The Behaviors of some Counting Functions of g-primes and g-integers as x goes to Infinity

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Abstract

In this article we focus on the behaviors of the generalised counting function of primes $\Pi_{\mathcal{P}}(x)$ and the counting function of integers $\mathcal{N}_{\mathcal{P}}(x)$ as well as the link between them as $x \longrightarrow \infty$. Here the Riemann zeta function $\zeta_{\mathcal{P}}(s)$ ($= \sum_n n^{-s}$, $\Re(s) > 1$) play an important role as a link between $\Pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$. This work will go through the method (not in details) adapted by Balanzario [Balanzario , 1998] and later generalised by AL- Maamori [AL- Maamori , 2013] . Finally we shall draw a diagram in order to determine the relation between α and β , (where α and β are the power of the error terms $H_1(x)$, $H_2(x)$ of $\Pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$ respectively) . The aim of this work is to analysis the behaviour of $\Pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$ as $x \longrightarrow \infty$.

Note that : ″ It's a beneficial to point out that our effort in this paper is not to exchange the values of some functions of Balanzario′ s method . Since , changing any small value of one of the functions of Balanzario′ s method may be leads to loss the aim of the work ″ . Therefore , in this article we show the ability of changing the values of some functions and in which places in the proof we should sort out .

Key words : Mathematical analysis and the generalization of prime systems .

الخلاصة

في هذا البحث نركز على تصرفات الدوال الحسابية الموسعة للأعداد الاولية $\Pi_{\mathcal{P}}(x)$ وللأعداد الصحيحة $\mathcal{N}_{\mathcal{P}}(x)$ وكذلك الرابط بينهما عندما x ← ∞ . هنا دالة ريمان زيتا
$\zeta_{\mathcal{P}}(s)$ ($= \sum_n n^{-s}$, $\Re(s) > 1$) , تلعب دورا مهما كرابط بين $\Pi_{\mathcal{P}}(x)$ و $\mathcal{N}_{\mathcal{P}}(x)$. هذا العمل سيتم من خلال سلوك طريقة العالم بلنزاريو [بلنزاريو، 1998] ليست بالتفاصيل والتي عممت من خلال المعموري [المعموري ، 2013] . بالنهاية سوف نرسم مخططا يحدد العلاقة بين α و β (حيث α و β هما القوى للحدود الخطأ $H_1(x)$ و $H_2(x)$ من $\Pi_{\mathcal{P}}(x)$ و $\mathcal{N}_{\mathcal{P}}(x)$ على التوالي . الغرض من هذا البحث هو تحليل تصرفات $\Pi_{\mathcal{P}}(x)$ و $\mathcal{N}_{\mathcal{P}}(x)$ عندما x ← ∞ .

ملاحظة : من المهم والنافع الاشارة بان جهدنا في هذا البحث ليست تغيير بعض قيم الدوال التي استخدمت في طريقة بلنزاريو حيث ان تغيير اي قيمة مهما كانت صغيرة لإحدى دوال طريقة بلنزاريو ربما تقودنا الى خسارة هدف الموضوع بأكمله . ولهذا نبين ايضا قابلية التغيير المسموح بها في قيم بعض الدوال . كذلك سوف نختم البحث بفتح باب لعمل مستقبلي .

الكلمات المفتاحية : التحليل الرياضي و الانظمة الاولية المعممة .

Introduction

Let $\mathcal{P} = \{ p_1 , p_2 , \dots \}$ be a set of real numbers satisfying the following conditions : $1 < p_1$, $p_n \leq p_{n+1}$ and $p_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Beurling [Beurling , 1937] called $\mathcal{P}$ the generalized primes (Beurling primes) . The generalised counting functions of primes and of integers are defined as follows :

$$\pi_{\mathcal{P}}(x) = \sum_{p \leq x , p \in \mathcal{P}} 1 \quad \text{and} \quad \mathcal{N}_{\mathcal{P}}(x) = \sum_{n \leq x , n \in \mathcal{N}} 1 .$$

We note that $\pi_{\mathcal{P}}(x)$ is defined as a discrete function .
The following definition is needed .

Definition : Let $\Pi_{\mathcal{P}}$, $\mathcal{N}_{\mathcal{P}}$ be functions such that ($\Pi_{\mathcal{P}} \in S_0^+$)$^{(*)}$ and ($\mathcal{N}_{\mathcal{P}} \in S_1^+$)$^{(**)}$ with ($\mathcal{N}_{\mathcal{P}} = \exp * \Pi_{\mathcal{P}}$)$^{(***)}$. Then ($\Pi_{\mathcal{P}}$, $\mathcal{N}_{\mathcal{P}}$) is called an outer g – prime system .

The generalised prime systems have been investigated by Beurling and later by many authors studied it such as Diamond [Diamond , 1969] , Hilberdink [Hilberdink, 2012] and so on . Beurling introduced the generalised prime theorem by showing :
If $\mathcal{N}_{\mathcal{P}}(x) = A(x) + O\left(\frac{x}{\log^{\gamma} x}\right)$ for $A > 0$ and $\gamma > \frac{3}{2}$, then $\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$.
This is an analogue of the prime number theorem (PNT) , also Beurling showed that the condition $\gamma > \frac{3}{2}$ is necessary in the sense that there is a continuous analogue of a g – prime system with $\gamma > \frac{3}{2}$ for which the PNT does not hold .

From a several papers in this field such as " Diamond [Diamond , 1970] , Diamond [Diamond , 1969] , Bateman [Bateman , 1969] , Ellison and Mends [Ellison and Mends , 1975] , Hilberdink [Hilberdink , 2009] and so on , We see that the main core (of the behaviors of $\pi_{\mathcal{P}}(x)$ and $\mathcal{N}_{\mathcal{P}}(x)$) is the size of their error terms where $\mathcal{N}_{\mathcal{P}}(x) = ax + H_1(x)$ (will use later in (6)) $\Longrightarrow$ $\pi_{\mathcal{P}}(x) = li(x) + H_2(x)$. Here $li(x) = \int_2^x \frac{dt}{\log t}$.

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(*) Here $S_0^+ = \{ f \in S$: S is the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ s.t. f is right-continuous and of local bounded variation with $f(1) = 0 \}$.

(**) Here $S_1^+ = \{ f \in S$: S is the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ s.t. f is right-continuous and of local bounded variation with $f(1) = 1 \}$.

(***) $f = \exp_* g$ iff $f * g_L = f_L$ where $f_L \in S$ defined for $x \geq 1$ by $f_L(x) = \int_1^x \log t \, d f(x)$.

In this article , we study the behaviors of $H_1(x)$ and $H_2(x)$ (as $x \longrightarrow \infty$) in deep as $\pi_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{P}}$ are counts g- prime functions . In order to see that taking the method of Balanzario in 1998 and later generalised by AL- Maamori in 2015 . We note that the error terms of the counting functions are mostly of the form :

(i) $O\left(\frac{x}{(\log x)^{\gamma}}\right)$ (ii) $O\left(x\, e^{c(\log x)^{\alpha}}\right)$ (iii) $O(x^{\alpha})$

we deal with the form (i) in our work .

Balanzario defined $\Pi_{\mathcal{P}}(x) = \int_1^x \frac{1-t^{-k}}{\log t} \gamma(t)\, dt$ where $k > 1$,

$$\gamma(t) = 1 - \sum_{n > n_0} \alpha_n \frac{\cos(b_n \log t)}{t^{a_n}} \quad \ldots\ldots (1)$$

and $\zeta_{\mathcal{P}}(s) = \int_1^{\infty} x^{-s}\, d\,\mathcal{N}_{\mathcal{P}}(x) = \int_1^{\infty} x^{-s}\, e^{d\,\Pi(x)} = \exp\{ \int_1^{\infty} x^{-s}\, d\,\Pi_{\mathcal{P}}(x)\}$ ….. (2)

for more details of (2) see [Hilberdink , 2012] .

Balanzario proved that : $\Pi_{\mathcal{P}}(x) = li(x) + O\left(x\, e^{-c\sqrt{\log x}}\right)$, $c > 0$ …. (3)

implies $\mathcal{N}_{\mathcal{P}}(x) = \rho x + \Omega^{(*)}_{\pm}\left(x\, e^{-k\sqrt{\log x}}\right)$, $\rho > o$, $k > 0$ …. (4)

If we assume that we have :

$\Pi_{\mathcal{P}}(x) = li(x) + O\left(x\, e^{-k(\log x)^{\alpha}}\right)$, $k > 0$. …… (5)

Balanzario showed that for $\alpha = \frac{1}{2}$ implies $\mathcal{N}_{\mathcal{P}}(x) = \rho x + O\left(x\, e^{-c(\log x)^{\beta}}\right)$ for some ρ , $c > 0$ and $\beta = \frac{1}{2}$ here .

Given (5) , Malliavin proved that : $\mathcal{N}_{\mathcal{P}}(x) = \rho x + O\left(x\, e^{-c(\log x)^{0.2}}\right)$ for some ρ , $c > 0$ (see [Malliavin , 1961]) . Diamond showed that with (5) holds , we could get : $\mathcal{N}_{\mathcal{P}}(x) = \rho x + O\left(x\, e^{-c(\log x)^{0.333\ldots}}\right)$ for some $\rho > 0$, $c > 0$.

This shows that $H_1(x) = (x\, e^{-k\sqrt{\log x}})$, k > 0 has fixed power at $\alpha = \frac{1}{2}$, but β varies into different values . Suppose that we get (5) with $\alpha = \frac{1}{3}$,

The question is : " What is the best possible value could get using Balanzario′ s method ? " .

(*) For F and G be functions defined on some interval (a , ∞) . We write F (x) = Ω (G (t)) , to mean that there exist a constant c > 0 such that $|F(t)| \geq c\,G(t)$ for some arbitrary large values of t . Further , we write F (t) = Ω_+ (G (t)) and F (t) = Ω_- (G (t)) if there exist a constant c > 0 such that F (t) ≥ c G(t) and F (t) ≤ - c G(t) hold respectively for some arbitrarily large values of t . We write F (t) = $\Omega_\pm$ (G (t) if both F (t) = Ω_+ (G (t)) and F (t) = Ω_- (G (t)) hold [Bateman , 1969] .

Suppose that we have $\Pi_{\mathcal{P}}(x) = \int_1^x \frac{1-t^{-k}}{\log t}\ \gamma(t)\ dt$, k > 1 where

$\Pi_{\mathcal{P}}(x) = li(x) + O(x\ e^{-c(\log x)^{\alpha}})$, for some c > 0 and take $\alpha = \frac{1}{3}$, K = 4 , $n_0 = 3$, $x = e^{10}$. Here changing α from $\frac{1}{2}$ into $\frac{1}{3}$ will leads to considerable work .

This means that :

$b_n = \exp\{ (\log x_n)^{1/3} \}$, $a_n = \frac{1}{\log b_n} = \frac{1}{(\log x_n)^{\frac{1}{3}}} = (\log x_n)^{-1/3}$,

$x_{n+1} = \exp\{ (\log x_n)^3 \} \implies \log x_{n+1} = (\log x_n)^3$ and

$T_n = \exp\{ (\log x_n)^{3/4} \}$, $\alpha_n = \frac{2}{n^2}$, $\alpha = \sum_{n>n_0} \alpha_n$.

Estimation of $\Pi_{\mathcal{P}}(x)$:

With the above new values (or condition) of the method , our aim is to avoid two important points which are :

(1) The loss of generality . (2) cut of some simple details .

For this , we keep tackling the curtail sectors of Balanzario′ s method . These curtail parts improved to be :

Proposition(1):(This is the modification of proposition (2) in [Balanzario , 1998])

If $\Pi_{\mathcal{P}}(x)$ is given by (1) , then $\Pi_{\mathcal{P}}(x) = li(x) + O(x\ e^{-c(\log x)^{\alpha}})$

here $\alpha = \frac{1}{3}$, c = 4 .

Proof : we have :

$$\Pi_{\mathcal{P}}(x) = \int_1^x \frac{1-t^{-k}}{\log t}\ \gamma(t)\ dt = \int_1^x \frac{1-t^{-k}}{\log t}\Big(1 - \sum_{n>n_0} \alpha_n \frac{\cos(b_n \log t)}{t^{a_n}}\ dt .$$

It's obvious that we could get $\Pi_{\mathcal{P}}(x) = li(x) - \sum_{n>n_0} \alpha_n \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}\ dt$.

It remains to estimate the summation part and show that :

$$\sum_{n>n_0} \alpha_n \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}\ dt = O(x\ e^{-4(\log x)^{\frac{1}{3}}}) .$$

New to estimate the integration in the summation part we get :

$\left| \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}\ dt \right| \leq 3\frac{x^{1-a_n}}{b_n}$. Its remains to calculate the magnitude $3\frac{x^{1-a_n}}{b_n}$, by definitions of b_n and a_n above , we have :

$$\frac{x^{1-a_n}}{b_n} = \frac{x\,.\,x^{-a_n}}{b_n} = \frac{x\,.\,x^{-\frac{1}{(\log x_n)^{\frac{1}{3}}}}}{e^{(\log x_n)^{\frac{1}{3}}}} = x \exp\Big\{ -\frac{\log x}{(\log x_n)^{\frac{1}{3}}} - (\log x_n)^{\frac{1}{3}} \Big\}$$

$$= x \exp\{ -(\log x)^{\frac{2}{3}} - (\log x_n)^{\frac{1}{3}} \}$$

$= x \ \exp\{ -(\log x)^{\frac{1}{3}} \ (1 + (\log x)^{\frac{1}{3}})\}$

$= x \ e^{-4(\log x)^{\frac{1}{3}}}$. Therefore ,

$\sum_{n>n_0} \alpha_n \left| \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t} dt \right| \leq \sum_{n>n_0} \alpha_n (3 x \ e^{-4(\log x)^{\frac{1}{3}}})$

$= 3\alpha \ x \ e^{-4(\log x)^{\frac{1}{3}}} = O(x \ e^{-4(\log x)^{\frac{1}{3}}})$.

Estimation of $\mathcal{N}_{\mathcal{P}}(x)$:

Here we calculate $\mathcal{N}_{\mathcal{P}}(x)$ in order to see the effecting of the error term of $\Pi_{\mathcal{P}}(x)$ on the behaviour of $\mathcal{N}_{\mathcal{P}}(x)$ in general . So we let $M_{\mathcal{P}}(x) = \int_1^x \mathcal{N}(t)dt$. The reason of doing this , is because dealing with $M_{\mathcal{P}}(x)$ is more easier than dealing with $\mathcal{N}_{\mathcal{P}}(x)$ in calculations . Therefore ,

$$M_{\mathcal{P}}(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)} ds \ , \ b > 1 . \qquad \ldots\ldots (6)$$

So , in order to calculate $M_{\mathcal{P}}(x)$ we have to calculate $\zeta_{\mathcal{P}}(s)$. Following same arguments as in [Balanzario , 1998] , we see that : $|\zeta_{\mathcal{P}}(s)| \leq 45$.

Now , the integration in (6) has a singularity pointes in 0 and 1 , therefore if we calculate the integration on [b - i∞ , b + i∞] , then we have to make a partition of the path on this interval in order to avoid these pointes by restricting the domain as follows :

Γ_1 : from b - i∞ to b – iT ,

Γ_2 : from b – iT to $-\frac{3}{2}$ – iT ,

Γ_3 : from $-\frac{3}{2}$ – iT to $-\frac{3}{2}$ + iT ,

Γ_4 : from $-\frac{3}{2}$ + iT to b + iT ,

Γ_5 : from b + iT to b + i∞ .

Thus , we can write (6) as follows :

$M_{\mathcal{P}}(x) = I_1 + \ldots + I_5 + J_{-n} + \ldots + J_n$ + residues { 0 , 1 } ,

where $I_m = \frac{1}{2\pi i}\int_{\Gamma_m} \zeta_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)} ds$, m = 1 , … , 5 ,

$J_m = \frac{1}{2\pi i}\int_{C_m} \zeta_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)} ds$, $n_0 < |m| \leq n$.

Here Γ_m is the path of integration in (6) and C_m is the *m* th horizontal loop with $\Im(s) = b_m$.

Now , we note that the estimation of I_1 is similar to I_5 and the estimation of I_2 is similar to I_4 .

Hence the estimation of I_1 and I_5 by [Balanzario , 1998] are : $O(\frac{x^2}{T_n})$,

the estimation of I_2 and I_4 are : $O(\frac{x}{T_n})^2$ and

the estimation of I_3 is : $O(\frac{1}{\sqrt{x}})$. Therefore , from above we get :

$|I_m| \leq O(\frac{x^2}{T_n})$, m = 1 , … , 5 and hence (6) can be written as follows :

$$M_{\mathcal{P}}(x) = O(\frac{x^2}{T_n}) + \sum_{m=-n}^{n} J_m + \text{residue } \{0, 1\} \qquad \ldots\ldots (7)$$

Now , if we calculate the residue { 0 , 1 } in (7) by [Balanzario , 1998] , then we get :

Residue { 0 , 1 } $= k\,\emptyset(1)\frac{x^2}{2} + (1-k)\,\emptyset(0)\,x$, and hence (7) can be written as follows :

$M_{\mathcal{P}}(x) = k\, \emptyset(1) \frac{x^2}{2} + (1-k)\, \emptyset(0)\, x + \sum_{m=-n}^{n} J_m + O\left(\frac{x^2}{T_n}\right)$ …… (8)

Now , if we calculate the expression $\frac{x^2}{T_n}$ appearing in (8) , then we get by [Balanzario , 1998] :

$\frac{x^2}{T_n} = x^2 \exp\{ -(\log x_n)^{3/4} \}$, and hence the equation (8) become as follows :

$M_{\mathcal{P}}(x) = k\, \emptyset(1) \frac{x^2}{2} + \sum_{m=-n}^{n} J_m + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$ …… (9)

Its remains to estimate the magnitude $\sum_{m=-n}^{n} J_m$ appearing in (9) as follows :

Proposition(2):(This is the modification of proposition (8) in [Balanzario,1998])

$\left| \sum_{n_0 < |m| \le n-1} J_m \right| \le 60\, x^2 e^{-(\log x)^{\frac{8}{9}}}$.

Proof : firstly , we estimate J_m as follows : $|J_m| = \left| \frac{1}{2\pi i} \int_{C_m} \zeta_{\mathcal{P}}(s) \frac{x^{s+1}}{s(s+1)}\, ds \right|$ and we see that by [Balanzario , 1998] we get : $|J_m| \le \frac{15}{b_m^2}\, x^2 e^{-a_m \log x}$

Secondly , we estimate $\sum_{n_0 < |m| \le n-1} J_m$ as follows :

If $|m| \le n-1$, then $e^{-a_m \log x} \le e^{-a_{n-1} \log x}$

$= \exp\left\{ -\frac{\log x}{\sqrt[3]{\log x_{n-1}}} \right\} = \exp\left\{ -\frac{\log x}{\sqrt[3]{(\log x_n)^{\frac{1}{3}}}} \right\} = \exp\left\{ -\frac{\log x}{(\log x_n)^{\frac{1}{9}}} \right\}$

$\le \exp\left\{ \frac{-\log x_n + \frac{2}{10^n}}{(\log x_n)^{\frac{1}{9}}} \right\} \le \exp\left\{ -(\log x_n)^{\frac{8}{9}} + \frac{2}{10^n} \right\} \le \exp\left\{ -(\log x_n)^{\frac{8}{9}} + \frac{4}{10^n} \right\} \le$

$2\, e^{-(\log x_n)^{\frac{8}{9}}}$, where $|\log x - \log x_n| \le \frac{2}{10^n}$, and hence

$\left| \sum_{n_0 < |m| \le n-1} J_m \right| \le 30\, x^2 e^{-(\log x)^{\frac{8}{9}}} \sum_{|m| > n_0} \frac{1}{b_m{}^2}$.

Now we finish the proof by noting that the last sum is finite :

$\sum_{|m| > n_0} \frac{1}{b_m{}^2} \le \sum_{|m| > n_0} \frac{1}{e^{2(\log x_m)^{\frac{1}{3}}}} \le 2 \sum_{m > n_0} e^{-2(\log x_m)^{\frac{1}{3}}}$

$\le 2 \sum_{m > n_0} e^{-2(10)^{\frac{m}{3}}} \le 2$.

Therefore , $\left| \sum_{n_0 < |m| \le n-1} J_m \right| \le 60\, x^2 e^{-(\log x)^{\frac{8}{9}}}$.

Since $e^{-(\log x)^{\frac{8}{9}}} \le e^{-(\log x)^{\frac{3}{4}}}$, then the equation (9) become as follows :

$M_{\mathcal{P}}(x) = k\, \emptyset(1) \frac{x^2}{2} + (J_{-n} + J_n) + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$ ….. (10)

It remains to study the expression : $J_{-n} + J_n$.

Here $J_n = J_n^{\square} + J_n^{\square}$ where $J_n^{\square}$, $J_n^{\square}$ refers to the integrals along the line segment $C_n^{\square}$ and $C_n^{\square}$ lying respectively above and below the branch cut C_n and suppose that $C_n^{\square}$ with its direction reversed [Balanzario , 1998] : $-C_n^{\square}$: $\begin{cases} \theta = -\pi \\ s = 1 - a_n + ib_n - t \\ ds = -dt \\ 0 \le t \le 1 - a_n + \frac{3}{2} \end{cases}$

Since $J_n^{\square} = \frac{-1}{2\pi i} \int_0^{1-a_n+\frac{3}{2}} \frac{\zeta_{\mathcal{P}}(1-a_n+ib_n-t)\, x^{2-a_n+ib_n-t}}{(1-a_n+ib_n-t)(2-a_n+ib_n-t)} (-dt)$, then

$J_n^{\square} = \frac{1}{2\pi i} \int_0^{(\log x)^{-\frac{1}{4}}} \frac{\zeta_{\mathcal{P}}(1-a_n+ib_n-t)\, x^{2-a_n+ib_n-t}}{(1-a_n+ib_n-t)(2-a_n+ib_n-t)}\, dt + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$ ….. (11)

Now , if we rewrite the integrand in (11) as follows :

$\frac{\zeta_{\mathcal{P}}(s)}{s(s+1)} = (s - 1 - a_n + ib_n)^{\frac{\alpha_n}{2}}\, f_n(s)$, where $f_n(s) = \frac{(s+k-1)\prod_{\substack{|m|>n_0 \\ m\neq n}}\left(1-\frac{k}{s-1+a_m-ib_m+k}\right)^{\frac{\alpha_m}{2}}}{s(s-1)(s+1)(s-1+a_n-ib_n+k)^{\frac{\alpha_n}{2}}}$, and we deduce that by [Balanzario , 1998] :

$J_n^{?} = \frac{1}{2\pi i}\, x^{2-a_n+ib_n}\, e^{-\pi i\frac{\alpha_n}{2}}\left(\frac{1}{\log x}\right)^{\frac{\alpha_n}{2}+1} s_n + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$,

with $S_n = \int_0^{(\log x)^{\frac{3}{4}}} e^{-t}\, t^{\frac{\alpha_n}{2}}\, f_n(1-a_n+ib_n-\frac{t}{\log x})\, dt$.

Similarly , we calculate $J_n^{?}$ in a similar way we obtain :

$J_n^{?} = \frac{-1}{2\pi i}\, x^{2-a_n+ib_n}\, e^{\pi i\frac{\alpha_n}{2}}\left(\frac{1}{\log x}\right)^{\frac{\alpha_n}{2}+1} S_n + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$ and from

$J_n = J_n^{?} + J_n^{?}$, we get $\Longrightarrow$

$J_n = \frac{\sin\frac{\pi}{2}\alpha_n}{\pi}\, x^{2-a_n+ib_n}\left(\frac{1}{\log x}\right)^{\frac{\alpha_n}{2}+1} S_n + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$ ….. (12)

Now , if we calculate J_{-n} we obtain the complex conjugate of J_n because $b_{-n} = -b_n$, therefore $J_n + \overline{J_n} = 2\,\Re(J_n)$.

Now , in order to estimate the integral S_n appearing in (12) , we first obtain the lower and upper bound for $f_n(s)$ which is appearing in S_n , we see that by [Balanzario , 1998] we get : $|f_n(s)| \leq \frac{64}{b_n^2}$ be the upper bound of $f_n(s)$ and

$|f_n(s)| \geq \frac{1}{800\, b_n^2}$ be the lower bound of $f_n(s)$.

Now , we can estimate the integral S_n appearing in (12) as follows :

We get by [Balanzario , 1998] that : $|S_n| \geq \frac{e^{-2(\log x_n)^{\frac{1}{3}}}}{1600}$ …. (13)

We shall use this lower bound for S_n appearing in (12) . Now consider the other factor in (12) , we get by [Balanzario , 1998] :

$\frac{\sin\frac{\pi}{2}\alpha_n}{\pi}\, x^{2-a_n}\left(\frac{1}{\log x}\right)^{\frac{\alpha_n}{2}+1} \geq \frac{x^2}{\pi}\, e^{-(\log x_n)^{\frac{1}{3}}}\, \frac{1}{2(\log x)^2(\log\log x_n)^2}$

Now , we can estimate the equation (12) as follows :

$|J_n| \geq \frac{x^2}{\pi\, 1600} \cdot \frac{e^{-3(\log x_n)^{\frac{1}{3}}}}{4(\log x)^4} \geq \frac{10^{-5}}{(\log x)^4}\, e^{-3(\log x)^{\frac{1}{3}}}$.

We already know that $|J_n|$ is large , but still it can be that $\Re(J_n) = 0$.

Now , let us recall here equation (12) , where $x = x_n(1+\frac{\theta_1}{\log x_n})$, $|\theta_1| < 1$. Then , we get by [Balanzario , 1998] :

$\Re(J_n) \geq \frac{10^{-5}}{2(\log x)^4}\, e^{-c(\log x)^{\frac{1}{3}}}$, $c > 0$ if $x \geq X_1$ and $\theta_1 = \theta(+)$ and

$\Re(J_n) \leq -\frac{10^{-5}}{2(\log x)^4}\, e^{-c(\log x)^{\frac{1}{3}}}$, $c > 0$ if $x \geq X_1$ and $\theta_1 = \theta(-)$.

These inequalities and the equation :

$M_{\mathcal{P}}(x) = 2\,\emptyset(1)\, x^2 + 2\,\Re(J_n) + O(x^2\, e^{-(\log x)^{\frac{3}{4}}})$,

imply relation : $M_{\mathcal{P}}(x) = 2\,\emptyset(1)\, x^2 + \Omega_{\pm}(x^2\, e^{-c_0(\log x)^{\frac{1}{3}}})$, $c_0 > 0$.

Now , we need the following trick to move from M(x) into $\mathcal{N}_{\mathcal{P}}(x)$.

Lemma (5) : Suppose that $\mathcal{N}_{\mathcal{P}}(x) \in S_1^+$. Let $M_{\mathcal{P}}(x) = \int_1^x N(t)dt$, then for every $0 < y < x$, we have :

$$\frac{M_{\mathcal{P}}(x)-M_{\mathcal{P}}(x-y)}{y} \le \mathcal{N}_{\mathcal{P}}(x) \le \frac{M_{\mathcal{P}}(x+y)-M_{\mathcal{P}}(x)}{y}$$

Proof :

Let $M(x) = c\,x^2 + O\left(x^2\, e^{-\lambda(\log x)^{\alpha}}\right)$ for some $\lambda > 0$, then
$M(x) - \frac{c}{2}x^2 = O\left(g(x)\right)$ such that $g(x) = x^2\, e^{-\lambda(\log x)^{\alpha}}$.

Therefore $\left|M(x) - \frac{c}{2}x^2\right| = O\left(g(x)\right)$. …… (14)

Since the function $\mathcal{N}_{\mathcal{P}}(x)$ is increasing function , so for every $0 < y < x$, we have :

$\int_0^x \mathcal{N}_{\mathcal{P}}(t)dt - \int_0^{x-y} \mathcal{N}_{\mathcal{P}}(t)dt = \int_{x-y}^{x} \mathcal{N}_{\mathcal{P}}(t)dt \le y\, \mathcal{N}_{\mathcal{P}}(x)$ …. (15)

On the other hand ,

$\int_0^{x+y} \mathcal{N}_{\mathcal{P}}(t)dt - \int_0^{x} \mathcal{N}_{\mathcal{P}}(t)dt = \int_{x}^{x+y} \mathcal{N}_{\mathcal{P}}(t)dt \ge y\, \mathcal{N}_{\mathcal{P}}(x)$ …. (16)

Therefore from (15) and (16) , we get :

$\frac{M_{\mathcal{P}}(x)-M_{\mathcal{P}}(x-y)}{y} \le \mathcal{N}_{\mathcal{P}}(x) \le \frac{M_{\mathcal{P}}(x+y)-M_{\mathcal{P}}(x)}{y}$ ….. (17)

This is sufficient to show that : if $M_{\mathcal{P}}(x) = \rho\, x^2 + E(x^2)$ for some $\rho > 0$,
then $M_{\mathcal{P}}(x) = \rho_1\, x + E(x)$, $\rho_1 > 0$.

Appendix :

Moreover , if we have $\alpha = \frac{2}{3}$, then we could get with the following setting :

$K = 4$, $n_0 = 3$, $x = e^{10}$,

$b_n = \exp\{(\log x_n)^{2/3}\}$, $a_n = \frac{1}{\log b_n} = \frac{1}{(\log x_n)^{\frac{2}{3}}} = (\log x_n)^{-2/3}$,

$x_{n+1} = \exp\{(\log x_n)^2\} \Longrightarrow \log x_{n+1} = (\log x_n)^2$ and

$T_n = \exp\{(\log x_n)^{3/4}\}$, $\alpha_n = \frac{2}{n^2}$, $\alpha = \sum_{n > n_0} \alpha_n$.

We see that by a previous steps that :

$\Pi_{\mathcal{P}}(x) = li(x) + O\left(x\, e^{-2(\log x)^{\frac{2}{3}}}\right)$,

$M(x) = 2\,\emptyset(1)\, x^2 + \Omega_{\pm}\left(x^2\, e^{-c_0(\log x)^{\frac{2}{3}}}\right)$, $c_0 > 0$

and hence $\mathcal{N}_{\mathcal{P}}(x) = \rho x + E(x)$, $\rho > 0$.

Apart from that if we draw a diagram of $\alpha - \beta$ space we would get :

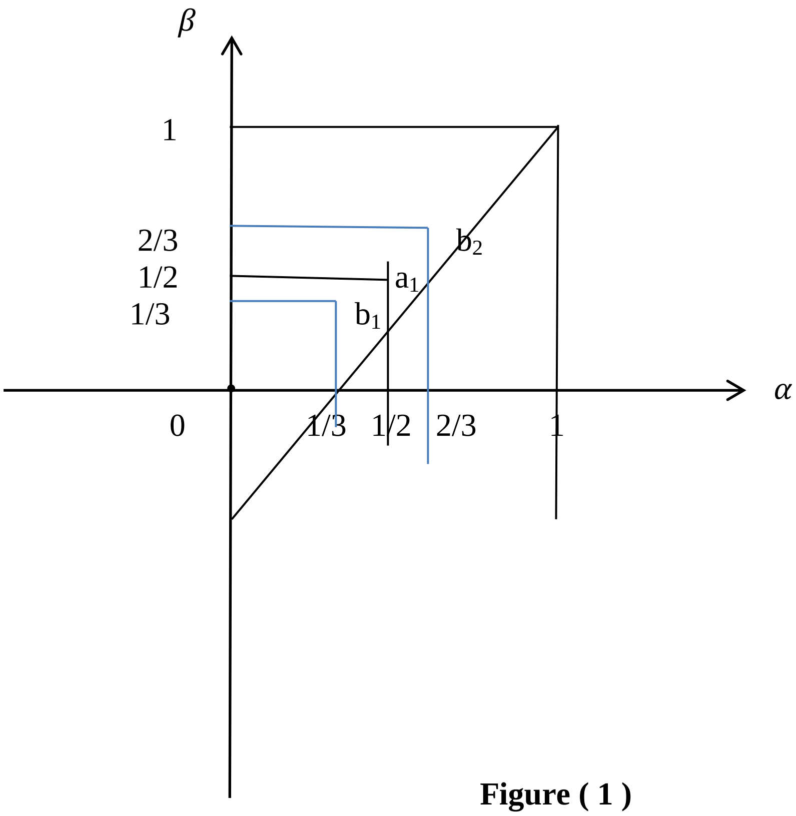

Figure (1)
Show the relation between α and β

a_1 related to Balanzario $\alpha = \beta = \frac{1}{2}$, $b_1 = \frac{1}{3}$ and $b_2 = \frac{2}{3}$, this means $\alpha = \beta = \frac{1}{3}$ and $\alpha = \beta = \frac{2}{3}$ respectively .

As a result we have seen the error term of $\Pi_{\mathcal{P}}(x)$ linked with the error term of $\mathcal{N}_{\mathcal{P}}(x)$ by zeta function $\zeta_{\mathcal{P}}(s)$.

Future work

Assume that (5) holds with $\frac{1}{2} \le \alpha \le 1$ (*)

Question (**) : given (*) could we get $\mathcal{N}_{\mathcal{P}}(x) = \rho x + O(x\, e^{-c(\log x)^{\beta}})$ $\beta \le \frac{1}{2}$ for any α ?

Note that if Question (**) holds this means that it is possible to get :

$\Pi_{\mathcal{P}}(x) = li(x) + O(x\, e^{-c(\log x)^{\frac{1}{2}}})$, $c > 0$

and $\mathcal{N}_{\mathcal{P}}(x) = \rho x + o(x)$, $\rho > 0$. This will show that Riemann Hypothesis is closed to be true .

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