

On generalized Szasz-Bernstein –Type Operators

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Abstract

Resently Dr.R.P. Pathak and Shiv Kumar Sahoo in 2012 intrudes a new modified Szasz-Bernstein –type operators, in the present paper, we introduce generalize Szasz- Bernstein- type operators $\varphi B_n(f; x)$, we proved that the operators are converge to the function being approximation. In addition, we establish a Voronovaskaja- type asymptotic formula for this operators .

Keywords: Linear positive operators, Korovkin theorem, Voronovaskaja- type asymptotic formula.

الخلاصة

مؤخرا باناك وكمرفي 2012 قدم مؤثرات جديدة من نوع زاز-برنستين المحسن، في البحث الحالي نقدم تعميم هذا البحث نقدم تعميم مؤثرات من نوع زاز برنستين $\varphi B_n(f; x)$ ، برهنا ان هذه المؤثرات تتقارب الى دالة التقريب بالإضافة الى ذلك نا قشنا صيغة فرونوفسكي لتلك المؤثرات.

الكلمات المفتاحية: مؤثرات خطية موجبة، مبرهنة كوروفكن، صيغة فرونوفسكي.

1. Introduction

In [Deo *et. al.*, 208 Introducing a new Bernstein type special operators $B_n(f; x)$

defined as : $B_n(f; x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right)$,

where, $p_{n,k}(x) = \left(1 + \frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}$, $0 \leq x \leq \frac{n}{n+1}$ (1.1)

Moreover, given the integral modification defined as:

$$L_x(f; x) = n \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^n p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt \quad (1.2)$$

In (Mortici, 2009) defines a new class of linear positive operators which generalize the Szasz-Mirakjan operators for the analytic function $\varphi: \mathbb{R} \rightarrow [0, \infty)$ and

$\varphi S_n: C^2([0, \infty)) \rightarrow C^\infty([0, \infty))$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ as:

$$\varphi S_n(f; x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k f\left(\frac{k}{n}\right) f \in C^2([0, \infty)). \quad (1.3)$$

Where $C^2([0, \infty)) = \left\{ f \in C([0, \infty)) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ exist and is finite} \right\}$

If $\varphi(y) = e^y$, we have the classical Szasz-Mirakjan operators.

In (Pathak and Shiv, 2012) introduce the new modified operators as:

$$B_n(f; x) = n \left(1 + \frac{1}{n}\right)^2 e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt. \quad (1.4)$$

Where, $x, t \in [0, \frac{n}{n+1})$.

In this paper, we investigate the new sequence of linear positive operators $\varphi B_n(f; x)$ define as:

$$\varphi B_n(f; x) = \frac{n}{\varphi(nx)} \left(n + \frac{1}{n} \right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt. \quad (1.5)$$

We will prove that the operators are converging to the function being approximation.

In addition, we discuss a Voronovaskaja- type asymptotic formula.

2. Preliminary Results

Lemma 1. [Morti2]

The φ Szasz-Mirakjan operators satisfies the following relation:

$$1) \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(1)}(nx)}{k!} (nx)^k k = nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \quad (2.1)$$

$$2) \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(1)}(nx)}{k!} (nx)^k k^2 = n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \quad (2.2)$$

$$3) \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(1)}(nx)}{k!} (nx)^k k^3 = n^3 x^3 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 3n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \quad (2.3)$$

$$4) \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(1)}(nx)}{k!} (nx)^k k^4 = n^4 x^4 \frac{\varphi^{(4)}(nx)}{\varphi(nx)} + 6n^3 x^3 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 7n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \quad (2.4)$$

Lemma 2.

For any $x \in [0, \infty)$, $n \in \mathbb{N}$ and $e_i = x^i, i = 1, 2, 3, 4$

the operators φB_n satisfies the following relations:

$$1) \varphi B_n(e_0; x) = 1 \quad (2.5)$$

$$2) \varphi B_n(e_1; x) = \frac{n}{(n+1)(n+2)} \left[nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right] \quad (2.6)$$

$$3) \varphi B_n(e_2; x) = \frac{n^2}{(n+1)^2(n+2)(n+3)} \left[n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 4nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 2 \right] \quad (2.7)$$

$$4) \varphi B_n(e_3; x) = \frac{n^3}{(n+1)^3(n+2)(n+3)(n+4)} \left[n^3 x^3 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 9n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 18nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 6 \right] \quad (2.8)$$

$$5) \varphi B_n(e_4; x) = \frac{n^4}{(n+1)^4(n+2)(n+3)(n+4)(n+5)} \left[n^4 x^4 \frac{\varphi^{(4)}(nx)}{\varphi(nx)} + 16n^3 x^3 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 72n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 96nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 24 \right] \quad (2.9)$$

Proof.

$$\begin{aligned}
 1) \quad \varphi B_n(e_0; x) &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t) dt \\
 &= \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \\
 &= 1 \\
 2) \quad \varphi B_n(e_1; x) &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t) t dt \\
 &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^{k+1} \\
 &\quad \times \left(\frac{n}{n+1} - t\right)^{n-k} dt \\
 &= \frac{n}{\varphi(nx)} \left(n + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \frac{k+1}{n(n+2)} \left(\frac{n}{n+1}\right)^3 \\
 &= \frac{n}{(n+1)(n+2)} \left[nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right] \\
 3) \quad \varphi B_n(e_2; x) &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t) t^2 dt \\
 &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^{k+2} \\
 &\quad \times \left(\frac{n}{n+1} - t\right)^{n-k} dt \\
 &= \frac{n}{\varphi(nx)} \left(n + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \frac{(k+1)(k+2)}{n(n+2)} \left(\frac{n}{n+1}\right)^4 \\
 &= \frac{n^2}{(n+1)^2(n+2)(n+3)} \left[n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 4nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 2 \right] \\
 4) \quad \varphi B_n(e_3; x) &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t) t^3 dt \\
 &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^{k+3} \\
 &\quad \times \left(\frac{n}{n+1} - t\right)^{n-k} dt \\
 &= \frac{n}{\varphi(nx)} \left(n + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \frac{(k+1)(k+2)(k+3)}{n(n+2)} \left(\frac{n}{n+1}\right)^5
 \end{aligned}$$

$$= \frac{n^3}{(n+1)^3(n+2)(n+3)(n+4)} \left[n^3 x^3 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 9n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 18nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 6 \right]$$

$$\begin{aligned} 5) \varphi B_n(e_4; x) &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t) t^4 dt \\ &= \frac{n}{\varphi(nx)} \left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^{k+4} \\ &\quad \times \left(\frac{n}{n+1} - t\right)^{n-k} dt \\ &= \frac{n}{\varphi(nx)} \left(n + \frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \frac{(k+1)(k+2)(k+3)(k+4)}{n(n+2)} \left(\frac{n}{n+1}\right)^6 \\ &= \frac{n^4}{(n+1)^4(n+2)(n+3)(n+4)(n+5)} \left[n^4 x^4 \frac{\varphi^{(4)}(nx)}{\varphi(nx)} + 16n^3 x^3 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 72n^2 x^2 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 96nx \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 24 \right] \end{aligned}$$

Lemma 3.

For the operators $\varphi B_n(f; x)$ get the following relation:

1) $\varphi B_n((t - x); x) =$

$$\frac{n \left[nx \left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) + (1 - 3x) \right] - 2x}{(n+1)(n+2)} \tag{2.10}$$

2) $\varphi B_n((t - x)^2; x) = \frac{n^2}{(n+1)^2(n+2)(n+3)} \left[n^2 x^2 \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) + n \left\{ \left(7 - 8 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) x^2 + \left(4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 2 \right) x \right\} + \left((17 - 6 \frac{\varphi^{(1)}(nx)}{\varphi(nx)}) x^2 - 8x + 2 \right) \right]$

$$+ n(17x^2 - 6x) + 6x^2 \tag{2.11}$$

3) $\varphi B_n((t - x)^3; x) =$

$$o\left(\frac{1}{n}\right) \tag{2.12}$$

4) $\varphi B_n((t - x)^4; x) =$

$$o\left(\frac{1}{n^2}\right) \tag{2.13}$$

Proof.

By little calculations, we get required results (2.10) to (2.13).

3. Main Results

If the function φ verifies

$$\lim_{y \rightarrow \infty} \frac{\varphi^{(1)}(y)}{\varphi(y)} = \lim_{y \rightarrow \infty} \frac{\varphi^{(2)}(y)}{\varphi(y)} = 1 \tag{3.1}$$

Then the following convergence theorem holds:

Theorem 1.

For $f \in C \left[0, \frac{n}{n+1}\right]$, the sequence of linear positive operators $\varphi B_n(f; x)$ is converges uniformly to f as $n \rightarrow \infty$

Proof.

From (2.5), (2.6) and (2.7) we have:

$$\begin{aligned} \varphi B_n(e_0; x) &\rightarrow e_0 && \text{Uniformly as } n \rightarrow \infty \\ \varphi B_n(e_1; x) &\rightarrow e_1 && \text{Uniformly as } n \rightarrow \infty \\ \varphi B_n(e_2; x) &\rightarrow e_2 && \text{Uniformly as } n \rightarrow \infty \end{aligned}$$

Then from Korovkin theorem, we get:

$$\varphi B_n(f; x) \rightarrow f \text{ Uniformly as } n \rightarrow \infty .$$

Remark. Taking (3.1) into account, for $x \in \left[0, \frac{n}{n+1}\right]$ we have

$$\lim_{n \rightarrow \infty} \left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) = 0$$

And

$$\lim_{n \rightarrow \infty} \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) = 0$$

In the following, we assume there exist γ so that $0 < \gamma \leq 1$ and the function φ verifies the conditions:

$$\lim_{n \rightarrow \infty} n^\gamma \left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) = \sigma_1(x) \tag{3.2}$$

$$\lim_{n \rightarrow \infty} n^\gamma \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) = \sigma_2(x) \tag{3.3}$$

Where σ_1, σ_2 are functions and $\sigma_1, \sigma_2: [0, \mathbb{R}) \rightarrow \mathbb{R}$

Theorem 2.

For f be an integrable bounded function on $\left[0, \frac{n}{n+1}\right]$ and for $f^{(2)}$ exist at a point

$x \in \left[0, \frac{n}{n+1}\right]$, then

$$\lim_{n \rightarrow \infty} n[\varphi B_n(f; x) - f(x)] = \left\{ \sigma_2(x)x^2 + \left(7 - 8 \frac{\varphi^{(1)}(nx)}{\varphi(nx)}\right) x^2 + \left(4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 2\right) x \right\} \frac{f^{(2)}(x)}{2} + \{\sigma_1(x)x + (1 - 3x)\} f^{(1)}(x)$$

Proof. By Tylor expansion for the function, $f(t)$ we get:

$$f(t) = f(x) + (t - x)f^{(1)}(x) + \frac{(t-x)^2}{2} f^{(2)}(x) + (t - x)^2 \xi(t - x) \quad (3.4)$$

Where $\xi(t - x) \rightarrow 0$ as $t \rightarrow x$

So to give $\varepsilon > 0$, there exist $\delta > 0$ such that $|\xi(t - x)| \leq \varepsilon$ whenever $|t - x| \leq \delta$

Weill, if $|x - t| > \delta$ then there exist $K > 0$ such that:

$$|\xi(t - x)| \leq K \leq K \frac{(t - x)^2}{\delta^2}$$

Then for any $t \in \left[0, \frac{n}{n+1}\right]$, we have

$$|\xi(t - x)| \leq \varepsilon + K \frac{(t - x)^2}{\delta^2} \quad (3.5)$$

Now, applying φB_n on (3.2), we have

$$\begin{aligned} \varphi B_n(f; x) &= f(x) + \varphi B_n((t - x),) f^{(1)}(x) + \varphi B_n((t - x)^2, x) \frac{f^{(2)}(x)}{2} \\ &\quad + \varphi B_n((t - x)^2 \xi(t - x); x) \\ &= f(x) + \frac{1}{(n+1)(n+2)} \left[n^2 \left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) + n(1 - 3x) - \right. \\ &\quad \left. 2x \right] f^{(1)}(x) \\ &\quad + \frac{1}{(n+1)^2(n+2)(n+3)} \left[n^4 \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + \right. \right. \\ &\quad \left. \left. 1 \right) x^2 \right. \end{aligned}$$

$$\begin{aligned}
 & +n^3 \left\{ \left(7 - 8 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) x^2 + \left(4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - \right. \right. \\
 & \left. \left. 2 \right) x \right\} \\
 & +n^2 \left\{ \left(17 - 6 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) - 8x + 2 \right\} \\
 & +n(17x^2 - 6x) + 6x^2 \left] \frac{f^{(2)}(x)}{2} \right. \\
 & \left. +n \left(1 + \frac{1}{n} \right)^2 \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2 \xi(t-x) dt \right.
 \end{aligned}$$

Multiplying by n , we get:

$$\begin{aligned}
 & n[\varphi B_n(x) - f(x)] \\
 & = \frac{n}{(n+1)(n+2)} \left[n^2 \left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) + n(1-3x) - 2x \right] f^{(1)}(x) \\
 & \quad + \frac{n}{(n+1)^2(n+2)(n+3)} \left[n^4 \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) x^2 \right. \\
 & \quad \left. + n^3 \left\{ \left(7 - 8 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) x^2 + \left(4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 2 \right) x \right\} \right. \\
 & \quad \left. + n^2 \left\{ \left(17 - 6 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) - 8x + 2 \right\} \right. \\
 & \quad \left. + n(17x^2 - 6x) + 6x^2 \right] \frac{f^{(2)}(x)}{2} + nE_n(t, x) \quad (3.6)
 \end{aligned}$$

$$\begin{aligned}
 E_n(t, x) & = n \left(1 + \frac{1}{n} \right)^2 \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2 \xi(t-x) dt \\
 |nE_n(t, x)| & = \left| n \left\{ n \left(1 + \frac{1}{n} \right)^2 \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2 \xi(t-x) dt \right\} \right|
 \end{aligned}$$

$$\left. n \left\{ n \left(1 + \frac{1}{n} \right)^2 \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2 |\xi(t-x)| dt \right\} \right\} \quad (3.7)$$

Then, from (3.3) we have:

$$\begin{aligned}
 |nE_n(t, x)| & \leq n\varepsilon \varphi B_n((t-x)^2; x) + \frac{K}{\delta^2} \varphi B_n((t-x)^4; x) \\
 & \leq n\varepsilon o\left(\frac{1}{n}\right) + \frac{nK}{\delta^2} o\left(\frac{1}{n^2}\right) \\
 & \leq \varepsilon + \frac{K}{\delta^2} o\left(\frac{1}{n}\right)
 \end{aligned}$$

Let $\delta = n^{-1/4}$ we get:

$$|nE_n(t, x)| \leq \varepsilon + Ko \left(\frac{1}{\sqrt{n}} \right)$$

Since ε is arbitrary and small, we get

$$|nE_n(t, x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.8)$$

Then,

$$\lim_{n \rightarrow \infty} n [\varphi B_n(x) - f(x)] = \lim_{n \rightarrow \infty} \frac{n}{(n+1)(n+2)} \left[n^2 \left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) + n(1 - 3x) - 2x \right] f^{(1)}(x)$$

$$+ \lim_{n \rightarrow \infty} \frac{n}{(n+1)^2(n+2)(n+3)} \left[n^4 \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) x^2 + n^3 \left\{ \left(7 - 8 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) x^2 + \left(4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 2 \right) x \right\} + n^2 \left\{ \left(17 - 6 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) - 8x + 2 \right\} \right]$$

$$+ n(17x^2 - 6x) + 6x^2 \left] \frac{f^{(2)}(x)}{2} + \lim_{n \rightarrow \infty} n E_n(t, x)$$

from (3.2), (3.3) and (3.8), we get :

$$\lim_{n \rightarrow \infty} n[\varphi B_n(f; x) - f(x)] = \left\{ \sigma_2(x)x^2 + \left(7 - 8 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} \right) x^2 + \left(4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 2 \right) x \right\} \frac{f^{(2)}(x)}{2} + \{ \sigma_1(x)x + (1 - 3x) \} f^{(1)}(x)$$

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