alpha \, .$$

Thus T is fuzzy continuous at $y \in Y$. As y is arbitrary, it follows that T is fuzzy continuous on $D(T)$. □

Corollary 18. *Let $T : X \to Y$ be a linear operator. Then T is fuzzy continuous on X, if and only if*

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\epsilon, \alpha) > 0, (\exists)\beta = \beta(\epsilon, \alpha) \in (0,1) \textit{such that}$$

$$(\forall)x \in X : N_1(x, \delta) > \beta \textit{ we have that } N_2(T(x), \varepsilon) > \alpha \, .$$

Theorem 19. *A linear operator $T : X \to Y$ is fuzzy continuous on X, if and only if $(\forall)\alpha \in (0,1), (\exists)\beta = \beta(\alpha) \in (0,1), (\exists)M = M(\alpha) > 0$ such that*

$$(\forall)t > 0, (\forall)x \in X \; : \; N_1(x, t) > \beta \Rightarrow N_2(T(x), Mt) > \alpha \, .$$

Proof: " $\Leftarrow$ " Let $\varepsilon > 0$, $\alpha \in (0,1)$ be arbitrary. Then there exist $\beta = \beta(\alpha) \in (0,1)$, $M = M(\alpha) > 0$ such that

$$(\forall)t > 0, (\forall)x \in X \; : \; N_1(x, t) > \beta \Rightarrow N_2(T(x), Mt) > \alpha \, .$$

In particular, for $t = \frac{\varepsilon}{M}$, we obtain

$$N_1\left(x, \frac{\varepsilon}{M}\right) > \beta \Rightarrow N_2(T(x), \varepsilon) > \alpha \, .$$

Applying Corollary 18, for $\delta = \frac{\varepsilon}{M} > 0$, we obtain that T is fuzzy continuous on X.

" $\Rightarrow$ " We suppose that $(\exists)\alpha_0 \in (0,1)$ such that

$$(\forall)\beta \in (0,1), (\forall)M > 0, (\exists)t_0 = t_0(\beta, M) > 0, (\exists)x_0 = x_0(\beta, M) \in X,$$

$$N_1(x_0, t_0) > \beta \text{ and } N_2(T(x), Mt_0) \le \alpha_0 \, .$$

The set $V_0 = \{y \in Y \; : \; N_2(y, t_0) > \alpha_0\}$ is an open neighborhood of 0_Y. We will prove that, for all neighborhood U of 0_X, we have $T(U) \not\subseteq V_0$, which contradicts the fuzzy continuity of T at 0_X. As $\{B(0, \beta, s)\}_{\beta \in (0,1), s > 0}$ is a fundamental system of neighborhoods of 0_X, it is enough to show that for all $\beta \in (0,1), s > 0$ we have $T(B(0, \beta, s)) \not\subseteq V_0$.

As $M > 0$ is arbitrary, we can chose $s = \frac{t_0}{M}$. We note that, for $z_0 = \frac{1}{M}x_0 \in X$, we have

$$N_1\left(z_0, \frac{t_0}{M}\right) = N_1\left(\frac{1}{M}x_0, \frac{t_0}{M}\right) = N_1(x_0, t_0) > \beta \, .$$

Hence $z_0 \in B\left(0,\beta,\frac{t_0}{M}\right)$. We will prove that $T(z_0) \notin V_0$, namely $N_2(T(z_0),t_0) \leq \alpha_0$. Indeed,

$$N_2(T(z_0),t_0) = N_2\left(T\left(\frac{1}{M}x_0\right),t_0\right) = N_2(T(x_0),Mt_0) \leq \alpha_0 \ .$$

□

Corollary 20. *A linear functional $f:(X,N_1,*) \to (\mathbb{C},N,\wedge)$ is fuzzy continuous, if and only if $(\exists)\beta \in (0,1), (\exists)M > 0$ such that*

$$(\forall)t > 0, (\forall)x \in X, N_1(x,t) > \beta \Rightarrow |f(x)| < Mt \ .$$

Proof: According to the previous theorem f is fuzzy continuous if and only if

$$(\forall)\alpha \in (0,1), (\exists)\beta \in (0,1), (\exists)M > 0 \text{ such that}$$

$$(\forall)t > 0, (\forall)x \in X \ : \ N_1(x,t) > \beta \Rightarrow N(f(x),Mt) > \alpha \ .$$

But

$$N(f(x),Mt) > \alpha \Leftrightarrow N(f(x),Mt) = 1 \Leftrightarrow |f(x)| < Mt \ .$$

Hence $(\exists)\beta \in (0,1), (\exists)M > 0$ such that

$$(\forall)t > 0, (\forall)x \in X, N_1(x,t) > \beta \Rightarrow |f(x)| < Mt \ .$$

□

Corollary 21. *Let $(X,N_1,*_1)$, $(Y,N_2,*_2)$ be FNLSs and*

$$p_\alpha(x) := \inf\{t > 0 \ : \ N_1(x,t) > \alpha\}, \alpha \in (0,1) \ ,$$

$$q_\alpha(x) := \inf\{t > 0 \ : \ N_2(x,t) > \alpha\}, \alpha \in (0,1) \ .$$

A linear operator $T: X \to Y$ is fuzzy continuous on X if and only if

$$(\forall)\alpha \in (0,1), (\exists)\beta = \beta(\alpha) \in (0,1), (\exists)M = M(\alpha) > 0$$

$$\textit{such that}\ q_\alpha(Tx) \leq Mp_\beta(x) \ , \ (\forall)x \in X \ .$$

Proof: According to the previous theorem,

$$(\forall)\alpha \in (0,1), (\exists)\beta = \beta(\alpha) \in (0,1), (\exists)M = M(\alpha) > 0 \text{ such that}$$

$$(\forall)t > 0, (\forall)x \in X \ : \ N_1(x,t) > \beta \Rightarrow N_2(T(x),Mt) > \alpha \ .$$

Thus, for $x \in X$, we have

$$\{t > 0 \ : \ N_1(x,t) > \beta\} \subseteq \{t > 0 \ : \ N_2(Tx,Mt) > \alpha\} \ .$$

Hence

$$\inf\{t > 0 \ : \ N_1(x,t) > \beta\} \geq \inf\{t > 0 \ : \ N_2(Tx,Mt) > \alpha\} \ ,$$

namely $\inf\{t > 0 \ : \ N_1(x,t) > \beta\} \geq \inf\left\{\frac{t}{M} > 0 \ : \ N_2(Tx,t) > \alpha\right\}$. Therefore

$$p_\beta(x) \geq \frac{1}{M}q_\alpha(Tx), \forall)x \in X \ .$$

□

Corollary 22. *A linear functional* $f : (X, N_1, *) \to (\mathbb{C}, N, \wedge)$ *is fuzzy continuous, if and only if* $(\exists)\beta \in (0,1), (\exists)M > 0$ *such that*

$$|f(x)| \leq M p_\beta(x), (\forall)x \in X .$$

Remark 23. We note that a subset A of a topological linear space X is said to be bounded if for every neighbourhood V of 0_X, there exists a positive number k such that $A \subset kV$. A linear operator $T : X \to Y$ is said to be bounded if T maps bounded sets into bounded sets. Based on this remark the following definitions are natural.

Definition 24. [12] A subset A of X is called fuzzy bounded, if $(\forall)\alpha \in (0,1)$, $(\exists)t_\alpha > 0$ such that $A \subset B(0, \alpha, t_\alpha)$.

Definition 25. [12] A linear operator $T : X \to Y$ is said to be fuzzy bounded if T maps fuzzy bounded sets of X into fuzzy bounded sets of Y.

We must note that the following result was established by I. Sadeqi and F.S. Kia [12] for FNLSs of type $(X, N, \wedge)$ which satisfy $(N7)$. Since the proof is entirely the same as in [12], it is omitted.

Theorem 26. *Let* $T : X \to Y$ *be a linear operator. The following sentences are equivalent:*

1. T *is fuzzy continuous;*
2. T *is topological continuous;*
3. T *is fuzzy bounded;*

4 Conclusion

As fuzzy continuity and topological continuity are equivalent and since FNLSs are metrizable topological linear spaces, all results and theorems in topological linear spaces hold for FNLSs. Particularly, we can obtain fuzzy versions for the classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem). This remark was made by I. Sadeqi and F.S. Kia [12] for FNLSs of type $(X, N, \wedge)$. Based on our results, these principles remain true without assuming $(N7)$ as in [12].

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