Asymptotic Solution of Singularly Pertubeted Problems with Quadratic Small Parameter

Methaq Hamza Geem
Dept. of Mathematics, college of Education,Al-Qadisiya university, Iraq
Methaq.Geem@qu.edu.iq

Abstract

In this paper we study singularly perturbed with initial problem of the system with two linear ordinary differential equations, each of which contains small parameter in its derivative and quadratic the same small parameter in one of them. A uniform asymptotic expansion is constructed on a time interval solution of the problem. obtained formulas for the terms of internal expansion, they allow us to find them only through algebraic operations, finally we give examples which related with our subject.

Keywords: Differential equations, asymptotic, singularly perturbed , boundary function, small parameeter,direct scheme.

الخلاصة

في هذا البحث درسنا مسألة الاضطراب المنفرد بالشروط الابتدائية لنظام من من المعادلات التفاضلية العادية الخطية، كل منها يحتوي على معلمة صغيرة في مشتقاتها والتربيعية من نفس المعلمة الصغيرة في واحد منهم. تم بناء توسيع منتظم تناظري على فترة من الزمن لحل المسألة. حصلنا على صيغة من الحدود للتوسيع الخارجي التي تسمح لنا بايجادها من خلال العمليات الجبرية، وأخيرا قدمنا أمثلة التي تتعلق بموضوعنا.

الكلمات المفتاحية:المعادلات التفاضلية،التقريب التناظري ، الاضطراب المنفرد، الدالة الحدودية، المعلمة الصغيرة، المخطط الامامي.

Mathematics Subject Classification : 34E15

I.Introduction:

The subject of differential equations with small parameeters high derivatives arise in the modeling and study of physical, biological, chemical phenomena and processes. This kind of equations are also found in automatic control theories, nonlinear oscillations, in gas dynamics, in the description of gyroscopic systems. These equations are called singularly perturbed. Their feature is that the order of the degenerate equation resulting from the initial with zero values of the parameeters, lower than the order of the original equations, as a consequence, the solution of the degenerate equation can not satisfy all the conditions specified for the initial equation. The problem as the formula:

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

Is called Cauchy problem with intial point x_0.

Primarilies of the theorey of singular perturbtions are the works of [Tikhonov ,1948,1950,1952], in which a general statement of the problem is given the Cuachy problem of systems of nonlinear ordinary differential equations with small parameeters for derivatives and the limiting transition from the solution for the orignal problem to the solution of the degenrate problem when the parameeters tend to zero.

Construction of approximation solutions of singularly perturbed problems is carried out in many ways, both numerically and asymptotically. The most widely accepted among the asymptotic methods are the method boundary functions [Vasil'eva ,1969,1959,1963,1973,1990,1962].

In this paper we consider the following singularly perturbed problem. The Cauchy problem of a system with two ordinary differential equations with quadratic small parameeters for derivatives. The behavior of the solution for the problem at a finite time interval is studied. The novelty of the problems of this study is that small parameeter that is in the two equations of the system each for its derivative, tend to zero independently of each other. In this way, we are talking about the construction of an asymptotics of the solution that is uniformly suitable for any relations between small parameeters.

II.view Problem

Consder the system:

$$\left.\begin{aligned} \varepsilon\dot{x} &= a(t)x + b(t)y \\ \varepsilon^2\dot{y} &= c(t)x + d(t)y \end{aligned}\right\} \dots\dots\dots..(1)$$

$$\left.\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \end{aligned}\right\} \dots\dots\dots\dots\dots..\dots..(2)$$

Such that a(t),b(t),c(t),d(t), are infinite differential functions on [0,T], satisfy the condtions a(t)<0 , d(t)<0 ,$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0\ , for\ every\ t \in [0,T]$.

Where ε > 0 is small parameeter.

III.The constraction of formula for solution of the problem

The asymptotic expansoin for the solution of problem (1),(2) will be constracted as type:

$$\left.\begin{aligned} x(t,\varepsilon) &= \bar{x}(t,\varepsilon) + \pi_x(\tau,\varepsilon) \\ y(t,\varepsilon) &= \bar{y}(t,\varepsilon) + \pi_y(\tau,\varepsilon) \end{aligned}\right\} \dots\dots\dots(3)$$

$\bar{x}(t,\varepsilon) = \sum_{i,j=0}^{\infty} \varepsilon^{i+2j}\, \bar{x}_{i,j}(t)$,

$\bar{y}(t,\varepsilon) = \sum_{i,j=0}^{\infty} \varepsilon^{i+2j}\, \bar{y}_{i,j}(t),\ \tau = \frac{t}{\varepsilon^3}$

Now by substituting the expansion (3) in the equations 1-2 and equating coefficients which the same power of ε we can determine the coffcients of series (3). In partcular, the system for finding $x_{0,0}, y_{0,0}$ coincides with a degenrate system of the system (1) (at ε=0).

The remaining coefficients of the expansion (3) are found from systems of linear algebraic equations of the form:

$$A(t)W_{m,n}(t) = T_{m,n}(t) \dots\dots\dots(4)$$

Where

$$A(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}, W_{m,n}(t) = \begin{bmatrix} x_{m,n}(t) \\ y_{m,n}(t) \end{bmatrix} \dots \dots (5)$$

A vector valued unctions $T_{m,n}$ can be expressed in terms of $W_{i,j}$, $(i + j < n + m)$. We note that the systems (4) have an uniquely solution , because from the condition (2) we have the relation $\det(A) \neq 0 , \forall\, t \in [0,T]$.

Further, we expand the functions a(t),b(t),c(t),d(t) in power sereis in neighborhoods of the point t=0 with coeffcients a_i ,b_i ,c_i ,d_i [i=0,1,2,…], respectively and new variable τ .

Without loss of generality view of condition $(D > 0)$ we assume that a_0=d_0=-1. The internal expansion π_x , π_y can be written by the following type:

$$\pi_x(\tau,\varepsilon) = \sum_{i,j=0}^{\infty} \varepsilon^{i+2j}\, \pi_x^{ij}(\tau), \pi_y(\tau,\varepsilon) = \sum_{i,j=0}^{\infty} \varepsilon^{i+2j}\, \pi_y^{ij}(\tau) \dots \dots (6)$$

Such that:

π_x^{ij}, π_y^{ij}, are a boundary functions in a neighborhoods of t=0 and it satisfies the following equality :

$$\left\| \pi_g^i(\tau) \right\| \le c e^{-\omega \tau_i}, \quad g = x, y \dots \dots \dots \dots (7)$$

Where c,ω are positive constants.

Substituting equation (6) in equation (1) we obtain :

$$\varepsilon^{-2} \sum_{i,j=0}^{\infty} \varepsilon^{i+2j} \frac{d\pi_x^{ij}}{d\tau} = \left(\sum_{i=0}^{\infty} a_i\, \tau^i \varepsilon^{3i} \right) \sum_{i,j=0}^{\infty} \varepsilon^{i+2j} \pi_x^{ij} + \left(\sum_{i=0}^{\infty} b_i\, \tau^i \varepsilon^{3i} \right) \sum_{i,j=0}^{\infty} \varepsilon^{i+2j} \pi_y^{ij}$$

$$\varepsilon^{-1} \sum_{i,j=0}^{\infty} \varepsilon^{i+2j} \frac{d\pi_y^{ij}}{d\tau} = \left(\sum_{i=0}^{\infty} c_i\, \tau^i \varepsilon^{3i} \right) \sum_{i,j=0}^{\infty} \varepsilon^{i+2j} \pi_x^{ij} + \left(\sum_{i=0}^{\infty} d_i\, \tau^i \varepsilon^{3i} \right) \sum_{i,j=0}^{\infty} \varepsilon^{i+2j} \pi_y^{ij}$$

We denote $\zeta_{m,n}(.) = \begin{bmatrix} \pi_x^{mn} \\ \pi_y^{mn} \end{bmatrix} \dots \dots (8)$

We will seek $\zeta_{m,n}(\tau,\varepsilon)$ by solving the following systems of differential equations with constant coefficients:

$$\frac{d}{d\tau} \zeta_{m,n} = A_0(\varepsilon) \zeta_{m,n}(\tau,\varepsilon) + S_{m,n}(\tau,\varepsilon) , m,n = 0,1, \dots \dots \dots (9)$$

Such that:

$$S_{m,n}(\tau,\varepsilon) = \sum_{i=0}^{\min\{m,n\}} \tau^i A_i \;, A_i = \begin{bmatrix} \varepsilon^2 a_i & \varepsilon^2 b_i \\ \varepsilon c_i & \varepsilon d_i \end{bmatrix}$$

With the conditions:

$$\zeta_{0,0}(0,\varepsilon) = W_0 - W_{0,0}(0) \dots \dots (10)$$

$$\zeta_{m,n}(0,\varepsilon) = -W_{m,n}(0) \dots \dots (11)$$

$$W_0 = \binom{x_0}{y_0}$$

Thus the formula of construction of the series (3) is complete to be proved that these series are inded an asymptotic expansoin solution of the problem on the closed interval [0,T].

IV. Estimate of e terms of internal expansion:

4.1.Definition:

Let λ_1, λ_2 are eigenvalues of matrix $A_0(\varepsilon)$ then:

$$\lambda_i = \frac{-\varepsilon - \varepsilon^2 \pm \sqrt{(\varepsilon + \varepsilon^2)^2 - 4\varepsilon^3 D_0}}{2}$$

$$\lambda_i = \frac{-\varepsilon - \varepsilon^2 \pm \sqrt{(\varepsilon - \varepsilon^2)^2 + 4\varepsilon^3 b_0 c_0}}{2}$$

$$\lambda_i = \frac{-\varepsilon - \varepsilon^2 \pm \varepsilon\sqrt{(1 - \varepsilon)^2 + 4\varepsilon b_0 c_0}}{2} \quad , i = 1,2.$$

Where $D_0 = 1 - b_0 c_0 > 0$

Depending on the values of b_0, c_0 in the matrix $A_0(\varepsilon)$ in definition 4.1, we can disscaus the cases:

1- λ_1, λ_2 are real ,different with note that $\lambda_2 < \lambda_1 < 0$.

In this case we realize its with the following relations:

i-$b_0 c_0 > 0$,$\forall\, \varepsilon > 0$.

ii-$b_0 c_0 = 0$ and $\varepsilon \neq 0 , \varepsilon \neq 1$

2-$\lambda_1 = \lambda_2 \in R$.

This case is realized at the conditions: $b_0 c_0 = 0$ and $\varepsilon = 0 ,1$

3-λ_1, λ_2 are complex with its conjugate.

This case is hold e condition: $Re(\lambda_i) = \frac{-\varepsilon(1+\varepsilon)}{2} < \frac{-\varepsilon^2}{1+\varepsilon}$

Since :

$$|\lambda_2| = \frac{\varepsilon + \varepsilon^2 + \varepsilon\sqrt{(1+\varepsilon)^2 - 4\varepsilon D_0}}{2} > \frac{\varepsilon(1+\varepsilon)}{2} > |\lambda_1|$$
$$= \frac{\varepsilon + \varepsilon^2 - \varepsilon\sqrt{(1+\varepsilon)^2 - 4\varepsilon D_0}}{2} = \frac{2\varepsilon^3 D_0}{\varepsilon + \varepsilon^2 + \varepsilon\sqrt{(1+\varepsilon)^2 - 4\varepsilon D_0}}$$
$$> D_0 \frac{\varepsilon^3}{\varepsilon + \varepsilon^2} = D_0 \rho \dots\dots\dots..(12)$$

$$\rho = \frac{\varepsilon^3}{\varepsilon + \varepsilon^2} = \frac{\varepsilon^2}{1+\varepsilon} \dots\dots(13)$$

4.2.Proposition:

The exponesial matrix $e^{A_0\tau}$ satisfies the estimate:

$$\|e^{A_0\tau}\| \le C_0 e^{-(k\rho(\varepsilon)\tau)}, \tau \ge 0$$

Such that C_0,k are positive constants , it does not depended on τ, ε

Proof:

By using [Lappo-Danilevsky I.A ., 1957 ,p.49) we have :

$$e^{A_0\tau} = e^{\lambda_2\tau} I + \frac{e^{\lambda_1\tau} - e^{\lambda_2\tau}}{\lambda_1 - \lambda_2}(A_0 - \lambda_2 I) \dots\dots\dots(14)$$

$$\|A_0 - \lambda_2 I\| \le k\varepsilon(1+\varepsilon) \dots\dots\dots\dots.(15)$$

In case(1) of estimate we have the inqualities:

$$e^{\lambda_2\tau} \le e^{\lambda_1\tau} \le e^{-D_0\rho\tau} \dots\dots\dots..(16)$$

By definition of λ_i and equation(15) we obtain:

$$\lambda_1 - \lambda_2 = \varepsilon\sqrt{(1-\varepsilon)^2 + 4\varepsilon b_0 c_0}$$

Thus:

$$\frac{\|A_0 - \lambda_2 I\|}{\lambda_1 - \lambda_2} \le k \frac{\varepsilon(1+\varepsilon)}{\varepsilon\sqrt{(1-\varepsilon)^2 + 4\varepsilon b_0 c_0}} = k \frac{(1+\varepsilon)}{\sqrt{(1-\varepsilon)^2 + 4\varepsilon b_0 c_0}} \le k_1 \dots\dots \quad (17)$$

Now we note :

$$\frac{e^{\lambda_1\tau} - e^{\lambda_2\tau}}{\lambda_1 - \lambda_2} = \tau \int_0^1 e^{(\lambda_2\tau + \theta(\lambda_1\tau - \lambda_2\tau))}\, d\theta$$

$$= \tau \int_{0}^{\frac{1}{2}} e^{(\lambda_2\tau + \theta(\lambda_1\tau - \lambda_2\tau))}\, d\theta + \tau \int_{\frac{1}{2}}^{1} e^{(\lambda_2\tau + \theta(\lambda_1\tau - \lambda_2\tau))}\, d\theta$$

$$\leq \tau e^{\frac{\lambda_2\tau}{2}} \int_{0}^{\frac{1}{2}} e^{\theta\lambda_1\tau}\, d\theta + \tau e^{\frac{\lambda_1\tau}{2}} \int_{\frac{1}{2}}^{1} e^{\theta\lambda_2\tau}\, d\theta$$

$$\leq \frac{k_2}{|\lambda_2|}\left[e^{\frac{\lambda_2\tau}{4}} + e^{\frac{\lambda_1\tau}{2}}\right] \ldots\ldots\ldots\ldots(18)$$

Now by (17) and (18) and by cases (1),(2) we have the eigenvalues of matrix A_0 satisfy $Re\lambda_i \leq -\lambda$, $\lambda > 0$ and thus obtain:

$$\|e^{A_0\tau}\| \leq C\frac{\|A_0\|}{\lambda}e^{-\frac{\lambda\tau}{2}}, \tau \geq 0$$

$$Re(\lambda_i) = \frac{-\varepsilon(1+\varepsilon)}{2}$$

Hence

$$\|e^{A_0\tau}\| \leq C_0 e^{-(k\rho(\varepsilon)\tau)}, \tau \geq 0$$

4.3.Example:

$$\varepsilon\frac{dx}{dt} = x - y\,, \varepsilon^2\frac{dy}{dt} = x + y \ldots\ldots..(19)$$

$$x(0) = x_0\,, y(0) = y_0 \ldots\ldots\ldots\ldots(20)$$

The exact solution of equation (19),(20) have a type:

$$x(t,\varepsilon) = (x_0\varepsilon^2 - y_0\varepsilon)(\varepsilon^2 + \varepsilon) + (x_0 - y_0)\left(\frac{\varepsilon}{\varepsilon + \varepsilon^2}\right)e^{-\frac{\varepsilon+\varepsilon^2}{\varepsilon^3}t}$$

$$y(t,\varepsilon) = (x_0\varepsilon^2 - y_0\varepsilon)(\varepsilon^2 + \varepsilon) - (x_0 - y_0)\left(\frac{\varepsilon^2}{\varepsilon + \varepsilon^2}\right)e^{-\frac{\varepsilon+\varepsilon^2}{\varepsilon^3}t}$$

These formulas show that in the case when the parameeters ε tend to zero, it is not possible to represent the solution of the problem in the form of a series of the power of the parameeters with coefficients that depend on the new scale $\frac{\varepsilon+\varepsilon^2}{\varepsilon^3}$ because the factor $\frac{\varepsilon+\varepsilon^2}{\varepsilon^3}$ does not decompose under exponentials in a power series of ε.

In calculating the asymptotic solution, we find that the external expansion is trivial and internal expansion contains only the principal terms:

$\pi_x(\tau,\varepsilon) = \pi_x^{00}(\tau), \pi_y(\tau,\varepsilon) = \pi_y^{00}(\tau)$, that means the internal solution identically coinciding with the exact solution by substitution $\tau = \frac{t}{\varepsilon^3}$.

References

Butuzov V.F, Vasilyeva A.B, Fedoryuk M.V , 1969,Asymptotic methods in the theory of ordinary differential equations , science. Mat. Analysis, 1967 ", VINITI, USSR Academy of Sciences, Moscow.

Demetri M.G. , 2006, singularly perturbed in control problem, Automatic and Telchinical, No.1, p.3-51.

Heping Ma,Bin Liu, 2016 , Linear-Quadratic Optimal Control Problem for Partially Observed Forward-Backward Stochastic Differential Equations of Mean-Field Type, Wiley press, Volume 18, Issue 6,Pages 2146–2157

Jingrui, 2017, Linear Quadratic Optimal Control Problems with Fixed Terminal States and Integral Quadratic Constraints, National University of Singapore, 119076, Singapore.

Lappo-Danilevsky I.A ., 1957,Application of functions from matrices to theory linear systems of ordinary differential equations. -M.

Tikhonov A.N. , 1950, On systems of differential equations containing parameters , Mat. Sat. - V. 27 (69), No. 1. - P. 147-156.

Tikhonov A.N., 1948, On the dependence of solutions of differential equations from a small parameter, Mat. Sat. . - V. 22 (64), No. 2. - P. 193-204.

Tikhonov A.N., 1952, The systems of differential equations containing small parameters for derivatives, Mat. Sat., V. 31(73), No. 3. - P. 575-586.

Vasil'eva A.V, Butuzov V.F , 1990,Asymptotic methods in the theory of singular perturbations. - Moscow: Higher School.

Vasilieva A.B. , 1959, Asymptotic formulas for solutions of systems of ordinary differential equations containing derivatives parameters of different orders of smallness, Dokl. AN SSSR., V. 28, No. 6. - P. 1110-1113.

Vasilieva A.B. , 1963,Asymptotics of solutions of some problems for ordinary nonlinear differential equations with a small parameter with the highest derivative // Uspekhi Mat., V. 18, No. 3. - P. 15-86.

Vasilyeva A.B, Butuzov V.F. , 1973 ,Asymptotic expansions of solutions singularly perturbed equations. - Moscow: Nauka.

Vasilyeva A.B. , 1963, Asymptotic methods in the theory of ordinary differential equations with small parameters at the leading derivatives , calculated. mathematics and mathematics. physics. J, V. 3, No. 4. - P. 611-642.

Vasilyeva A.B. ,1962,Asymptotic formulas for solutions of ordinary differential equations with a small factor at the highest derivative, which are valid on an infinite interval, Dokl. AN THE USSR., V. 142, № 4. - p. 709-712.