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Spherical Approximation on Unit Sphere

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Abstract

In this paper we introduce a Jackson type theorem for functions in L_p spaces on sphere And study on best approximation of functions in L_p spaces defined on unit sphere.

our central problem is to describe the approximation behavior of functions in L_p spaces for $p < 1$ by modulus of smoothness of functions.

Keywords : Modulus of smoothness , orthogonal matrices on R^d , approximation of functions in L_p spaces defined on unit sphere .

الخلاصة

في هذا البحث نقدم نظرية جاكسون للدوال في الفضاءات L_p عندما $0 < p < 1$ على كرات الوحدة و دراسة افضل تقريب للدوال في الفضاءات L_p المعرفه على كرات الوحدة مشكلتنا الاساسية هنا هي وصف عملية التقريب لسلوك الدوال في الفضاءات L_p, $p < 1$ بأستخدام مقاييس معامل النعومة للدالة

الكلمات المفتاحيه: مقاييس النعومة , المصفوفات المتعامدة في الفضاء $\mathbf{R}^d$, تقريب الدوال في الفضاءات L_p المعرفة على كرات الوحدة.

We need the following definitions from Ditzian,2008

Definitions 1.1

For $L_p(S^{d-1})$, we denote thespace of functions on the sphere
$(S^{d-1}) = \{x = (x_1, x_2, \ldots, x_d): x_1^2 + x_2^2 + \cdots + x_d^2 = 1\}$.

Let $f: S^d \to R$,

For functions on S^{d-1} in the function spaces $L_p(S^{d-1})$, $p < 1$, we define the quasinorm

$$\|f\|_{L_p(S^{d-1})} = \left(\int_{S^d} |f(x_1, x_2, \ldots, x_d)|^p \, dx_1 dx_2 \ldots dx_d\right)^{\frac{1}{p}}$$

Nawmodulus of smoothness $\omega_r(f,t)_{L_p(S^{d-1})}$ where is recently introduced inDitzian,1999. $\omega_r(f,t)_{L_p(S^{d-1})}$ is given by

$$\omega_r(f,t)_{L_p(S^{d-1})} = \sup\left\{\left\|\Delta_\rho^r f\right\|_{L_p(S^{d-1})} : \rho \in O_t\right\}, t \geqq 0$$

Such that $\Delta_\rho f(x) = f(\rho x) - f(x), \Delta_\rho^r f(x) = \Delta_\rho\left(\Delta_\rho^{r-1} f(x)\right)$

$\Delta_\theta^r f = (TQ - I)^r f$

If $r = 1 \Delta_\theta^1 f = (TQ - I) f$

$TQf(x) = f\left(Q(x)\right), Q \epsilon So(d)$

$O_t = \{\rho \in So(d): \max_{x \epsilon S^{d-1}} [\rho x . x \geq cost\}]$

And $\mathbf{So}(\square)$ is the groupof orthogonal matrices of $d \times d$ real entries with determinant 1

Let M_0 be the $d \times d$ (even) matrix given by

$$M\theta = \begin{bmatrix} \cos\theta\sin\theta & & 0 \\ -\sin\theta\cos\theta & & \\ & \ddots & \\ 0 & & \cos\theta\sin\theta \\ & & -\sin\theta\cos\theta \end{bmatrix}$$

Clearly $M_0 = I, (M_\theta)^j = M_{j\theta}$ and $(M_\theta)^{-1} = M_{-\theta}$

$$S_{(\theta,p)}f(x) = \frac{1}{m_\theta}\left(\int_{xy=\cos\theta} |f(y)|^p \, d\gamma(y)\right)^{\frac{1}{p}}, \qquad S_\theta I = I,$$

where $d\gamma$ is the measure on the set $\{y \in S^{d-1}: x.y = \cos\theta\}$ that it causes by the Lebesgue measure on $S^{\intercal}(d-2)$ $(\{y: x.y = \cos\theta\}$ is an isometric map of dilation on $S^{\intercal}(d-2)\})$.

and m_θ is given by $S_\theta I = I;$

$\tilde{\Delta}$ is the Laplace - Beltrami differential operator given, using the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

$\tilde{\Delta} f(x) = \Delta F(x)$ for $x \in S^{d-1}$,where $F(x) = f\left(\frac{x}{|x|}\right)$

If $f \in L_p(S^{d-1})$ The K- functional of f defined as

$$K_{2\alpha}(f, \tilde{\Delta}, t^{2\alpha})_{L_p(S^{d-1})} \cong \inf\left(\|f-g\|_{L_p(S^{d-1})} + t^{2\alpha}\left\|(-\tilde{\Delta})^{\alpha} g\right\|_{L_p(S^{d-1})} : (-\tilde{\Delta})^{\alpha} g \in L_p(S^{d-1})\right)$$

$$S_\theta f(x) = \frac{1}{m(\theta)}\int_{\{y \in S^{d-1}: x.y=\cos\theta\}} f(y)\,d\gamma(y),$$

$$S_\theta I = I, x \in S^{d-1}$$

$$m_\ell(u) = \frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j} m(ju)$$

2.AuxiliaryResults

In this papper we need the following lemmas.

Lemma 2.1 Ditzian,2004

For $f \in L_p(S^{d-1}), p < 1$

We have $\int \quad \left|f\left(Q^{-1}M_\theta Q_x\right|^p dQ\right)^{1/\square}$

Where $S_{\theta,p}(f) = \frac{1}{m\theta}\left(\int_{xy=\cos\theta} |f(y)|^p \, d\gamma(y)\right)^{1/p}$

Lemma 2.2

For $f \in L_p(S^{d-1}), p < 1$ then

$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$, for any $\rho \in So(d)$

Proof

Since , ,so that
$\rho \in S$,then
$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f\|_{L_p(S^{d-1})}$
and
$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ ■

Lemma 2 .3 Ditzian,2008

An operation by an element of $so(d)$is an isometry and in most situations under thecondition
$\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0 as \|\rho - I\| \to 0$,
where $\|\rho - \eta\|^2 = \max_{x \in S^{d-1}} \left((\rho x - \eta x).(\rho x - \eta x)\right)$. (Note that $\max(\rho x . x) \geqq cos$ is equivalent to $\|\rho - I\| \leqq 2\left|\frac{\sin t}{2}\right|$)

Lemma 2.4 Ditzian ,2004
For an integer ℓ we have

$$\binom{2\ell}{\ell} + 2\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell - j} cosj\theta = 4^\ell sin^{2\ell}\frac{\theta}{2} .$$

Lemma 2.5 Ditzian,2004

$$S_\theta f(x) = \frac{1}{m(\theta)}\int_{\{y \in S^{d-1}: x.y = cos\theta\}} f(y)d\gamma(y), S_\theta I = I,$$

$x \in S^{d-1}$

For $S_{j,\theta}f$ given in

$$S_{j,\theta}f(x) = \frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell - j}S_{j\theta}f(x)$$

$(S_{j,\theta}f)^\wedge(x) \equiv m_l(2\pi\ell\,|x|)\hat{f}(x)$ and

$$1 - m_l(u) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\frac{4^l}{\binom{2l}{l}}\int_0^1\left(\frac{\sin us}{2}\right)^{2\ell}(1 - s^2)^{\frac{d-3}{2}}ds$$

.

Lemma 2.6 Ditzian ,2004
For $0 < u \leq \pi$, $0 < C_1u^{2l} \leq 1 - m_l(u) \leq C_2u^{2\ell}$,
For $u \geq \pi$, $0 < m_l(u) \leq S_{j,\theta} < 1.$
Lemma2.7 Ditzian,2004

For

$$\left|\left(\frac{d}{du}\right)^j m_l(u)\right| \leq C_{\ell,j}\left(\frac{1}{1+u}\right)^{\frac{d-1}{2}}$$

Lemma2.8

Proof

Following the proof of lemma 3.17 step by step in Stein,G.Weiss,1971 we can get the proof ■

3. The Main Results

In this paper we introduce our main results

Theorem 3.1

For $f \in L_p(S^{d-1}), p < 1,$ we have

(i) $S_{(\theta,p)} f \in L_p(S^{d-1})$

(ii) $\left\| S_{(\theta,p)} f \right\|_{L_p(S^{d-1})} \le C(p) \|f\|_{L_p(S^{d-1})}$

(iii) $\left\| S_{(\theta,p)} f - f \right\|_{L_p(S^{d-1})} \le C(p) \omega_r(f,\theta)_{L_p(S^{d-1})}$

Proof of

(i) Using $\|f(\rho)\|_{L_p(S^{d-1})} = \|f\|_{L_p(S^{d-1})}$

We directly get $S_{(\theta,p)} f \in L_p(S^{d-1})$

(ii) Now we use $\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ and hence $\|f(Q^{-1} M_\theta Q_x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ to obtain

$$\left\| S_{(\theta,p)} f \right\|_{L_p(S^{d-1})} = \left\| \int_{so(d)} |f(Q^{-1} M_\theta Q_x)|^{pr} dQ \right\|_{L_p(S^{d-1})}$$

For $Q^{-1} M_\theta Q = \rho$

$$\left(\int_{so(d)} \left| f\left(Q^{-1} M_\theta Q_x \right|^{pr} dQ \right)^{\frac{1}{p}} \right|^{p} \Bigg)^{\frac{1}{\square}})$$

$$= \left(\int_{S^{d-1}} \int_{so(d)} |f(Q^{-1} M_\theta Q_x)|^{pr} dQ\, dx \right)^{\frac{1}{p}}$$

$$\le C(p) \left(\int_{S^{d-1}} |f(\rho x)|^{pr} dx \right)^{1/p}$$

$$\le C(p) \|f^r\|_{L_p(S^{d-1})}$$

(iii) The inequality $\|S_\theta f - f\|_{L_p(S^{d-1})} \le c(p) \omega_r(f,\theta)_{L_p(S^{d-1})}$ follows from to abovewhere

$\Delta_p f(x) = f(\rho x) - f(x)$

for $r = 1$ to show that the remainder of

$\|S_\theta f - f\|_{L_p(S^{d-1})} \le C(p) \omega_r(f,\theta)_{L_p(S^{d-1})}$

we note that

$$\int_{So(d)} f(Q^{-1} M_\theta Q_x) dQ = \int_{So(d)} f(Q^{-1} M_{-\theta} Q_x) dQ$$

$$= \int_{So(d)} f(Q^{-1} M_{-\theta} Q_x^{-1}) dQ$$

and use $\left\|\Delta_\rho^r f\right\|_{L_p(S^{d-1})} = \|(TQ - I)^r\|_{L_p(S^{d-1})}$

$$= \left\|(T-I)^s \sum_{m=0}^{r-s} \binom{r-s}{m} (T-I)^m T^{r-s-m}\right\|_{L_p(S^{d-1})}$$

For $T_\rho f(x) = f(\rho x), \rho = Q^{-1} M_\theta Q$ and $\rho^{-1} = Q M_{-\theta} Q^{-1}$ ■

Theorem 3.2

If $f \in L_p(S^{d-1}), p \le 1$ and $\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$,for any $\rho \in So(d)$, $\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0$ as $|\rho - I| \to 0$

where $|\rho - \eta|^2 = \max_{x \in S^{d-1}} \left((\rho x - \eta x).(\rho x - \eta x)\right).$ (Not that $\max(\rho x - x) \ge cos\square$ is equivalent to $|\rho - I| \le 2\left|\frac{\sin t}{2}\right|$)

then $\left\|C_n^\delta(f,x)\right\|_{L_p(S^{d-1})} \le C(p)\|f\|_{L_p(S^{d-1})}$

Proof

In fact $C_n^\delta(f,x) = \int_0^\pi M_n^\delta(\theta) S_\theta(f,x) d\theta$ is known for $f \in L_p(S^{d-1})$

$$\left\|C_n^\delta(f,x)\right\|_{L_p(S^{d-1})} = \left\|\int_0^\pi M_n^\delta(\theta) S_\theta(f,x) d\theta\right\|_{L_p(S^{d-1})}$$

$$\le \left\|\int_0^\pi M_n^\delta(\theta) S_\theta(f,x) d\theta\right\|_{L_1(S^{d-1})}$$

$$\le \int_0^\pi M_n^\delta(\theta) \|S_\theta(f,x)\|_{L_1(S^{d-1})} d\theta$$

$$= \|S_\theta(f,x)\|_{L_1(S^{d-1})}$$

$$= \left(\int_{S^{d-1}} [|S_\theta(f,x)|dx)]\right.$$

$$= \left(\int_{S^{d-1}} |S_\theta|^{1-p+p}\right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{S^{d-1}} |S_\theta|^p |S_\theta|^{1-p}\right)^{\frac{1}{p}} \left(\int_{S^{d-1}} |S_\theta|^{1-p+p}\right)^{1-\frac{1}{p}}$$

$$\le C(p)\left(\int_{S^{d-1}} |S_\theta|^p\right)^{\frac{1}{p}}$$ using

$S_{(\theta,p)} f = \frac{1}{m_\theta}\left(\int |f(y)|^p d\gamma(y)\right)^{\frac{1}{p}}$ we get

$\left\|C_n^\delta(f,x)\right\|_{L_p(S^{d-1})} \le C(p)\|f\|_{L_p(S^{d-1})}$ ■

Theorem 3.3

Let $f \in L_p(S^{d-1})$ for even $d > 3$, with $\|f(\rho x)\|_{L_p(S^{d-1})} = \|f\|_{L_p(S^{d-1})}$ for each $\rho \in So(d)$ and $\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0$ as $|\rho - I| \to 0$ then

$$\left\|C_n^\delta - f\right\|_{L_p(S^{d-1})} \le C(p,\theta)\omega_r(f,\mu)_{L_p(S^{d-1})}.\omega_r([f,\mu)]_\infty.\omega_r(f,\mu)_\infty^{\frac{-1}{p}}$$

Proof

$$\left\|C_n^\delta - f\right\|_{L_p(S^{d-1})} \leq \left\|\int_0^\pi \mu_n^\delta(\theta)\left|[(S]_\theta f - f)d\theta\right|\right\|_{L_1(S^{d-1})}$$

$$\leq \int_0^\pi \mu_n^\delta(\theta)\|S_\theta(f) - f\|_{L_1(S^{d-1})}d\theta$$

Then using the fact that $\int_0^\pi M_n^\delta(\theta)d\theta = 1$ We get

$$\left\|C_n^\delta - f\right\|_{L_p(S^{d-1})} \leq \left(\int_{S^{d-1}}|S_\theta(f) - f|^{p-p+1}\,dx\right)^{\frac{1}{p}-\frac{1}{p}+1}$$

$$\leq \left(\int_{S^{d-1}}|S_\theta(f) - f|^p|S_\theta(f) - f|^{p+1}\right)^{\frac{1}{p}}\left([\int_{S^{d-1}}|S_\theta(f) - f|\,d\theta\right)]^{1-\frac{1}{p}}$$

$$\int_{S^{d-1}}|S_\theta(f) - f|^p\,\|S_\theta(f) - f\|_\infty\|S_\theta(f) - f\|_\infty$$

$$\leq C(p,\theta)\|f - s_\theta(f)\|_{Lp(S^{d-1})}\omega_r(f,\mu)_\infty\omega_r(f,\mu)_\infty^{\frac{-1}{p}}$$

$$\leq C(p,\theta)\omega_r(f,\mu)_p\,\omega_r(f,\mu)_\infty\omega_r(f,\mu)_\infty^{\frac{-1}{p}}\ \blacksquare$$

Theorem3.4

If$\in L_p(S^{d-1})$, then

$$E_n([f)]_{L_p(S^{d-1})} \leq c(p)K_{2\alpha}(f,\tilde{\Delta},n^{-2\alpha})_{L_p(S^{d-1})},\ \alpha > 0$$

proof

$$E_n(f) \leq \|f - g\|_{L_p(S^{d-1})}$$

$$\leq c(p)\|f - g\|_{L_p(S^{d-1})} + t^{2\alpha}\left\|\left(-\tilde{\Delta}\right)^\alpha g\right\|_{L_p(S^{d-1})} : \left(-\tilde{\Delta}\right)^\alpha g \in L_p(S^{d-1})$$

$$\leq c(p)K_{2\alpha}(f,\tilde{\Delta},n^{-2\alpha})_{L_p(S^{d-1})}\qquad\blacksquare$$

Theorem 3.5

If $f \in L_p(S^{d-1})$ with even $d > 3$ with

$\|f(\rho)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ for any $\rho \in So(d)$

$\|f(\rho) - f(x)\|_{L_p(S^{d-1})} \to 0\ as|\rho - I| \to 0\ for\theta \leq \frac{\pi}{2\ell}$

Then $$\left\|f + \frac{2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell}S_{j,\theta}f\right\|_{L_p(S^{d-1})} \cong K_{2\ell}(f,\tilde{\Delta},\theta^{2\ell})_{L_p(S^{d-1})}$$

Proof

by definition of the K—functional with

$$\left\|f + \frac{2}{\binom{2\ell}{\ell}\Sigma(-1)^j\binom{2\ell}{\ell}S_{j,\theta}f}\right\|_{L_p(S^{d-1})}$$

$$\|f - S_j\theta\|_{L_p(S^{d-1})} + t\|-\tilde{\Delta}S_\theta f\|_{L_p(S^{d-1})})$$

We realize the operator$\eta_{\alpha\theta}(f)$ using the function$\eta(x)$ satisfying $\eta(x)\epsilon C^\infty(R_+)$,

for$0 \leq x \leq 1$, and$\eta(x) = 0$ for$x \geq 2$ the operator$\eta_{\alpha\theta}(f)$ is given by

$$\eta_{\alpha\theta}(f) = \sum_{k=0}^\infty \eta(\alpha\,\theta k)P_k(f)$$,where $$f \sim \sum_{k=0}^\infty P_k(f)$$

for

and the definition of the K- functional $K_\ell\left(f,\Delta,t^{2\ell}\right)_{L_p(S^{d-1})}$ we just have to show for all $f \in L_p(R^d)$ andsome fixed $\propto > 0$

(as $K_\ell\left(f,\Delta,t^{2\ell}\right)_{L_p(S^{d-1})} \approx K_\ell(f,\Delta,\propto^{-2\ell}t^{2\ell})_{L_p(S^{d-1})}$

displaythat

(i) $\left\|f - S_{j,\theta}f\right\|_{L_p(S^{d-1})} \geq C_1 \left\|f - \eta_{\propto\square/t}f\right\|_{L_p(S^{d-1})}$

(ii) $\left\|f - S_{j,\theta}f\right\|_{L_p(S^{d-1})} \geq C_2\, t^{2\ell} \left\|\Delta^\ell \eta_{\propto\square/t}f\right\|_{L_p(S^{d-1})}$

Toprove i- it sufficient to show

$f - S_{j,\theta}f\|_{L_p(S^{d-1})})$

$\leq C_4 \left\|f - S_{j,\theta}f\right\|_{L_p(S^{d-1})}$ (1)

since' $as\eta_{1/t}andS_{j,\theta}$ arebounded multiplier operators on $L_p(R^d)$,we have

$$\left\|\left(I - \eta_{\propto\square/t}\right)\left(I + S_{j,\theta} + S_{j,\theta}^2 + S_{j,\theta}^3 + S_{j,\theta}^4\right)\left(f - S_{j,\theta}f\right)\right\|_{L_p(S^{d-1})} \leq C_5 \left\|f - S_{j,\theta}f\right\|_{L_p(S^{d-1})}$$

,

where I is the identity operator. To prove(1) wehave to show that

$$\phi(u) = \frac{(1-\eta(u/\propto))m_\ell(u)^5}{1-m_\ell(u)}$$

is abounded multiplier on $L_p(R^d) or \|D^v\phi(u)\|_{L_p(S^{d-1})} \leq \frac{C}{[(1+|u|)|\square]^{d+\propto}}, \propto > 0$

(At least for $|v| \leq d+1$, but here that restriction does not metter) while the above iswell known and used manytimes.We show it to below to help the reader.For $\Phi^\vee(x)$ given by $\Phi^\vee(x) = \int_{R^d} \square\, \Phi(y)e^{2\pi\square xy}dy$

Which may be considered as aFourier transform, and prove the lemma2.8

which implies the sufficiency ofshowing that $\|D^v\phi(u)\|_{L_p(S^{d-1})} \leq \frac{C}{(1+|u|)^{d+2}}$ for $\propto > 0\ and |v| \leq d+1$ .We observation that for $|u| \leq 1, \phi(u) = 0$. for $|u| \geq 1$,then using Lemma2.7, we recall that the multipliers,and we get that

$$\|D^v\phi(u)\|_{L_p(S^{d-1})} \leq C(v)\left(\frac{1}{1+|u|}\right)^{5\left(\frac{d-1}{2}\right)} = C(v)\left(\frac{1}{1+|u|}\right)^{d+\frac{3}{2}d-\frac{5}{2}}$$

and for

The proof of (ii) is step by step of the relation (3.9) of Ditzian,2004■

Theorem 3.6

For $f \in L_p(S^{d-1})$ for $d = 2n, n = 2,3,4,\ldots$ with

$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ for any $\rho \in So(d)$

$\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0$ as $|\rho - I| \to 0$,

where $|\rho - \eta|^2 = \max_{x \in S^{d-1}} \left((\rho x - \eta x).(\rho x - \eta x)\right)$ (Note that $\max(\rho x.x) \geq cos$ is equivalent to $|\rho - I| \leq 2\left|sin\frac{t}{2}\right|$) then

$$E_n(f)_{L_p(S^{d-1})} \leq c(p)\omega^2\left(f,\frac{1}{n}\right)_{L_p(S^{d-1})}$$

Proof

We have $E(n)_{L_p(S^{d-1})} \leq C(p)K_{2\alpha}(f,\tilde{\Delta},n^{-2\alpha})_{L_p(S^{d-1})}$ by

$$\left\|f + \frac{2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell j}S_{j\theta}f\right\|_{L_p(S^{d-1})} \approx K_{2\ell}(f,\tilde{\Delta},\theta^{2\ell})_{L_p(S^{d-1})}$$

We have

$$E_n(f)_{L_p(S^{d-1})} \leq C(p)K_{2\alpha}(f,\tilde{\Delta},n^{-2\alpha})_{L_p(S^{d-1})} \leq C(p)\left\|f + \frac{2}{\binom{2\ell}{\ell}\sum(-1)^j\binom{2\ell}{\ell-j}S_{j\theta}f}\right\|_{L_p(S^{d-1})}$$

And by $\|S_\theta f - f\|_{L_p(S^{d-1})} \leq C(p)\omega^2(f,\theta)$

we get

$$E_n(f)_{L_p(S^{d-1})} \leq C(p)\omega^2(f,\theta)_{L_p(S^{d-1})} \blacksquare$$

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