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On Guaranteed Global Exponential Stability Of Polynomial Singularly Perturbed Control Systems

Hajer Bouzaouache, Naceur Benhadj Braiek

Abstract: The problem of global exponential stability for a class of nonlinear singularly perturbed systems is examined in this paper. The stability analysis is based on the use of basic results of integral manifold of nonlinear singularly perturbed systems, the composite Lyapunov method and the notations and properties of Tensoriel algebra. Some of the derived results are presented as linear matrix inequalities (LMIs) feasibility tests. Moreover, we pointed out that if the global exponential stability of the reduced order subsystem is established this is equivalent to guarantee the global exponential stability of the original high order closed loop system. An upper bound ε_1 of the small parameter ε, can also be determined up to which established stability conditions via LMI's are maintained verified. A numerical example is given to illustrate the proposed approach.

Keywords: Nonlinear singularly perturbed system, Integral manifold, Lyapunov stability, Kronecker product, Linear matrix inequalities (LMIs).

1 Introduction

Stability analysis and control of nonlinear singularly perturbed systems have been widely studied in the literature [2], [6], [7], [12], [13]. In a two time scale framework, the stability study of the controlled systems using the Lyapunov stability method [15] and the integral manifold approach as a means for the control of nonlinear systems based on the singular perturbation method have been developed in recent years [10], [11], [14], [16], [17]. The approaches proposed in this direction. differ by imposing different conditions on the smoothness properties of the used functions, different assumptions and different classes of Lyapunov functions.

In this paper, we are concerned with the global exponential stability of polynomial singularly perturbed systems when the chosen design manifold is an exact integral one. Further extension of some previous results [11], [17] are suggested and leads to effective global exponential stability conditions via LMIs [8] which can be easily verified when using LMI toolbox of Matlab.

The contribution of the present paper is based, on one hand, on the use of the Lyapunov method which is a powerful tool for combined controller design and stability analysis, the definition of appropriate Lyapunov functions for the reduced systems and the corrected system via the integral manifold approach and on the other hand, on the notations and properties of the tensoriel product [9].

Our paper is organized as follows: in section 2 we present the considered description of the studied systems which allows important algebraic manipulations and some results from the literature on integral manifolds for nonlinear singularly perturbed systems. Some useful notations and needed assumptions are introduced in section 3. Exploiting the stability statements about singularly perturbed systems possessing integral manifolds and using the composite Lyapunov technique, we propose in section 4 an appropriate control law that insures the existence of an attractive integral manifold and furthermore insures stability of the studied systems when the dynamics are restrictive to the integral manifold. The stability results proving the global exponential stability of polynomial singularly perturbed systems are also given and presented as linear matrix inequalities feasibility tests. Finally an illustrative example is treated and some conclusions are drawn.

2 Studied systems and integral manifolds

The class of systems to be considered in this paper are described by the following state equations:

$$\begin{cases} \dot{x} = f(x,z) & \text{(a)} \\ \varepsilon\dot{z} = g(x,z) + l(x,z)u & \text{(b)} \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_1}$ is the state of the slow subsystem (1-a), $z \in \mathbb{R}^{n_2}$ is the state of the fast subsystem, $u \in \mathbb{R}^{p}$is the input control. ε is a small positive parameter. f, g and l are analytic vector fields which are sufficiently many times continuously differentiable functions of their arguments. Using the Kronecker power of vectors, these functions can be written in the polynomial form as [7]:

$$\begin{cases} f(x,z,\varepsilon) = \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{i+1} F_{ij}x^{[i+1-j]} \otimes z^{[j-1]} \\ g(x,z) = \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{i+1} G_{ij}x^{[i+1-j]} \otimes z^{[j-1]} \\ l(x,z) = \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{i+1} L_{ij}(I_m \otimes (x^{[i+1-j]} \otimes z^{[j-1]})) \end{cases} \tag{2}$$

In general the stability of the reduced order subsystems for a class of nonlinear singularly perturbed systems cannot guarantee the stability of the original full order system even with the additional stability of the boundary layer subsystem but when an attractive manifold is designed, the stability problem of the original system reduces to a stability problem of a low dimensional system on the manifold. Subsequently, in the context of control system design, our goal is to find an appropriate control law that insures the existence of an attractive integral manifold and furthermore insures stability of the studied systems (1) when the dynamics are restrictive to the integral manifold.

The basic ideas of exploiting the integral manifold method are:

- If an integral manifold Σ of systems described by (1) is established, so that if the initial states start on Σ, the trajectory of the system remains on Σ thereafter.
- When restricted to the integral manifold Σ, the dynamics of the system should insure stability of the equilibrium.
- The integral manifold should be attractive so that if the initial conditions are off Σ, the solution trajectory asymptotically converges to Σ.

According to these important issues of the integral manifold method, let's present the definition, and the properties of integral manifold of nonlinear systems.

Definition 1. [16] *The set $\Sigma \subset \mathbb{R} \times \mathbb{R}^n$ is said to be an integral manifold (invariant manifold) for the differential equation: $\dot{X} = N(t,X)$, $X, N \in \mathbb{R}^n$ if for $(t_0, X_0) \in \Sigma$, the solution $(t, X(t)), X(t_0) = X_0$, is in Σ for $t \in \mathbb{R}$. If $(t, X(t)) \in \Sigma$ for only a finite interval of time, then Σ is said to be a local integral manifold.*

Lemma 1. [11] *Consider the following system:*

$$\begin{cases} \dot{x} = f(t,x,y,\varepsilon) \\ \varepsilon\dot{y} = g(t,x,y,\varepsilon) \end{cases} \tag{3}$$

$x, f \in \mathbb{R}^n, y, g \in \mathbb{R}^m, t \in \mathbb{R}$, ε a small parameter. And suppose the following hypotheses hold:

- *The algebraic equation $g(t,x,y,0) = 0$ has an isolated solution $y = h_0(t,x)$, $\forall t \in \mathbb{R}$, $\forall x \in \mathrm{B}_x$*
- *The functions f, g, and h_0 are twice continuously differentiable ($\in C^2$) $\forall t \in \mathbb{R}, \forall x \in \mathrm{B}_x$. $\forall \varepsilon \in [0, \varepsilon_0)$ and for $\|y - h_0(t,x)\| \le \overline{\varphi}_y$ where ε_0 and $\overline{\varphi}_y$ are positive real constants.*

– *The eigenvalues* $\lambda_i = \lambda_i(t,x), i = 1,2,\ldots,m$ *of the matrix* $Z(t,x) := \left(\frac{\partial g}{\partial y}\right)(t,x,h_0(t,x),0)$ *satisfy the inequality*

$$\mathrm{Re}\,[\lambda_i] \leq -2\beta < 0 \;\forall t \in \mathbb{R}, \forall x \in B_x \tag{4}$$

Then there exists $\varepsilon \leq \varepsilon_1$ *such that* $\forall \varepsilon \in [0, \varepsilon_1)$*, the singularly perturbed system (3) has an m-dimensional local integral manifold*

$$\Sigma_\varepsilon : y = h_0(t,x) + H(t,x,\varepsilon) = h(t,x,\varepsilon) \tag{5}$$

where $h(t,x,\varepsilon)$ *is defined for all* $x \in \mathrm{B}_x$ *and* $\varepsilon \leq \varepsilon_1$ *and is continuously differentiable* $(\in C^1)$

The function $h(t,x,\varepsilon) \in C^1$ *satisfies the so-called manifold equation :*

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial x} f(t,x,h,\varepsilon) g(t,x,h,\varepsilon) \tag{6}$$

which is obtained by substituting y by h in equation (3).

On this manifold, the flow of systems (3) is governed by the n-dimensional reduced system

$$\dot{x} = f(t,x,h(t,x,\varepsilon),\varepsilon) \tag{7}$$

Furthermore, if for $x \in \mathrm{B}_x$ *and p integer we have* $f(t,x,y,\varepsilon) \in C^{p+1}, g(t,x,y,\varepsilon) \in C^{p+2}$ *and* $h_0(t,x) \in C^{p+2}$ *then* $h \in C^p$

3 Useful notations and assumptions

In our study we make use of the following lemma 2 and Assumptions 1-2. The lemma 2 is concerned with a Kronecker transformation of vectors. More properties of the Kronecker product are given in the Appendix.

Lemma 2. [6] *Given* $X = \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^n$*;* $x \in \mathbb{R}^{n_1},\ z \in \mathbb{R}^{n_2}$ *and* $n = n_1 + n_2$ *there exists a matrix* $\overset{(i)}{M} \in \mathbb{R}^{n^i \times n_i}$ *making possible a transformation which introduces the change of coordinates that forms the new following Kronecker power of vector:*

$$\widehat{\begin{pmatrix} x \\ z \end{pmatrix}}^{[i]} = \begin{bmatrix} x^{[i]} \\ \vdots \\ x^{[i-j]} \otimes z^{[j]} \\ \vdots \\ z^{[i]} \end{bmatrix} \in \mathbb{R}^{n_i}$$

such that

$$X^{[i]} = \begin{pmatrix} x \\ z \end{pmatrix}^{[i]} = \overset{(i)}{M} \widehat{\begin{pmatrix} x \\ z \end{pmatrix}}^{[i]} \tag{8}$$

with

$$\left\{ \begin{array}{l} \overset{(i)}{M} = (\overset{(i-1)}{M} \otimes I_n) \overset{(\mathrm{i})}{U} \overset{(\mathrm{i})}{V} \\ \overset{(1)}{M} = I_n \end{array} \right. \left| \begin{array}{l} 1 \leq j \leq (i-1) \\ n_i = \sum\limits_{j=0}^{i} n_1^{i-j} n_2^j \end{array} \right. \tag{9}$$

and

$$\overset{(\mathrm{i})}{V} = \underbrace{\begin{bmatrix} I_{n_1^i} & & & & \\ & \frac{a}{b} & & 0 & \\ & & \ddots & & \\ & 0 & & \frac{a}{b} & \\ & & & & \ddots & \\ & & & & & I_{n_2^i} \end{bmatrix}}_{\substack{(i+1) \text{ blocs columns} \\ 2i \text{ blocs rows}}}$$

$$\overset{(\mathrm{i})}{U} = \begin{bmatrix} U_{n_1^{(i-1)} \times n} & & & & \\ & \ddots & & 0 & \\ & & U_{n_1^{(i-k)} n_2^{(k-1)} \times n} & & \\ & 0 & & \ddots & \\ & & & & U_{n_2^{(i-1)} \times n} \end{bmatrix} \tag{10}$$

$for\ j \in \{2, \ldots i\}$

$$\begin{cases} a = U_{n_2 \times n_1^{(i-j+1)}} \otimes I_{n_2^{(j-2)}} \\ b = I_{n_1^{(i-j+1)} \times n_2^{(j-1)}} \end{cases}$$

The permutation matrix denoted $U_{n \times m}$ is defined in [9]

$$U_{n \times m} = \sum_{i=1}^{n} \sum_{k=1}^{m} (\underset{(n)}{e_i}\ \underset{(m)}{e_k^T}) \otimes (\underset{(m)}{e_k^T}\ \underset{(n)}{e_i}) \tag{11}$$

This matrix is square $(nm \times nm)$ and has precisely a single 1 in each row and in each column.

To clarify the meaning of $\overset{(i)}{M}$, consider the following example:

For $n = 3$ $(n_1 = 2, n_2 = 1)$, $i = 2$; $X^{[2]} = \begin{pmatrix} x \\ z \end{pmatrix}^{[2]} = \overset{(2)}{M} \widehat{\begin{pmatrix} x \\ z \end{pmatrix}}^{[2]}$

$$X^{[2]} = \begin{pmatrix} x \\ z \end{pmatrix}^{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x^{[2]} \\ x \otimes z \\ z^{[2]} \end{pmatrix}$$

Assumption 1. *There exists a continuously differentiable function* $V_1(t,x) : \mathbb{R} \times \mathbb{R}^{n_1} \to \mathbb{R}^+$ *such that the following inequalities hold:*

$$\forall \mathrm{t} \in \mathbb{R}, x \in \mathbb{R}^{n_1}$$

$$\alpha_1 \|x\|^2 \leq V_1(t,x) \leq \alpha_2 \|x\|^2 \tag{12}$$

$$\|\nabla_x V_1(t,x)\| \leq \alpha_3 \|x\|^2 \tag{13}$$

$$\nabla_t V_1(t,x) + (\nabla_x V_1(t,x))^T f(t,x,0,0) \leq -2\gamma_1 V_1 \tag{14}$$

where $\alpha_1, \alpha_2, \alpha_3$, and γ_1 are positive constants.

Assumption 2. *There exists a continuously differentiable function $V_2(t,z) : \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}^+$ such that the following inequalities hold:*

$$\forall t \in \mathbb{R}, z \in \mathbb{R}^{n_2}$$

$$\beta_1 \|z\|^2 \leq V_2(t,z) \leq \beta_2 \|z\|^2 \tag{15}$$

$$\frac{dV_2(t,z)}{dt} \leq -\frac{2}{\varepsilon}\gamma_2 V_2(t,z) \tag{16}$$

$$\tag{17}$$

where β_1, β_2 and γ_2 are positive constants.

4 Main results

Given the system (1), (2), we have to determine an adequate feedback control u that, starting from any initial states, will attract exponentially the trajectories of the closed loop system along the chosen design manifold to the equilibrium point at the origin.

In what follows, we assume that hypothesis of lemma 1 are satisfied, and hence the singularly perturbed system (1) has an n_2 dimensional integral manifold:

$$z = h_0(x) \tag{18}$$

satisfying the equation (1-b). The flow of system (1), on this manifold, is governed by the n_1 dimensional reduced system:

$$\dot{x} = f(t,x,h_0(t,x,\varepsilon),\varepsilon) \tag{19}$$

This result can be reached by the design of a desired control u satisfying:

$$l(x,z)u = -g(x,z) + A(z-h_0) + \varepsilon \frac{dh_0(x)}{dx} f(x,z) \tag{20}$$

where A is a Hurwitz matrix. Specifically, we choose the design manifold in this paper to be equal to $z = h_0(x) = 0$. Then, the control in equation (20) becomes:

$$l(x,z)u = -g(x,z) + Az \tag{21}$$

and the nonlinear singularly perturbed system (1) can be written as:

$$\begin{cases} \dot{x} = \sum\limits_{i=1}^{r} \sum\limits_{j=1}^{i+1} F_{ij} x^{[i+1-j]} \otimes z^{[j-1]} & \text{(a)} \\ \varepsilon \dot{z} = Az & \text{(b)} \end{cases} \tag{22}$$

It is then clear, that the fast subsystem and the fast states of the system (22) are attracted toward the manifold as quickly as desired by the choice of the Hurwitz matrix A.

The reduced order system of (22) is obtained by setting $\varepsilon = 0$ as:

$$\dot{x} = f(x,0) = \sum_{i=1}^{r} F_{ii} x^{[i]} \tag{23}$$

The boundary layer system is given, in the fast time scale $\tau \frac{t}{\varepsilon}$, by:

$$\frac{dz^*(\tau)}{d\tau} = Az^*(\tau) \tag{24}$$

Now to study the stability of the system (22), let's consider that the reduced order system (23) and the boundary layer system (24) have respectively $V_1(x)$ and $V_2(z)$ as quadratic Lyapunov candidate functions verifying Assumptions 1-2 and defined as follows :

$$V_1(x) = x^T P_1 x \tag{25}$$

$$V_2(z) = z^T P_2 z \tag{26}$$

where P_1, P_2 are symmetric positive definite matrices solutions of the following Lyapunov equations:

$$\dot{V}_1(x) \leq -x^T Q_1 x \tag{27}$$

$$\dot{V}_2(z) \leq -z^T Q_2 z \tag{28}$$

Q_1, Q_2 are also positive definite matrices.

Based on results of the stability theory [15] and others derived in previous work [1], [5], (27) and (28) are formulated as follows:

$$\tau_1^T (\mathrm{M}_1^T \mathrm{P}_1 + \mathrm{P}_1 \mathrm{M}_1) \tau_1 \leq -Q_1 \tag{29}$$

$$\left(\frac{A}{\varepsilon}\right)^T P_2 + P_2 \left(\frac{A}{\varepsilon}\right) \leq -Q_2 \tag{30}$$

where

$$\mathrm{P}_1 = \begin{bmatrix} P_1 & & & 0 \\ & P_1 \otimes I_{n_1} & & \\ & & \ddots & \\ 0 & & & P_1 \otimes I_{n_1^{s-1}} \end{bmatrix} \tag{31}$$

and

$$M_1 = \begin{bmatrix} \lambda_{11}\Pi_{11} & \lambda_{12}\Pi_{12} & \cdots & \lambda_{1s}\Pi_{1s} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda_{s1}\Pi_{s1} & \cdots & \cdots & \lambda_{ss}\Pi_{ss} \end{bmatrix} \tag{32}$$

with

$$\Pi_{k-j+1,j} = \begin{bmatrix} \underset{(n^{k-j},n^j)}{mat} (F_{kk}^{1^T}) \\ \underset{(n^{k-j},n^j)}{mat} (F_{kk}^{2^T}) \\ \vdots \\ \underset{(n^{k-j},n^j)}{mat} (F_{kk}^{n^T}) \end{bmatrix} \tag{33}$$

For the corrected system (22), we define the following Lyapunov functions:

$$V(x,z,\varepsilon) = X^T E_\varepsilon P X \tag{34}$$

where

$$X = \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^n; P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad \text{and} \quad E_\varepsilon = \begin{pmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{pmatrix} \tag{35}$$

The derivative of $V(x,z,\varepsilon)$ along the trajectories of (22) is given by:

$$\dot{V}(x,z,\varepsilon) = X^T E_\varepsilon P \dot{X} + \dot{X}^T E_\varepsilon P X \tag{36}$$

In the above equation, we need to explicit the derivative of the state vector X. So, we begin by writing the state equations (22) in the following form:

$$\dot{X} = \dot{\begin{pmatrix} x \\ z \end{pmatrix}} = \sum_{i=1}^{r} \Lambda_i \widehat{\begin{pmatrix} x \\ z \end{pmatrix}}^{[i]} \tag{37}$$

with

$$\Lambda_1 = \begin{pmatrix} F_{11} & F_{12} \\ 0 & \frac{A}{\varepsilon} \end{pmatrix}$$

and for $i > 1$:

$$\Lambda_i = \begin{pmatrix} F_{i1} \cdots F_{ij} \cdots F_{i(i+1)} \\ O_{n_2 \times \alpha_i} \end{pmatrix} \tag{38}$$

Using the results given by the lemma 2, it follows from equation (37) that

$$\dot{X} = \sum_{i=1}^{r} \Lambda_i \overset{(i)}{M} + X^{[i]} \tag{39}$$

where $\overset{(i)+}{M}$ is the Moore-Penrose pseudo inverse of $\overset{(i)}{M}$ defined in (A.4).

The derivative of the composite Lyapunov function (34) is then written:

$$\dot{V}(x,z,\varepsilon) = 2\sum_{k=1}^{r} X^T (E_\varepsilon P \Lambda_k \overset{(k)}{M} +) X^k \tag{40}$$

Using the property of the vec-function (1), we have:

$$\dot{V}(x,z,\varepsilon) = 2\sum_{k=1}^{r} V_k^T X^{[k+1]} \tag{41}$$

where

$$V_k = vec(E_\varepsilon P \Lambda_k \overset{(k)}{M} +) \tag{42}$$

Knowing that all polynomials with even degree $(2s)$ can be represented as a symmetric quadratic form. Thus, we assume in the following development that r is odd: $r = 2s-1$, and it comes out:

$$V_k^T X^{[k+1]} = \sum_{j=g_k}^{h_k} \lambda_{k-j+1,j} X^{[k+1-j]} N_{k-j+1,j} X^{[j]} \tag{43}$$

where $\lambda_{k-j+1,j}$ are arbitrary reals verifying:

$$\sum_{j=g_k}^{h_k} \lambda_{k-j+1,j} = 1 \tag{44}$$

and for $k = 1, \ldots, 2s-1$:

$$g_k = \sup(0, k+1-s) \text{ and } h_k = \inf(s,k) \tag{45}$$

for $j = g_k, \ldots, h_k$:

$$N_{k-j+1,j} = \underset{(n^{k-j+1}, n^j)}{mat} (V_k) \tag{46}$$

Applying the properties [1], one obtains:

$$N_{k-j+1,j} = \underset{(n^{k-j+1}, n^j)}{mat} \left(vec(E_\varepsilon P \Lambda_k \overset{(k)}{M} +) \right) = U_{n^{k-j} \times n} (E_\varepsilon P \otimes I_{n^{k-j}}).M_{k-j+1,j} \tag{47}$$

with

$$M_{k-j+1,j} = \begin{bmatrix} \underset{(n^{k-j},n^j)}{mat}(\mathrm{B}_k^{1^T}) \\ \underset{(n^{k-j},n^j)}{mat}(\mathrm{B}_k^{2^T}) \\ \vdots \\ \underset{(n^{k-j},n^j)}{mat}(\mathrm{B}_k^{n^T}) \end{bmatrix} \text{and} \mathrm{B}_k = \Lambda_k \overset{(k)}{M} + \tag{48}$$

where B_k^i is the i-th row of the matrix B_k:

$$\mathrm{B}_k = \begin{bmatrix} \mathrm{B}_k^1 \\ \mathrm{B}_k^2 \\ \vdots \\ \mathrm{B}_k^n \end{bmatrix} \tag{49}$$

By (47) and from the relation (48), we obtain:

$$\begin{aligned} & X^{[k-j+1]^T} N_{k-j+1,j} X^{[j]} \\ &= X^{[k-j+1]^T} U_{n^{k-j}\times n}(E_\varepsilon P \otimes I_{n^{k-j}})\mathrm{M}_{k-j+1,j} X^{[j]} \\ &= X^{[k-j+1]^T} (E_\varepsilon P \otimes I_{n^{k-j}})\mathrm{M}_{k-j+1,j} X^{[j]} \end{aligned} \tag{50}$$

Consequently, we have:

$$V_k^T X^{[k+1]} = \sum_{j=g_k}^{h_k} \lambda_{k-j+1,j} X^{[k-j+1]^T} N_{k-j+1,j} X^{[j]} = X^T (P_\varepsilon M_k) X \tag{51}$$

with

$$X = \begin{bmatrix} X \\ X^{[2]} \\ \vdots \\ X^{[s]} \end{bmatrix} \tag{52}$$

and

$$P_\varepsilon = \begin{bmatrix} E_\varepsilon P & & & 0 \\ & E_\varepsilon P \otimes I_n & & \\ & & \ddots & \\ 0 & & & E_\varepsilon P \otimes I_{n^{s-1}} \end{bmatrix} \tag{53}$$

Let's note that P_ε is a symmetric positive matrix, and $\dot{V}(X,\varepsilon)$ (41) can be written as:

$$\dot{V}(X,\varepsilon) = 2\sum_{k=1}^{2s-1} V_k^T X^{[k+1]} = X^T (P_\varepsilon M_\varepsilon + M_\varepsilon^T P_\varepsilon) X \tag{54}$$

with

$$M_\varepsilon = \sum_{k=1}^{2s-1} M_k = \begin{bmatrix} \lambda_{11}M_{11} & \lambda_{12}M_{12} & \cdots & \lambda_{1s}M_{1s} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda_{s1}M_{s1} & \cdots & \cdots & \lambda_{ss}M_{ss} \end{bmatrix} \tag{55}$$

When considering the nun-redundant form, the vector X can be written as:

$$X = \tau \tilde{X} \tag{56}$$

where

$$\tau = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_s \end{bmatrix} \text{ and } \tilde{X} = \begin{bmatrix} \tilde{X} \\ \vdots \\ \tilde{X}^{[s]} \end{bmatrix} \tag{57}$$

>From (54) and (56) we easily obtain:

$$\dot{V}(X,\varepsilon) = \tilde{X}^T \tau^T (P_\varepsilon M_\varepsilon + M_\varepsilon^T P_\varepsilon) \tau \tilde{X} \tag{58}$$

Let us denote the largest eigenvalue of the matrix P_ε by $\lambda_{\max}(P_\varepsilon)$, the smallest eigenvalue of Q by $\lambda_{\min}(Q)$. Where the matrix Q verifies: $\dot{V}(X,\varepsilon) \leq -X^T QX$. The positive definiteness of P_ε and Q implies that these scalars are all strictly positive. Since matrix theory shows that:

$$P_\varepsilon \leq \lambda_{\max}(P_\varepsilon)I; \; \lambda_{\min}(Q)I \leq Q \tag{59}$$

We have

$$X^T QX \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P_\varepsilon)} X^T \left[\lambda_{\max}(P_\varepsilon)I\right] X \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P_\varepsilon)} X^T \left[\lambda_{\max}(E_\varepsilon P)I\right] X \tag{60}$$

Otherwise

$$X^T X \geq \|X\|^2 \text{ and } E_\varepsilon P \leq \lambda_{\max}(E_\varepsilon P)\mathrm{I} \tag{61}$$

Hence, (54) will satisfies the following condition:

$$\dot{V}(X,\varepsilon) \leq -2\gamma V(X,\varepsilon) \text{ where } \gamma = \frac{1}{2} \cdot \frac{\lambda_{\min}(Q)}{\lambda_{\max}(\mathrm{P}_\varepsilon)} \tag{62}$$

It comes out

$$V(X,\varepsilon) \leq V(X_0) e^{-2\gamma(t-t_0)} \tag{63}$$

Considering the previous developments, we state now our main result:

Theorem 1. *Assume the following assumptions hold:*

(i) Lemma 1 satisfied

(ii) Assumptions 1 - 2 are satisfied

The system (1) is globally exponentially stable (GES), if there is, for all $\varepsilon < \varepsilon_1$, $\varepsilon_1 > 0$ a feasible solution to the LMI :

$$\begin{cases} \varepsilon > 0 \\ \exists P_1^T = P_1 > 0 \\ \exists P_2^T = P_2 > 0 \\ \exists P^T = P > 0 \\ \left(\frac{A}{\varepsilon}\right)^T P_2 + P_2 \left(\frac{A}{\varepsilon}\right) \leq -Q_2 \\ \tau_1^T (\mathrm{M}_1^T \mathrm{P}_1 + \mathrm{P}_1 \mathrm{M}_1) \tau_1 \leq -Q_1 \\ \tau^T (\mathrm{M}_\varepsilon^T \mathrm{P}_\varepsilon + \mathrm{P}_\varepsilon \mathrm{M}_\varepsilon) \tau \leq -Q \end{cases} \tag{64}$$

M_1, τ_1 and P_1 are given by (32), (57) and (31). M_ε, τ and P_ε are given by (55), (57) and (53). Moreover, the Lyapunov function that demonstrates the G.E.S is given by: $V(x) = X^T E_\varepsilon P_\varepsilon X$

Now, let's evaluate the convergence rate of the full order system (22). In view of (12), (15), and (27), we have for all $t \in \mathbb{R}$, $x \in \mathbb{R}^{n_1}$ and $z \in \mathbb{R}^{n_2}$:

$$V_1(x) \leq (\alpha_2 \|x_0\|^2 + \varepsilon\beta_2 \|z_0\|^2) e^{-2\gamma(t-t_0)} \tag{65}$$

>From (12) and (15) we have:

$$\|x\| \leq \left(\left(\sqrt{\frac{\alpha_2}{\alpha_1}} \|x_0\| \right)^2 + \left(\sqrt{\frac{\varepsilon\beta_2}{\alpha_1}} \|z_0\| \right)^2 \right)^{1/2} e^{-\gamma(t-t_0)} \leq \left(\sqrt{\frac{\alpha_2}{\alpha_1}} \|x_0\| + \sqrt{\frac{\varepsilon\beta_2}{\alpha_1}} \|z_0\| \right) e^{-\gamma(t-t_0)} \tag{66}$$

Identically, using (15) and (16), we obtain:

$$\beta_1 \left\|z^2\right\| \leq V_2(z) \leq \beta_2 \left\|z_0^2\right\| e^{-2(\gamma_2/\varepsilon)(t-t_0)} \tag{67}$$

then

$$\|z\| \leq \sqrt{\frac{\beta_2}{\beta_1}} \|z_0\| e^{-(\gamma_2/\varepsilon)(t-t_0)} \tag{68}$$

>From (66) and (68), we can write in the case $\gamma \leq (\gamma_2/\varepsilon)$ that

$$\|X\|^2 \leq 2\eta e^{-2\gamma(t-t_0)}$$

where

$$\eta = \max\left(\left(\sqrt{\frac{\alpha_2}{\alpha_1}}\|x_0\| + \sqrt{\frac{\varepsilon\beta_2}{\alpha_1}}\|z_0\|\right)^2, \left(\sqrt{\frac{\beta_2}{\beta_1}}\|z_0\|\right)^2\right) \tag{69}$$

which implies that the norm $\|X\|$ of the state vector converges to zero exponentially, with a rate γ. The convergence rates of the reduced systems can be calculated:

$$\gamma_1 = \frac{1}{2} \cdot \frac{\lambda_{\min}(Q_1)}{\alpha_2}, \qquad \gamma_2 = \frac{\lambda_{\min}(Q_2)}{2\beta_2}$$

>From above, we state the following second result:

Theorem 2. *Assume the following assumptions hold:*
(i) Lemma 1 satisfied
(ii) Assumptions 1 - 2 are satisfied
(iii) There exists a Lyapunov function $V(t, X, \varepsilon)$ that satisfies equation ((34))
Then the original nonlinear singularly perturbed system ((1)) is globally exponentially stable under the proposed control ((21)) and with the convergence rate γ ((63)).

Moreover, note that when we proves that the limit of γ as $\varepsilon \to 0$, tends to the convergence rate of the reduced order system. This implies that under the proposed control (21), the global exponential stability of the initial studied system is equivalent to that of the reduced order system.

5 Illustrative Example

To illustrate the previous derived results, we consider a third order nonlinear singularly perturbed system defined by the following equations:

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + 0.1x_1 z \\ \dot{x}_2 = -x_1 - 0.09x_2 + 2z + 0.05x_1 z \\ \varepsilon\dot{z} = 4x_1 - 4x_2 + z + 0.5x_1^2 - x_2^2 + 10u \end{cases}$$

This system can be described by the following model using the Kronecker product and the power of vectors which allowed important algebraic manipulations.

$$\begin{cases} \dot{x} = F_{11}x + F_{12}z + F_{21}x^{[2]} + F_{22}x \otimes z + F_{23}z^{[2]} \\ \varepsilon\dot{z} = G_{11}x + G_{12}z + G_{21}x^{[2]} + G_{22}x \otimes z + G_{23}z^{[2]} + Bu \end{cases}$$

where

$$F_{11} = \begin{bmatrix} -1 & 1 \\ -1 & -0.09 \end{bmatrix}, F_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, G_{11} = \begin{bmatrix} 4 & -4 \end{bmatrix}, G_{12} = 1,$$

$$F_{21} = 0_{2\times 4}, F_{22} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}, F_{23} = 0_{2\times 1}$$

$$G_{21} = \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix}, G_{22} = \begin{bmatrix} 0 & 0 \end{bmatrix}, G_{23} = 0, B = 10.$$

When implemented using the LMI toolbox of Matlab, the proposed LMI's conditions proves that the numerical studied system which is initially instable can be globally exponentially stabilised by the given controller with the considered $A = -1$ for all $\varepsilon < \varepsilon_1 = 0.5$ in view of theorem 1:

$$u = -0.4x_1 + 0.4x_2 - 0.1z - 0.05x_1^2 + 0.1x_2^2$$

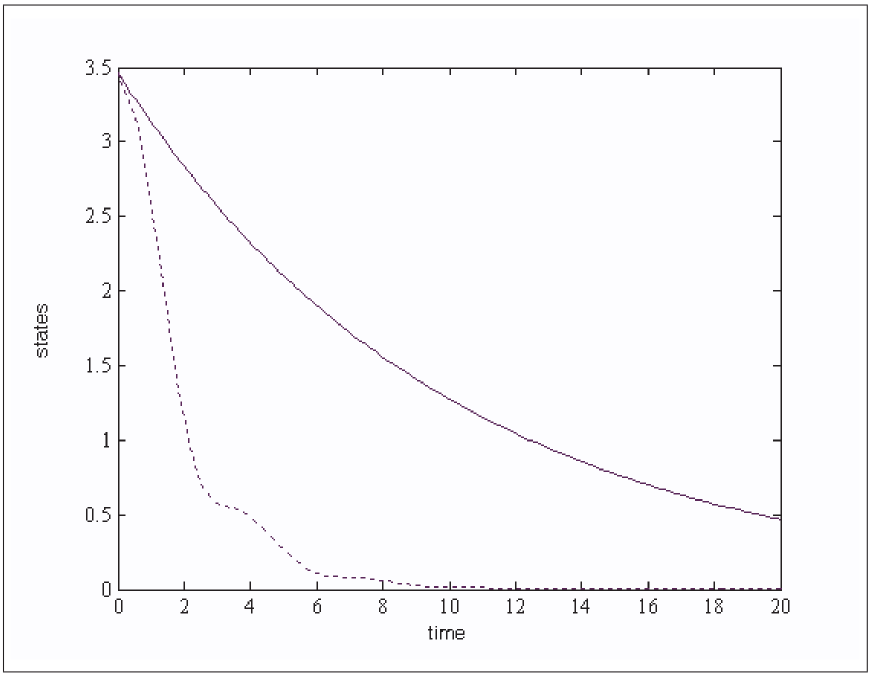

Figure 1: State trajectories of the controlled studied system

Hence for all $\varepsilon < \varepsilon_1 = 0.5$ and from any initial states, the trajectories of the system are steered to the origin along the integral manifold with the convergence rate 0.1 in view of (62). Indeed, it is shown in Fig. 1 that the state trajectories of the controlled system (– – –) are bounded by the function (—).

6 Conclusion

In this paper, the global exponential stabilisation for nonlinear singularly perturbed control systems is investigated. In the stability study, the composite Lyapunov method was applied and the global exponential stability of the equilibrium of the full control system was established for all $\varepsilon < \varepsilon_1$. The upper bound ε_1 for which the stability properties are guaranteed can be reached after a number of iterations on ε when resolving the proposed LMI's conditions via the LMI Toolbox of Matlab. A numerical example has been provided to illustrate the proposed results.

7 Appendix

Notations: The dimensions of the matrices used here are the following:
$A(p\times q)$, $B(r\times s)$, $C(s\times h)$, $D(s\times h)$, $E(n\times p)$,
$P(n\times n)$, $X(n\times 1)\in\mathbb{R}^m$, $Y(m\times 1)\in\mathbb{R}^m$, $Z(q\times 1)\in\mathbb{R}^q$
Let's consider the following notations:
I_n: $n\times n$ identity matrix;
$0_{n\times m}$: $n\times n$ zero matrix;
0zero matrix of convenient dimensions ;
A^T: transpose of matrix A;
$A>0$ ($A\geq 0$): symetric positive definite (semi definite matrix A);
$\underset{(q)}{e_k}$: q dimensional unit vector which has 1 in the kth element and zero elsewhere.
The kth row of a matrix such as A is denoted $A_{k.}$ and the kth column is denoted $A_{.k}$. The ik element of A will be denoted a_{ik}.
The Kronecker product of A and B is denoted $A\otimes B$ a $(p.r\times q.s)$ matrix, and the $i-th$ Kronecker's power of A denoted i $A^{[i]}=A\otimes A\otimes\cdots\otimes A$ s a $(p^i\times q^i)$ matrix.

The nun-redundant j-power $\tilde{X}^{[j]}$ of the state vector X introduced in [9] is defined as:

$$\tilde{X}^{[1]}=X^{[1]}=X$$
$$\begin{cases}\forall j\geq 2 \quad \tilde{X}^{[j]}=\Big[x_1^j, x_1^{j-1}x_2,\cdots,x_1^{j-1}x_n, x_1^{j-2}x_2^2\\ x_1^{j-2}x_2x_3,\cdots,x_1^{j-2}x_2x_n,\\ \cdots x_1^{j-2}x_2^n,\cdots,x_1^{j-3}x_2^3,\cdots,x_n^j\Big]\end{cases} \tag{A.1}$$

where the repeated components of the redundant j-power are omitted. Then we have the following relation:

$$\begin{cases}\forall j\in\mathbb{N} \quad \exists! T_j\in\mathbb{R}^{n^j\times\alpha_j}; \alpha_j=\begin{pmatrix} n+j-1\\ j\end{pmatrix}\\ X^{[j]}=T_j\tilde{X}^{[j]}\end{cases} \tag{A.2}$$

thus, one possible solution for the inversion can be written as:

$$\tilde{X}^{[j]}=T_j^+X^{[j]} \tag{A.3}$$

where T_j^+ is the Moore-Penrose pseudo inverse of T_j given by:

$$T_j^+=\left(T_j^TT_j\right)^{-1}T_j^T \tag{A.4}$$

and α_j stands for the binomial coefficients.

An important vector valued function of matrix denoted $vec(.)$ was defined as [9]:

$$vec_{pq\times 1}(A)=\begin{bmatrix} A_{.1}\\ A_{.2}\\ \vdots\\ A_{.q}\end{bmatrix} \tag{A.5}$$

A matrix valued function is a vector denoted $mat_{(n,m)}(.)$ was defined in [1] as follows: If V is a vector of dimension $p=n\times n$ then $Mmat_{(n,m)}(V)$ is the $n\times m$ matrix verifying:

$$V=vec(M) \tag{A.6}$$

Among the main properties of this product presented in [9], [1], we recall the following useful ones:

$$(A\otimes B)(C\otimes D)=(AC)\otimes(BD) \tag{A.7}$$
$$(A\otimes B)^T=A^T\otimes B^T \tag{A.8}$$
$$B\otimes A=U_{r\times p}(A\otimes B)U_{q\times s} \tag{A.9}$$
$$X\otimes Y=U_{n\times m}(Y\otimes X) \tag{A.10}$$
$$vec(EAC)=\left(C^T\otimes E\right)vec(A) \tag{A.11}$$
$$vec\left(A^T\right)=U_{p\times q}vec(A) \tag{A.12}$$
$$\forall i\leq k \quad X^{[k]}=U_{n^i\times n^{k-i}}X^{[k]} \tag{A.13}$$

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Hajer Bouzaouache, Naceur Benhadj Braiek
Laboratory of Study and Automatic Control of Processes
Address: Polytechnic School of Tunisia (EPT) BP.743, 2078 La Marsa, Tunisia.
E-mail: hajer.bouzaouache,naceur.benhadj@ept.rnu.tn

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