Journal of Babylon University/Pure and Applied Sciences/ No.(1)/ Vol.(26): 2018

On monotone Rational Approximation

Eman S. Bhaya **Dhay Basim Ahmed**
University of Babylon College of Education for pure science, Department of mathematics .
emanbhaya@itnet.uobabylon.edu.iq dhay-20052006@yahoo.com

Abstract

In this chapter, we focus on the shape preserving approximation using rational polynomials, for rational in L_p _spaces, $p < 1$. In our work, we solve a problem raised by R. DeVore in several lectures for many years ago. It mean we discuss the order of monotone rational approximation for function in L_p, spaces for $p < 1$.

Keywords. ,monotone functions, the shape preserving approximation using rational polynomials, rational polynomials in L_p space for $p < 1$.

الخلاصة

ركزنا في عملنا هنا على التقريب الحافظ للشكل باستخدام متعددات الحدود النسبية للدوال في فضاءات L_p عندما $p < 1$. حيث قمنا بحل مشكله مقدمه من قبل الرياضاتي رونالد ديفور من خلال الكثير من محاضراته وسيمناراتيه لسنوات كثيره . اي اننا ناقشنا رتبه التقريب الرتيب باستخدام متعددات الحدود النسبية للدوال في فضاءات L_p عندما $p < 1$ عن طريق برهان مبرهنه مباشره للتقريب الرتيب النسبي للدوال في فضاءات L_p عندما $p < 1$.

الكلمات المفتاحية : الدوال الرتيبة, التقريب الحافظ للشكل, متعددات الحدود النسبية في فضاءات L_p عندما $p < 1$.

1. Introduction.

In1987, P. P. Petrushev and V. A. Popov, proved an estimation for rational approximation of real function.
In 1984, Newman, and V. A Popov, proved direct estimates for convex rational approximation.
Estimates about monotone rational approximate were obtained in 1999, It seems that not exact results are known until the present time.

In this chapter, we solve a problem raised by R. A. Devore in several knots during the last 25 years. It means we find exact order of $L_p -$ monotone rational *approximation for functions in the* L'_p *spaces p-1,* $0 < p < 1$.

Let P_n be the space of all algebraic polynomials of degree at most n, and Q_n be the set of all rational functions r=s/q,$q \neq 0$ where the error of the best uniform rational approximation of a continuous *function* f on [0,1] is defined by
$P_n(f) = \inf \|f - r\|_{c[0,1]}$.

In this chapter, we use rational approximation to prove a direct theorem for monotone approximation on L_p spaces for $p<1$.

Let π_n denote the space of algebrauic polynomials of degree not exceeding n. We say that r is an (n,m) rational function if $r = p/q$ where $p \in \pi_n$ and $q \in \pi_n$ if f is analytic in some neighborhood U of zero then the (n, m) rational function $s = p/q$.

Let $\|\cdot\|_k$ denote the supremum norm on a set k in the complex plan c . We call the rational function r is a best (n, m) rational approximation to f on k if r satisfies

$$\|f - r\|_k = \inf \|f - p/q\|_k$$

So we shall use some definitions to prove these problem , these are:

$$R_m(x) = \frac{N(x) - N(-x)}{N(x) + N(-x)}$$

$$N(x) = \prod_{i=1}^{m} (x + a^i)\ , a = e^{-1/\sqrt{m}}$$

$$H_{n,m}(x) = \frac{\hat{R}_m(x)}{1 + 4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt$$

$$X_+^\circ = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$R_{n,m}(x) = \frac{2}{A}\int_0^x T_n^m\,(t)\,dt + H_{n,m}(x) \cdot \frac{R_m^*(x)}{1 + b}$$

$$Q_{n,m}(x) = \frac{1 + R_{[n/m]+1,m}(x)}{2}$$

2. The auxiliary Results

This section consists of lemmas that we need in our main results.

Lemma. 2.1

Let us take the polynomial

$$T_n(x) = \sum_{k=0}^{3n} (-1)^k\, n^{2k+1}\, \frac{x^{2k}}{(2k+1)!} \qquad (2.2)$$

have $6n$ degree

$$A = A_{m,n} = \int_{-1}^{1} T_n^m\,(x)\,dx \qquad (2.3)$$

Satisfy

$$\frac{1}{2}\left(\frac{\sin nx}{x}\right)^m \leq T_n^m(x) \leq 2\left(\frac{\sin nx}{x}\right)^m \quad , \quad x \in \left(0, \frac{3}{n}\right]$$

So,

$$\int_x^1 T_n^m\,(t)\,dt \leq \frac{1}{(m-2)x^{m-1}}, \quad x \in [\tfrac{2}{n}, 1] \qquad (2.4)$$

Then, $A \geq \frac{1}{2^m} n^{m-1}$, $A \leq 6n^{m-1}$ (2.5)

Notation 2.6

Let

$$R_m(x) = \frac{N(x) - N(-x)}{N(x) + N(-x)} \qquad (2.7)$$

Be a rational function

$$N(x) = \prod_{i=1}^{m} (x + a^i), and\ \ a = e^{-1/\sqrt{m}} \qquad (2.8)$$

Therefore

$$R'_n(x) = \frac{\big(N(x) + N(-x)\big)\big(N'(x) - N'(-x)\big) - \big(N(x) - N(-x)\big)\big(N'(x) + N'(-x)\big)}{(N(x) + N(-x))^2} \qquad (2.9)$$

Lemma 2.10 Let R_m be function that satisfy

1) $\|1 - R_m(x)\|p \leq 2^{p-1}\big(1 - e^{-\sqrt{m}}\big)[1 + k]$, $x \in \left[e^{-\sqrt{m}}, 1\right]$ (2.11)

2) Let $\boldsymbol{R_m}(\boldsymbol{x}) = \frac{\boldsymbol{N(x)-N(-x)}}{\boldsymbol{N(x)+N(-x)}}$ be a function , such that

$$N(x) = \prod_{i=1}^{m}(x + a^i) \qquad , \quad a = e^{-1/\sqrt{m}} \quad (2.12)$$

and let $R'_m(x) \geq 0,\ x \in [0, e^{-\sqrt{m}}]$.

Proof (1) :

Let $R_m(x) = \frac{N(x)-N(-x)}{N(x)+N(-x)}$ rational function

such that $N(x) = \prod_{i=1}^{m}(x + a^i)$, $a = {-1}/{\sqrt{m}}$, so we have

$$\|1 - R_m\|_p = \left(\int_{e^{-\sqrt{m}}}^{1} |1 - R_m(x)|^p\, dx\right)^{1/p}$$

$$\leq \left(\int_{e^{-\sqrt{m}}}^{1} 1^p\, dx - \int_{e^{-\sqrt{m}}}^{1} |R_m(x)|^p\, dx\right)^{1/p} \leq 2^{p-1}\left(\int_{e^{-\sqrt{m}}}^{1} dx\right)^{1/p} +$$

$$\left(\int_{e^{-\sqrt{m}}}^{1} |R_m(x)|^p\, dx\right)^{1/p} \quad \leq 2^{p-1}\left(\left|\, [x]_{e^{-\sqrt{m}}}^{1}\,\right|^p\right)^{1/p} + \left(\int_{e^{-\sqrt{m}}}^{1}\left|\frac{N(x)-N(-x)}{N(x)+N(-x)}\right|^p dx\right)^{1/p},$$

where, $N(x) = \prod_{i=1}^{m}(x + a^i)$.

So we obtain,

$$\|I - R_m\| \leq 2^{p-1}\left(\left|1 - e^{-\sqrt{m}}\right|^p\right)^{1/p}$$

$$-\left(\int_{e^{-\sqrt{m}}}^{1}\left(\frac{\prod_{i=1}^{m}(x + a^i) - \prod_{i=1}^{m}(-x + a^i)}{\prod_{i=1}^{m}(x + a^i) + \prod_{i=1}^{m}(-x + a^i)}\right)^p dx\right)^{1/p}$$

Since $x \in [e^{-\sqrt{m}}, 1]$. so we have, $\|1 - R_m\|_p \leq 2^{p-1}\left(\left|1 - e^{-\sqrt{m}}\right|^p\right)^{1/p} +$

$$\left(\int_{e^{-\sqrt{m}}}^{1}\left|\frac{\prod_{i=1}^{m}(1+1) + \prod_{i=1}^{m}(1+1) + \cdots}{\prod_{i=1}^{m}\left(e^{-\sqrt{m}} + e^{-\sqrt{m}}\right) + a^i}\right|\, dx^{\,p}\right)^{1/p}$$

$$\leq 2^{p-1}\left(\left|1 - e^{-\sqrt{m}}\right|^p\right)^{1/p}$$

$$+\left(\int_{e^{-\sqrt{m}}}^{1}\left|\frac{2^m + 2^m}{\prod_{i=1}^{m}\left(e^{-\sqrt{m}} + e^{-\sqrt{m}}\right)} + \frac{2^m + \cdots}{\prod_{i=1}^{m}\left(e^{-\sqrt{m}} + e^{-\sqrt{m}}\right)}\right|^p dx\right)^{1/p}$$

$$\leq 2^{p-1}\left(\left|1 - e^{-\sqrt{m}}\right|^p\right)^{1/p} + \left(\int_{e^{-\sqrt{m}}}^{1}\left|\frac{2^{m+1}}{2e^{-m\sqrt{m}}}\right|^p dx\right)^{1/p}$$

$$\leq 2^{p-1}\left(1 - e^{-\sqrt{m}}\right) + \left(\frac{2^{m+1}}{2e^{-m\sqrt{m}}}\int_{e^{-\sqrt{m}}}^{1} dx\right)^{1/p}$$

Let K be constant, such that, $K = \frac{2^{m+1}}{2e^{-m\sqrt{m}}}$

So we obtain

$$\|I - R_m\| \leq 2^{p-1}\left(1 - e^{-\sqrt{m}}\right) + K\left(\left|[x]_{e^{-\sqrt{m}}}^{1}\right|^p\right)^{1/p}$$

$$\leq 2^{p-1}\left[\left(\left|\left(1 - e^{-\sqrt{m}}\right) + K\left(1 - e^{-\sqrt{m}}\right)\right|^p\right)^{1/p}\right] \leq 2^{p-1}\left(\left|1 - e^{-\sqrt{m}}\right|^p\right)^{1/p}[1 + K]$$

$$\|I-R_m\|_p \le 2^{p-1}\left(1-e^{-\sqrt{m}}\right)[1+K]$$
$$\|I-R_m\|_p \le 2^{p-1}\left(1-e^{-\sqrt{m}}\right)\left[1+\frac{2^{m+1}}{2e^{-m\sqrt{m}}}\right]$$

Proof (2) :

We must find $\|R'_m\|_p$, and $x \in \left[e^{-\sqrt{m}}, \infty\right]$

$$\|R'_m\|_p = \left(\int_0^{e^{-\sqrt{m}}} |R'_m(x)|^p \, dx\right)^{1/p}$$

When $R_m(x) = \frac{N(x)-N(-x)}{N(x)+N(-x)}$

So $R'_m(x) = \frac{(N(x)+N(-x))\left(N'(x)+N'(-x)\right)+(N(x)+N(-x))\left(\left(N'(x)+N'(-X)\right)\right)}{[N(x)+N(-x)]^2}$

$R'_m(x) \le \frac{(N(x)+N(-x))\left(N'(x)+N'(-X)\right)+(N(x)+N(-x))\left(N'(x)+N'(-x)\right)}{[N(x)+N(-x)]^2}$

Since $N(x) = \prod_{i=1}^m (x+a^i)$, $a = e^{-1/\sqrt{m}}$, $a = e^{-1/\sqrt{m}}$

$$N'(x) = \prod_{i=1}^m \left(x+a^i\right)' = \left((x+a^1)'(x+a^2)'(x+a^3)' \ldots (x+a^m)\right)' \quad (2.13)$$

Since by properties of derivative such that if $f(x) < g(x)$

Then $f'(x) < g'(x)$ *we obtain,*

$R'_m(x) \le$

$$\frac{\left(\prod_{i=1}^m (x+a^i)+\prod_{i=1}^m (-x+a^i)\right)\left(\prod_{i=1}^m (x+a^i)'+\prod_{i=1}^m (-x+a^i)'\right)+\left(\prod_{i=1}^m (x+a^i)+\prod_{i=1}^m (x+a^i)\right)\left(\prod_{i=1}^m \begin{matrix}(x+a^i)'+\\ \prod_{i=1}^m (-x+a^i)'\end{matrix}\right)}{\left[\prod_{i=1}^m (x+a^i)+\prod_{i=1}^m (-x+a^i)\right]^2}$$

Since $x > 0$ and derivative is monotone ,we obtain

$R'_m(x) \le \frac{\left(\prod_{i=1}^m (x+a^i)+\prod_{i=1}^m (-x+a^i)\right)((x^m)'+(-x^m)'+\prod_{i=1}^m (x+a^i)+\prod_{i=1}^m (-x+a^i)(x^m)'+(-x^m)'}{\left(\prod_{i=1}^m (x+a^i)+(-x+a^i)\right)^2}$

$$\le \frac{\prod_{i=1}^m \left(x+a^i\right)+\left(-x+a^i\right)[mx^{m-1}+0]+\prod_{i=1}^m \left(\left(x+a^i\right)+\left(-x+a^i\right)(mx)^{m-1}\right)+0}{\left(\prod_{i=1}^m (x+a^i)+(-x+a^i)\right)^2}$$

$$= \frac{\prod_{i=1}^m 2a^i(mx^{m-1})+\prod_{i=1}^m 2a^i(mx^{m-1})}{\left(\prod_{i=1}^m 2a^i\right)^2} = \frac{\prod_{i=1}^m 2a^i[mx^{m-1}+mx^{m-1}]}{\left[\prod_{i=1}^m 2a^i\right]^2}$$

Since $a^i = e^{-1/\sqrt{m}}$, $\le \frac{\prod_{i=1}^m 2e^{-1/\sqrt{m}}[mx^{m-1}+mx^{m-1}]}{\left[\prod_{i=1}^m 2e^{-1/\sqrt{m}}\right]^2} = \frac{mx^{m-1}+mx^{m-1}}{\prod_{i=1}^m 2e^{-1/\sqrt{m}}}$

Let C is constant, such that

$$C = \frac{1}{\prod_{i=1}^m 2e^{-1/\sqrt{m}}}$$

$$R'_m(x) \le c2mx^{m-1} = \frac{1}{2\prod_{i=1}^m e^{-1/\sqrt{m}}} * 2mx^{m-1}$$

$$R'_m(x) \le \frac{mx^{m-1}}{\prod_{i=1}^m e^{-1/\sqrt{m}}} = Cm\,x^{m-1} \quad (2.14)$$

$$\|R'_m\|_p = \left(\int_0^{e^{-\sqrt{m}}} |C(mx^{m-1} + mx^{m-1})|^p \, dx \right)^{1/p}$$

$$\le C \left(\int_0^{e^{-\sqrt{m}}} |mx^{m-1}|^p \, dx \right)^{1/P} + \left(\int_0^{e^{-\sqrt{m}}} |mx^{m-1}|^p \, dx \right)^{1/p}$$

$$\le C \left(\left| [x^m]_0^{e^{-\sqrt{m}}} \right|^p \right)^{1/p} + \left(\left| [x^m]_0^{e^{-\sqrt{m}}} \right|^p \right)^{1/p} \le C \left[\left(e^{-\sqrt{m}}\right)^m - 0 + \left(e^{-\sqrt{m}}\right)^m - 0 \right]^{p/p}$$

$$\le C \left[\left(e^{-\frac{1}{m}}\right)^m + \left(e^{-\frac{1}{m}}\right)^m \right] \le C[e^{-1} + e^{-1}] \le C(\frac{1}{e} + \frac{1}{e})$$

We obtain,

$$\|R'_m\|_p \le C \frac{2}{e}$$

$$\|R'_m\|_p \le \frac{1}{\prod_{i=1}^m 2e^{-1/\sqrt{m}}} * \frac{2}{e} \le \frac{2}{e * 2 \prod_{i=1}^m e^{-1/\sqrt{m}}}$$

$\|R'_m\|_p \le \frac{1}{e \prod_{i=1}^m e^{-1/\sqrt{m}}}$, $where\ x \in \left[0, e^{-\sqrt{m}}\right]$ ▯

Lemma. 2.15

$\hat{R}_m$ is odd and $\hat{R}_m(0) = 0$,

$\hat{R}_m\left(\frac{1}{m}\right) = 1,\ \hat{R}_m(x) = \frac{R_m\left(nxe^{-\sqrt{m}/2}\right) + bnx}{R_m\left(e^{-\sqrt{m}/2}\right) + b}$ and $b = 32m^{3/2}\, e^{-\sqrt{m}/2}$

So we obtain, $\hat{R}_m(x) \le 1 + \frac{3b}{2} n \left(x - \frac{1}{n}\right), x \in \left[\frac{1}{n}, 1\right]$

$\hat{R}_m\left(\frac{e^{-\sqrt{m}/2}}{n}\right) \ge 1 - \frac{3b}{2}$.   2.16

Lemma. 2.17

If, $H_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A} \int_0^X P_m^m(t)\, dt$, then following satistied

$H_{n,m}(x) < 1,\quad when\ x \in \left[0, \frac{1}{2n}\right]$

$H_{n,m}(x) \le 0,\quad when\ x \in \left[\frac{1}{2n}, \frac{3}{n}\right]$

we obtain,

$H_{n,m}(x) \le -\frac{1}{A} \int_0^x P_m^m(t)\, dt, x \in \left[\frac{3}{n}, 1\right]$ (2.19)

Lemma. 2.18

$R^*{}_m(\mathrm{x}) = \frac{1}{1+(nx)^{m^{5}+m}}$, even function satisfying

$R^{*\prime}{}_m(\mathrm{x}) \le 0, \mathrm{x} \in [0,1]$, $0 < R^*{}_m(x) \le 1, x \in [0,1]$, $1 - R^*{}_m(x) \le (nx)^{m^5}, x \in [0, \frac{1}{2n}]$, $R^*{}_m(x) \le \frac{1}{(nx)^{m^5}}, \left[\frac{2}{n}, 1\right]$, and $R^{*\prime}{}_m(x) \le \frac{n}{2^{m^5}}, x \in \left[0, \frac{1}{2n}\right]$

Hence $\frac{-R^{*\prime}{}_{m(x)}}{R^*{}_{m(x)}} \ge \frac{m^5}{x}, x \in [\frac{2}{n}, 1]$.

Remark. 2.20

Denote by

$$X_+^\circ = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

So put $R_{n,m}(x) = \frac{2}{A}\int_0^x T_m^m(t)\,dt + H_{n,m}(x) * \frac{R^*m(X)}{1+b}$ (2.21)

and $Q_{n,m}(x) = \frac{1+R_{[n/m]+1,m}(x)}{2}$.

Theorem 2.22

a) If $m > N_\circ$ and integer $n > m^2$ are even then $Q'_{n,m}(x) \geq 0, x \in [-1,1]$, $Q_{n,m} = p_{n,m} + q_{n,m}$, where $p_{n,m} \in \rho_{7n}$, $q_{n,m} \in Q_{2m^6}$

$$\left\|Q_{n,m} - X_+^\circ\right\|_p, and \quad \frac{e^{-\sqrt{m}/4}}{n} \leq |x| \leq 1$$

b) $\left\|Q_{,n,m}(x) - X_+^\circ\right\|$ when $\frac{j-1}{n} \leq |x| \leq \frac{j}{n}$ for each even $m > N_\circ$ and integer $n > m^2$ we have, $Q'_{n,m} \geq 0$, and $x \in [-1,1]$

$$Q_{n,m} = p_{n,m} + q_{n,m.} where\ p_{n,m} \in P_{7n}, q_{n,m} \in Q_{2m^6}$$

To find $\left\|Q_{n,m} - X_+^\circ\right\|_p$ $when\ \frac{j-1}{n} \leq |x| \leq \frac{j}{n}, j = 2, \dots, n$

Proof (a) :

Let $X_+^\circ = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$R_{n,m}(x) = \frac{2}{A}\int_0^X T_n^m(t)\,dt + H_{n,m}(x) * \frac{R^*m(x)}{1+b}$ (2.23)

and $Q_{n,m}(x) = \frac{1+R_{[n/m]+1,m(x)}}{2}$

$\ni H_{n,m}(x) = \frac{\hat{R}_{m(x)}}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt$ (2.24)

and $\hat{R}_m(x) = \frac{R_{m\left(nxe^{-\sqrt{m}/2}\right)} + bnx}{R_{m\left(e^{-\sqrt{m}/2}\right)} + b}$

$$\left\|Q_{n,m}(x) - 1\right\|_p \quad \text{where } x \geq 0$$

$$\left\|\frac{1 + R_{\left[\frac{n}{m}\right]+1,m(x)}}{2} - 1\right\|_p \quad \text{where } \frac{e^{-\sqrt{m}/4}}{n} \leq |x| \leq 1$$

$$= \left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1} \left|\frac{1 + R_{\left[\frac{n}{m}\right]+1,m}(x)}{2}\right|^p dx\right)^{1/p} + \left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1} |1|^p\,dx\right)^{1/p}$$

$$\leq 2^{p-1}\left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1}\left|1+R_{\left[\frac{n}{m}\right]+1,m}(x)\right|^{p}dx\right)^{1/p}+[x]^{1}_{\frac{e^{-\sqrt{m}/4}}{n}}$$

$$\leq 2^{p-1}*\frac{1}{2}\left[\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1}\left|1+R_{\left[\frac{n}{m}\right]+1,m}(x)\right|^{p}dx\right]^{1/p}+[x]^{1}_{\frac{e^{-\sqrt{m}/4}}{n}}$$

$$\leq 2^{p-1}*\frac{1}{2}\left\{\left([x]^{1}_{\frac{e^{-\sqrt{m}/4}}{n}}\right)^{1/p}\right.$$

$$+\left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1}\left|\frac{2}{A}\int_{0}^{x}T^{m}_{\left[\frac{n}{m}\right]+1}(t)\,dt+H_{\left[\frac{n}{m}\right]+1,m}(x)*\frac{R^{*}{}_{m}(x)}{1+b}\right|^{p}dx\right)^{1/p}$$

$$\left.+\left([x]^{1}_{\frac{e^{-\sqrt{m}/4}}{n}}\right)^{1/p}\right\}$$

$$\leq 2^{p-1}*\frac{1}{2}*2\left([x]^{1}_{\frac{e^{-\sqrt{m}/4}}{n}}\right)^{1/p}$$

$$+\left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1}\left|\frac{2}{A}\int_{0}^{x}T^{m}_{\left[\frac{n}{m}\right]+1,m}(t)\,dt+H_{\left[\frac{n}{m}\right]+1,m}(x)*\frac{R^{*}{}_{m}(x)}{1+b}\right|^{p}dx\right)^{1/p}$$

Since $\int_{x}^{1}T^{m}_{m}(t)\,dt\leq\frac{1}{(m-2)x^{m-1}}$, $x\in\left[\frac{2}{n},1\right]$

So we obtain,

$$\left\|Q_{n,m}(x)-1\right\|_{p}$$

$$\leq 2^{p-1}\left([x]^{1}_{\frac{e^{-\sqrt{m}/4}}{n}}\right)^{1/p}$$

$$+\left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^{1}\left|\frac{2}{A}\frac{1}{(m-2)x^{m-1}}+H_{\left[\frac{n}{m}\right]+1,m}(x)*\frac{R^{*}{}_{m}(x)}{1+b}\right|^{p}dx\right)^{1/p}$$

By using $R_{\left[\frac{n}{m}\right]+1,m}(x)$ *and* $\int_x^1 T_n^m(t)\,dt$

Since, $H_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt$

So we obtain,

$$\left\|Q_{n,m}(x)-1\right\|_p \le 2^{p-1}\left[\left(\left|[x]^1_{\frac{e^{-\sqrt{m}/4}}{n}}\right|^p\right)^{1/p} + \left(\int_{\frac{e^{-\sqrt{m}/4}}{n}}^1 \left|\frac{2}{A}*\frac{1}{(m-2)x^{m-1}} + \left(\frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt * \frac{R^*{}_m(x)}{1+b}\right)\right|^p dx\right)^{1/p}\right]$$

Then,

$$\left\|Q_{n,m}(x)-1\right\|_p \le 2^{p-1}\|X\|_p + \frac{2}{A}\left\|\frac{1}{(m-2)x^{m-1}}\right\| + \frac{\|\hat{R}_m(x)\|}{1+4b} + \frac{2}{A}\left\|P_m^m(t) * \frac{R^*{}_m(x)}{1+b}\right\|$$

$$\le 2^{p-1}\|X\|_p + \frac{2}{A}\left\|\frac{1}{(m-2)x^{m-1}}\right\| + \frac{\|\hat{R}_m(x)\|}{1+4b} + m^m(nx)^{2m^2+1} * \frac{1}{1+(nx)^{m^5+m}},$$

where $R^*{}_m(x) = \frac{1}{1+(nx)^{m^5+m}}$ and we have :

$$\frac{2}{A}\int_0^x P_m^m(t)\,dt \le m^m(nx)^{2m^2+1} \qquad (2.25)$$

so we obtain,

$$\left\|Q_{n,m}(x)-X_+^\circ\right\|_p \le 2^{p-1}\|X\|_p + \frac{2}{A}*\frac{1}{(m-2)}\,\frac{1}{\|X^{-m+1}\|} + \frac{\|\hat{R}_m\|}{1+4b} + \frac{m^m(nx)^{2m^2+1}}{1+(nx)^{m^5+m}}$$

$$\le 2^{p-1}\|X\| + C\frac{1}{\|X^{-m+1}\|} + \frac{1}{1+4b}\|\hat{R}_m\| + \frac{m^m(nx)^{2m^2+1}}{1+(nx)^{m^5+m}}$$

Since $\frac{e^{-\sqrt{m}/4}}{n} \subset \frac{e^{-\sqrt{m}/2}}{n}$, *we obtain*, $\hat{R}_m(\frac{e^{-\sqrt{m}/2}}{n}) \ge \frac{1-3b}{2}$

$$\le 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + \frac{1-3b}{2+8b} + \frac{m^m(nx)^{2m^2+1}}{1+(nx)^{m^5+m}}$$

$$\le 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + C_2 + \frac{m^m (nx)^{2m^2} * (nx)}{1+(nx)^{m(m)^4} * (nx)}$$
$$\le 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + C_2 + \frac{m^m (nx)^{2m^2}}{1+(nx)^{m(m^4)}}$$

$$\le 2^{p-1}\left(\left[\left|\frac{x^2}{2}\right|^p\right]^1_{\frac{e^{-\sqrt{m}/4}}{n}}\right)^{1/p} + \frac{C_1}{\left(\left|[X^{-m+1+1}]^1_{\frac{e^{-\sqrt{m}/4}}{n}}\right|^p\right)^{1/p}} + C_2 + \frac{m^m (nx)^{2m^2}}{1+(nx)^{m(m^4)}}$$
$$\le 2^{p-1} * \frac{1}{2}[x^2]^1_{\frac{e^{-\sqrt{m}/4}}{n}} + \frac{C_1}{[X^{-m+2}]^1_{\frac{e^{-\sqrt{m}/4}}{n}}} + C_2 + \frac{m^m (n(1))^{2m^2}}{1+(n(1))^{m(m^4)}}$$
$$\le 2^{p-1} * \frac{1}{2}[x^2]^1_{\frac{e^{-\sqrt{m}/4}}{n}} + \frac{C_1}{[X^{-m+2}]^1_{\frac{e^{-\sqrt{m}/4}}{n}}} + C_2 + \frac{m^m (n)^{2m^2}}{1+(n)^{m^5}}$$
$$\le 2^{p-1} * \frac{1}{2}\left(1-(\frac{e^{-\sqrt{m}/4}}{n})^2\right) + \frac{C_1}{1^{-m+2}-(\frac{e^{-\sqrt{m}/4}}{n})^{-m+2}} + C_2 \frac{m^m}{1+n^m}$$
$$\le 2^{p-1} * \frac{1}{2}\left(1-\frac{e^{-\sqrt{m}/2}}{n^2}\right) + \frac{C_1}{1^{-m+2}-(\frac{e^{-\sqrt{m}/4}}{n})^{-m+2}} + k$$

We put, $K = C_2 \frac{m^m}{1+n^m}$, $C = \frac{C_1}{1^{-m+2}-\left(\frac{e^{-\sqrt{m}/4}}{n}\right)^{-m+2}}$

We obtain,

$$\left\|Q_{n,m} - X_+^\circ\right\|_p \le 2^{p-1} * \frac{1}{2}(1-\frac{e^{-\sqrt{m}/2}}{n^2} + C + K$$

When $-1 < x < -\frac{e^{-\sqrt{m}/4}}{n}$ then we obtain,

$$\left\|Q_{n,m-X_+^\circ}\right\|_p \le 2^{p-1}\left(\int_{-1}^{-\frac{e^{-\sqrt{m}/4}}{n}} |x|^p\,dx\right)^{1/p} + \frac{C_1}{\left(\int_{-1}^{-\frac{e^{-\sqrt{m}/4}}{n}} |X^{-m+1}|^p\,dx\right)^{1/p}} + C_2$$
$$+ \frac{m^m (nx)^{2m^2}}{1+(nx)^{m(m^4)}}$$
$$\le 2^{p-1}\left[\frac{x^2}{2}\right]_{-1}^{\frac{-e^{-\sqrt{m}/4}}{n}} + \frac{C_1}{[X^{-m+2}]_{-1}^{\frac{-e^{-\sqrt{m}/4}}{n}}} + C_2 + \frac{m^m (n(-1))^{2m^2}}{1+(n(-1)^{m(m^4)}}$$

When $x \in [-1,1]$.

$$\left\|Q_{n,m} - X_{+}^{\circ}\right\|_p \leq 2^{p-1} * \frac{1}{2}\left(\frac{-e^{\sqrt{m}/_4}}{n} + 1\right) + \frac{C_1}{(\frac{-e^{\sqrt{m}/_4}}{n})^{-m+2} - (-1)^{-m+2}}$$

$$+ C_2 \frac{m^m(-n)^{2m^2}}{1+(-n)^{m^5}} \leq 2^{p-1} * \frac{1}{2}(\frac{-e^{-\sqrt{m}/_2}}{n} + 1 + K_{1+}k_2$$

Put $k_1 = \frac{C_1}{(-\frac{e^{-\sqrt{m}/_4}}{n})^{-m+2} - (-1)^{-m+2}}, K_2 = \frac{m^m}{1+(-n)^m}$ ▯ .

Proof (b) :

let $X_{+}^{\circ} = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$R_{n,m}(x) = \frac{2}{A}\int_0^x T_n^m\,(t)\,dt + H_{n,m}(x) * \frac{R^*m(x)}{1+b}$, and

$Q_{n,m}(x) = \frac{1+R_{[\frac{n}{m}]+1,m(x)}}{2}$. ∋ $H_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m\,(t)\,dt$

and , $\hat{R}_m(x) = \frac{R_{m\left(nxe^{-\sqrt{m}/_2}\right)} + bnx}{R_{m\left(e^{-\sqrt{m}/_2}\right)} + b}$

$\left\|Q_{n,m}(x) - 1\right\|_p$ where $x \geq 0$

$\left\|\frac{1+R_{[\frac{n}{m}]+1,m}(x)}{2} - 1\right\|_p$ where $\frac{j-1}{n} \leq |x| \leq \frac{j}{n}$

$$\left\|Q_{n,m} - X_{+}^{\circ}\right\|_p = \left(\int_{\frac{j-1}{n}}^{\frac{j}{n}} \left|\frac{1+R_{[\frac{n}{m}]+1,m}(x)}{2} - 1\right|^p dx\right)^{1/p}$$

$$\leq 2^{p-1}\left(\int_{\frac{j-1}{n}}^{\frac{j}{n}} \left|\frac{1+R_{[\frac{n}{m}]+1,m}(x)}{2}\right|^p dx\right)^{1/p} + \left(\int_{\frac{j-1}{n}}^{\frac{j}{n}} 1^p\,dx\right)^{1/p}$$

$$\leq 2^{p-1}\left(\int_{\frac{j-1}{n}}^{\frac{j}{n}} \left|\frac{1+R_{[\frac{n}{m}]+1,m}(x)}{2}\right|^p dx\right)^{1/p} + [x]_{\frac{j-1}{n}}^{\frac{j}{n}}$$

$$\leq 2^{p-1} * \frac{1}{2}\left(\int_{\frac{j-1}{n}}^{\frac{j}{n}} \left|1+R_{[\frac{n}{m}]+1,m}(x)\right|^p dx\right)^{1/p} + [x]_{\frac{j-1}{n}}^{\frac{j}{n}}$$

$$\leq 2^{p-1} * \frac{1}{2}\left(\left|[x]_{\frac{j-1}{n}}^{\frac{j}{n}}\right|^{p}\right)^{1/p} + \int_{\frac{j-1}{n}}^{\frac{j}{n}}\left(\left|\frac{2}{A}\int_{0}^{x} T_{\left[\frac{n}{m}\right]+1}^{m}(t)\,dt + H_{\left[\frac{n}{m}\right]+1,m}(x) * \frac{R^{*}{}_{m}(x)}{1+b}\right|^{p} dx\right)^{1/p} + [x]_{\frac{j-1}{n}}^{\frac{j}{n}}$$

$since \int_{x}^{1} T_{n}^{m}(t)\,dt \leq \frac{1}{(m-2)X^{m-1}}, when\ x \in \left[\frac{2}{n}, 1\right]$

So we obtain,

$$\left\|Q_{n,m}(x) - 1\right\|_{p} \leq 2^{p-1} * \frac{1}{2} * 2\left(\left|[x]_{\frac{j-1}{n}}^{\frac{j}{n}}\right|^{p}\right)^{1/p} + \left(\int_{\frac{j-1}{n}}^{\frac{j}{n}}\left|\frac{2}{A} * \frac{1}{(m-2)X^{m-1}} + H_{\left[\frac{n}{m}\right]+1,m}(x) * \frac{R^{*}{}_{m}(x)}{1+b}\right|^{p} dx\right)^{1/p}$$

Since, $H_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt$

So we have,

$$\left\|Q_{n,m}(x) - X_{+}^{\circ}\right\|_{p} \leq 2^{p-1} * \frac{1}{2}2\left(\left|[x]_{\frac{j-1}{n}}^{\frac{j}{n}}\right|^{p}\right)^{1/p} + \left(\int_{\frac{j-1}{n}}^{\frac{j}{n}}\left|\frac{2}{A} * \frac{1}{(m-2)X^{m-1}} + \left(\frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt\right) * \frac{R^{*}{}_{m}(x)}{1+b}\right|^{p} dx\right)^{1/p}$$

Then, $\left\|Q_{n,m}(x) - X_{+}^{\circ}\right\|_{p} \leq 2^{p-1}\|X\|_p + \frac{2}{A}\left\|\frac{1}{(m-2)X^{m-1}}\right\| + \frac{\|\hat{R}_m(x)\|}{1+4b} + \frac{2}{A}\left\|P_m^m(t) * \frac{R^{*}{}_{m}(x)}{1+b}\right\|$

$$\leq 2^{p-1}\|X\|_p + \frac{2}{A}\left\|\frac{1}{(m-2)X^{m-1}}\right\| + \frac{\left\|\hat{R}_m(x)\right\|}{1+4b} + m^m(nx)^{2m^2+1} * \frac{1}{1+(nx)^{m^5+m}}$$

where $R^{*}{}_{m}(x) = \frac{1}{1+(nx)^{m^5+m}}$, and we have, $\frac{2}{A}\int_0^x P_m^m(t)\,dt \leq m^m(nx)^{2m^2+1}$

So we obtain,

$$\left\|Q_{n,m}(x) - X_{+}^{\circ}\right\|_{p} \leq 2^{p-1}\|X\| + C\frac{1}{\|X^{-m+1}\|} + \frac{1}{1+4b}\left\|\hat{R}\right\| + \frac{m^m(nx)^{2m^2+1}}{1+(nx)^{m^5+m}}$$

$$\left\|Q_{n,m} - X_{+}^{\circ}\right\|_{p} \leq 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + \frac{\frac{1-3b}{2}}{1+4b} + \frac{m^m(nx)^{2m^2+1}}{1+(nx)^{m^5+m}}$$

$$\leq 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + \frac{1-3b}{2+8b} + \frac{m^m(nx)^{2m^2+1}}{1+(nx)^{m^5+m}}$$

$$\leq 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + C_2 + \frac{m^m(nx)^{2m^2} * (nx)}{1+(nx)^{m(m^4)} * (nx)}$$

$$\leq 2^{p-1}\|X\| + \frac{C_1}{\|X^{-m+1}\|} + C_2 + \frac{m^m(nx)^{2m^2}}{1+(nx)^{m(m^4)}}$$

$$when \frac{j-1}{n} \leq |x| \leq \frac{j}{n}, j = 2, \dots n,$$

so we obtain,

$$\|Q_{n,m}(x)-X_+^\circ\| \le 2^{p-1}\left(\int_{\frac{j-1}{n}}^{\frac{j}{n}}|X|^p\,dx\right)^{1/p}+\frac{C_1}{\left(\int_{\frac{j-1}{n}}^{\frac{j}{n}}|X^{-m+1}|^p\,dx\right)^{1/p}}+C_2\frac{m^m\left(n\frac{j}{n}\right)^{2m^2}}{1+(n\frac{j}{n})^{m(m^4)}}=$$

$$2^{p-1}*\frac{1}{2}(x^2)_{\frac{j-1}{n}}^{\frac{j}{n}}+\frac{\frac{C_1}{(X^{-m+2})_{\frac{j-1}{n}}^{\frac{j}{n}}}}{-m+2}+C_{2\frac{m^m}{1+j^m}}$$

$$\le 2^{p-1}*\frac{1}{2}\left((\frac{j}{n})^2-\left(\frac{j-1}{n}\right)^2\right)+(-m+1)\frac{C_1}{(\frac{j}{n})^{-m+2}-(\frac{j-1}{n})^{-m+1}}+C_2+\frac{m^m}{1+j^m}$$

$$\le 2^{p-1}*\frac{1}{2}(\frac{j^2-(j-1)^2}{n^2}+K_1+K_2$$

Where $K_1=(-m+1)\frac{C_1}{\left(\frac{j}{n}\right)^{-m+2}-(\frac{j-1}{n})^{-m+2}}$, And $K_2=C_2+\frac{m^m}{1+j^m}$

$\|Q_{n,m}-X_+^\circ\|_p \le \frac{2^{p-1}}{2}\left(\frac{j^2-j^2+2j-1}{n^2}\right)+K_1+k_2 \quad =\frac{2^{p-1}}{2}\left(\frac{2j-1}{n^2}\right)+K_1+K_2$ ▯

When $-\frac{j}{n}\le x<-\left(\frac{j-1}{n}\right)$

3. The main result

In this section we introduce our theorem for rational Monotone approximation.

Theorem 3.1

If f is monotone function in $L'_p[0,1]$ satisfying $f'\in L'_p[0,1]$

Then $E_n^{(1)}(f)\le\frac{c(p)}{n}\|f'\|_p, n=1,2,\dots and\ p<1$

Proof:

Assume f non decreasing on[0,1] satisfies $\|f'\|_p=1$, and

$0=x_\circ<x_1<\cdots<x_k=1$. Operation satisfies

$f(x_{i+1})^p-f(x_i)^p=\frac{1}{n}$, for i=0,1,2,…,k-1

$$\left(\int_{x_i}^{x_{i+1}}|f'(x)|^p\,dx\right)^{1/p}(x_{i+1}-x_i)^{\frac{p-1}{p}}\ge f(x_{i+1})^p-f(x_i)^p=\frac{1}{n}$$

Therefore, $\|f'\|_{L_p[x_i,x_{i+1}]}\ge\frac{1}{n^p(x_{i+1}-x_i)^{p-1}}$, $i=0,1,\dots,k-1$.

Then , $\sum_{i=0}^{n-1}\|f'\|_{L_{p[x_i,x_{i+1}]}}^p\ge\sum_{i=0}^{k-1}\frac{1}{n^p(x_{i+1}-x_i)^{p-1}}$, then

$\|f'\|_{L_p}^p\ge\sum_{i=0}^{k-1}\frac{1}{n^p(x_{i+1}-x_i)^{p-1}}$, therefore

$$\sum_{i=0}^{k-1}\frac{1}{n^p(x_{i+1}-x_i)^{p-1}}\le 1$$

By Eq.(2.14) we obtain,

$\sum_{i=0}^{k-1}\frac{1}{(n(x_{i+1}-x_i)^{p-1}}\le$n, and then, $\frac{1}{n\,n^{p-1}(x_{i+1}-x_i)^{p-1}}\le 1$

$$\frac{1}{(n(x_{i+1}-x_i)^{p-1})}\le n\frac{1}{n(x_{i+1}-x_i)}\le\frac{1}{n^{p-1}}$$

Since by Eq.(2.19)

$$\frac{1}{n(x_{i+1,} - x_i)} \leq \frac{1}{n^{p-1}}, i = 0,1, \dots, k-1$$

$$\frac{k}{n} \leq f(0)^p - f(1)^p = \|f'\|_p^p = 1$$

So which implies, $\frac{k}{p} < 1 \; and \; k \leq n$

So define, $s(\mathrm{x}) = \mathrm{f(x_\circ)} + \frac{1}{\mathrm{n}} \sum_{\mathrm{i=1}}^{\mathrm{k-1}} (\mathrm{x - x_i})_+^\circ$

It is clear, $\mathrm{s}(x_i) - f(x_i), i = 0,1, \dots, k-1$

And, $\|s - f\|_{L_p[0,1]} \leq sup|s - f| \leq \frac{2}{n}$

Define the even number greater then no

$$m_i = 2n_\circ + 16[ln^2 + (n^{-1} \max\{(x_{i+1} - x_i)^{-1}, (x_i - x_{i-1})^{-1}\})]$$

Since m_i are even $m_i > n_\circ$ We have, $\left\|Q_{n,m}(x) - X_+^\circ\right\| < e^{-\sqrt{m}}/_4$

Thus, $e^{-\sqrt{m}}/_4 < n \min\left\{x_{i+1} - x_i, x_i - x_{i-1}, \frac{1}{n}\right\}$

So, $m_i \leq 2N_\circ + 16ln^2 \; \frac{1}{n^{p-1}}$, so, $\left(2N_\circ + 16 \; ln^2 n^{\frac{1}{p-1}}\right)^2 < n$

Since $n > m^2$, $when \; p < 1$, For all i=1,2,…,k-1

We may use the lemma for the rational function $\mathrm{Q_{n,m}}$

Such that $Q_{n,m}(x) = \frac{1 + R_{[\frac{n}{m}]+1,m}(x)}{2}$

Put $\mathrm{R(x) = f(x_\circ)} + \frac{1}{\mathrm{n}} \sum_{\mathrm{i=1}}^{\mathrm{k-1}} \mathrm{Q_{n,m_i}(x - x_i)}$, and since each $\mathrm{Q_{n,m_i}}$ is

Anon decreasing function.so we have by

$\mathrm{n}\|f(x_i) - R(x_i)\|_p \leq c(p)n\|f(x_i) - S(x_i)\|_p + n\|S(x_i) - R(x_i)\|_p$

$2\mathrm{c(p)n} * \frac{2}{\mathrm{n}} + \mathrm{n} \left\|\mathrm{f(x_\circ)} + \frac{1}{\mathrm{n}} \sum_{\mathrm{i=1}}^{\mathrm{k-1}} (\mathrm{x - x_i})_+^\circ - (\mathrm{f(x_\circ)} + \frac{1}{\mathrm{n}} \sum_{\mathrm{i=1}}^{\mathrm{k-1}} \mathrm{Q_{n,m_i}}(\mathrm{x} - \mathrm{x_i})\right\|_\mathrm{p} \leq \sum_{\mathrm{i=1}}^{\mathrm{k-1}} \left\|(\mathrm{x - x_i})_+^\circ - \mathrm{Q_{n,m_i}(x - x_i)}\right\|_\mathrm{p} \leq 2\mathrm{c(p)} + \sum_{\mathrm{i=1}}^{\mathrm{k-1}} \left\|(\mathrm{x} - \mathrm{x_\circ})_+^\circ - \mathrm{Q_{n,m_i}(x - x_i)}\right\|_\mathrm{p}$ ▯

References

Gao, B.; Newman, D.J.; Popov, V.A. Convex Approximation By Rational Function, SIAMJ. Math. Anal. 26 (1995)

Live, G. I.; Opitz, U. CoMonotone Approximation Of [x], Serdica Math.J. 10 (1984) 88-105.

Lorenz, G.G.; Golitschek, M.V.; Makovoz, Y.; Constructive Approximation, Springer , Berlin, 1996.

Lorenz, G.G.; Zeller, K. Degree Of Approximation By Monotone Polynomials, J. Approx. Theory 1 (1968) 501-504.

Petrushev, P.P.; Popov, V.A. Rational Approximation Of Real Runction, Cambridge University Press, Cambridge, 1987.

Zhou, S.P. On monotone Rational Approximation: A new Approach, Analysis (Munich) 19 (1999) 391-395.