RICH DYNAMICS OF A SIMPLE SI MODEL WITH TWO AGE GROUPS

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Abstract- The objective of this paper is to study the global dynamics of a simple SI model with two explicit age groups. We divide the population into juvenile and adult groups. Only adults are assumed to be sexually active and we implicitly assume that the sex ratio is constant. We also assume that infected adults may produce both susceptible newborns and infected newborns. We show that such models can produce the often observed and feared scenario of susceptible extinction. This suggests that age structure alone will not remove the total extinction dynamics in a typical ratio-dependent predator-prey model. An important feature of our work is that our analytical results are all presented in terms of the biologically meaningful threshold values.

KeyWords: Structured SI model, global stability, STDs epidemics.

1. INTRODUCTION

Throughout the history, mankind has suffered many devastating infectious diseases and continues to be fearful of such diseases. In the 1918-1919 influenza pandemic, over 20 million people perished including more than half a million Americans. Indeed, many believe infectious diseases had greatly reduced the human population growth rates in the world prior to 18th century (page 302, Brauer and Castillo-Chavez 2001).

Sexually Transmitted Diseases such as HIV epidemic exert extremely heavy burden in some populations, to say the least. Many developing countries are confronted with the devastating impact of the epidemics on the young and productive population groups. There is a growing number of children and adolescents suffering HIV in these countries, worsening the situation among the poor and marginalized groups. This inevitably results in the further diversion of resources from other health, welfare and educational priorities. This vicious cycle prompts the question such as how much worse this situation can become before it is over. Specifically one wonders how the population levels in such countries will be if all the demographical and epidemiological parameters stay constant and how to find more effective ways to reduce or stop the spread of these devastating epidemics. To this end, it is important to better understand the interaction of infectious diseases transmission process and population growth dynamics. This task is frequently carried out by formulating and studying plausible epidemiological models with vital dynamics in the form of differential equations (Thieme 2003).

Our main objective here is to study the global dynamics of a simple SI model with two explicit age groups and apply the findings to the HIV dynamics in the US. Specifically, we would like to explore the long term HIV dynamics to answer questions such as what will happen to human population level if all the demographical and epidemiological parameters stay constant; if necessary, what can be done to slow the spread of HIV before effective cures or vaccines are found; what are the effects of treatments.

2. A SI MODEL WITH TWO AGE GROUPS

Our model deals with a closed population of heterosexual individuals. We divide the population into juvenile (J) (0-14 years) and adult (A) (15 years or older) groups. Only adults are assumed to be sexually active and we implicitly assume that the sex ratio is constant. We also assume that infected adults may produce both susceptible newborns and infected newborns. The epidemics divides the population into two classes: susceptible class(1) and infected class (2). The juvenile and adult groups are each further divided into two classes. In the following, we define J_1 as the size of the susceptible juvenile class, J_2 as the size of the infected juvenile class, A_1 as the size of the susceptible adult class, and A_2 as the size of the infected adult class. We assume that infected individuals do not recover (such as HIV), juveniles are not sexually active and only adults can reproduce. Our model takes the following form.

$$\begin{cases} J_1' &= \beta_1 A_1 + (1-\xi)\beta_2 A_2 - \eta_1 J_1 - \mu J_1 - m J_1 N, \\ A_1' &= \eta_1 J_1 - \dfrac{\upsilon A_1 A_2}{A} - \alpha A_1 - m A_1 N, \\ J_2' &= \xi\beta_2 A_2 - \eta_2 J_2 - \mu J_2 - \gamma J_2 - m J_2 N, \\ A_2' &= \eta_2 J_2 + \dfrac{\upsilon A_1 A_2}{A} - \alpha A_2 - \gamma A_2 - m A_2 N, \end{cases} \tag{1}$$

where $A = A(t) = A_1(t) + A_2(t)$ and $N = N(t) = J_1(t) + J_2(t) + A_1(t) + A_2(t)$. Notice that $f(A_1, A_2) \equiv$

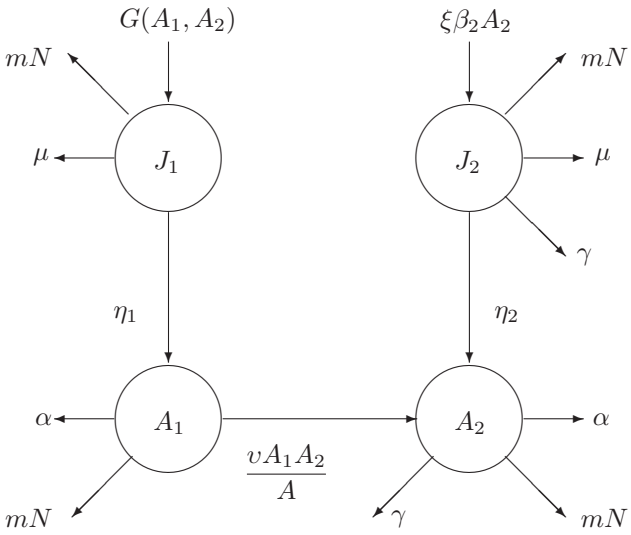

Figura 1: Flow diagram for the structured SI model with two age. Consider $G(A_1, A_2) = \beta_1 A_1 + (1 - \xi)\beta_2 A_2$

$A_1A_2/(A_1+A_2) \leq \min\{A_1, A_2\}$, we see that if we define $f(0, 0) = 0$, then it is continuous at $(A_1, A_2) = (0, 0)$. In fact, $f(A_1, A_2)$ is Lipschitzian for $A_1 \geq 0$ and $A_2 \geq 0$. However, $f(A_1, A_2)$ is not differentiable at $(A_1, A_2) = (0, 0)$.

Most of the terms in system (**??**) are standard. The exception is the additional mortality term due to crowdedness (intraspecific competition) in each classes. Crowdedness or intraspecific competition occurs when individuals of the same population negatively affect each other's fitness by reducing the availability of one or more limiting resources. At the individual level, competition for resources can affect development, fertility and survival. At the population level, intraspecific competition for resources can give rise to logistic population growth. It is widely known that intraspecific competition enables the coexistence of competing species.

3. PRELIMINARY ANALYSIS

The key value governing the time evolution of these equations is the so-called reproduction number (ratio) R_0. It is defined as the number of secondary infections caused by a single primary infection. For our model, we can see that

$$R_0 = \left(\frac{\xi\beta_2}{\alpha+\gamma}\right)\left(\frac{\eta_2}{\eta_2+\mu+\gamma}\right).$$

When $R_0 < 1$, each infected individual will infect fewer than one person before dying or recovering, so the disease will die out. When $R_0 > 1$, each infected individual will infect more than one person, so the disease will spread. R_0 is probably the single most important quantity in epidemiology research and applications. The following biologically meaningful threshold value is also useful in our mathematical presentation.

$$R_1 = \left(\frac{\beta_1}{\alpha}\right)\left(\frac{\eta_1}{\eta_1+\mu}\right), \qquad R_2 = \frac{\upsilon}{\gamma+\alpha}.$$

Biologically, R_1 is the average amount of new susceptible juveniles produced by one susceptible adult during its adulthood times the probability of surviving the susceptible juvenile stage. R_2 is the average number of contacts per infective adult per day times average period of infectivity. R_1 and R_0 are called *reproduction ratios* of the susceptible and infected classes respectively. R_2 is called the *infectious contact number*. We can establish the following result on the existence of the disease free equilibrium.

Proposition 1 *For system* (**??**)*, if* $R_1 > 1$*, then the system has a unique disease-free equilibrium given by the expression*

$$J_1^* = \frac{(\alpha + mN_1^*)N_1^*}{\alpha+\eta_1+mN_1^*}, \qquad A_1^* = \frac{\eta_1 N_1^*}{\alpha+\eta_1+mN_1^*},$$

where

$$N_1^* \equiv N_1 \equiv \frac{-(\eta_1+\mu+\alpha)+\sqrt{(\eta_1+\mu+\alpha)^2-4\alpha(\eta_1+\mu)(1-R_1)}}{2m}.$$

For a susceptible extinction equilibrium to exist, we must have $\xi = 1$. We have the following result.

Proposition 2 *For system* (**??**)*, if* $R_0 > 1$ *and* $\xi = 1$*, then the system has a unique susceptible extinction equilibrium given by the expression*

$$J_2^* = \frac{(\alpha+\gamma+mN_2^*)N_2^*}{\alpha+\eta_2+\gamma+mN_2^*}, \qquad A_2^* = \frac{\eta_2 N_2^*}{\alpha+\eta_2+\gamma+mN_2^*},$$

where

$$N_2^* \equiv N_2 \equiv \frac{-(\eta_2+\mu+2\gamma+\alpha)+\sqrt{\Delta}}{2m}$$

with

$$\Delta = (\eta_1+\mu+2\gamma+\alpha)^2 - 4(\alpha+\gamma)(\eta_2+\mu+\gamma)(1-R_2).$$

4. LOCAL STABILITY

In this sections, we present local stability results for the nontrivial equilibria of the system (**??**). All our conditions are in terms of the biologically meaningful threshold values defined in the previous section.

We present first the local stability result for the disease-free equilibrium.

Theorem 1 *If* $R_1 > 1$ *and* $R_0 + R_2 \leq 1$*, then the disease-free equilibrium* $(J_1^*, A_1^*, 0, 0)$ *is locally asymptotically stable in* R_+^4 *for the system* (**??**).

Proof 1 *From the Proposition 3.1, we see that the condition* $R_1 > 1$ *ensures the uniqueness of the disease-free equilibrium.* $R_0 + R_2 \leq 1$ *implies that* $R_0 < 1$.

The Jacobian matrix of the system (**??**) *evaluated at the disease-free equilibrium is of the form*

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_3 \\ \mathbf{0} & \mathbf{B}_2 \end{pmatrix}$$

where $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{0}$ *are* 2×2 *matrices and*

$$\mathbf{B}_1 = \begin{pmatrix} -(\eta_1 + \mu + m(J_1^* + N_1)) & \beta_1 - mJ_1^* \\ \eta_1 - mA_1^* & -(\alpha + m(A_1^* + N_1)) \end{pmatrix},$$

$$\mathbf{B}_2 = \begin{pmatrix} -(\eta_2 + \mu + \gamma + mN_1) & \xi\beta_2 \\ \eta_2 & -(\alpha + \gamma + mN_1) \end{pmatrix},$$

$$\mathbf{B}_3 = \begin{pmatrix} -mJ_1^* & (1-\xi)\beta_2 - mJ_1^* \\ -mA_1^* & -(+mA_1^*) \end{pmatrix}.$$

We shall show that $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *have negative eigenvalues. Clearly* $tr(\mathbf{B}_1) = -(\eta_1 + \mu + m(J_1^* + N_1)) - (\alpha + m(A_1^* + N_1)) < 0$. *From the* J_1 *and* A_1 *equations, we see that* $\beta_1\eta_1 = (\alpha + mN_1)(\eta_1 + \mu + mN_1)$. *Hence, we have*

$$\det(\mathbf{B}_1) > 0.$$

In consequence, $\mathbf{B}_1$ *has negative eigenvalues.*

We now study $\mathbf{B}_2$ *under the assumption that* $R_2 < 1$ *which implies that* $\gamma + \alpha >$. *We have* $tr(\mathbf{B}_2) = -\gamma - (\eta_2 + \mu) - 2mN_1 + (-\gamma - \alpha) < 0$ *and* $\det(\mathbf{B}_2) > 0$, *since* $R_2 + R_0 \leq 1$. *In consequence,* $\mathbf{B}_2$ *has negative eigenvalues.*

The above arguments show that the matrix $\mathbf{B}$ *has negative eigenvalues. Therefore, the disease-free equilibrium is locally asymptotically stable.*

Now we present our local stability result for the susceptible extinction equilibrium.

Theorem 2 *If* $R_1 \leq 1$, $R_0 > 1$ *and* $\xi = 1$, *then the susceptible extinction equilibrium* $(0, 0, J_2^*, A_2^*)$ *is locally asymptotically stable in* R_+^4 *for the system* (**??**).

Proof 2 *The Jacobian matrix of the system* (**??**) *evaluated at the susceptible extinction equilibrium is of the form*

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{D}_3 & \mathbf{D}_2 \end{pmatrix}$$

where $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{0}$ *are* 2×2 *matrices and*

$$\mathbf{D}_1 = \begin{pmatrix} -(\eta_1 + \mu + mN_2) & \beta_1 \\ \eta_1 & -(+\alpha + mN_2) \end{pmatrix},$$

$$\mathbf{D}_2 = \begin{pmatrix} -(\eta_2 + \mu + \gamma + m(J_2^* + N_2)) & \beta_2 - mJ_2^* \\ \eta_2 - mA_2^* & -(\alpha + \gamma + m(A_2^* + N_2)) \end{pmatrix},$$

$$\mathbf{D}_3 = \begin{pmatrix} -mJ_2^* & -mJ_2 \\ -mA_2^* & -mA_2^* \end{pmatrix}.$$

We will show that $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *have negative eigenvalues.*

Clearly, $tr(\mathbf{D}_1) = -(\eta_1 + \mu + mN_2) - (+\alpha + mN_2) < 0$ *and* $\det(\mathbf{D}_1) > 0$,

since $R_1 \geq 1$. *In consequence,* $\mathbf{D}_1$ *has negative eigenvalues.*

For $\mathbf{D}_2$, *we have* $tr(\mathbf{D}_2) = -(\eta_2 + \mu + \gamma + m(J_2^* + N_2)) - (\alpha + \gamma + m(A_2^* + N_2))) < 0$. *From the last two equations of system* (**??**), *we have*

$$\frac{\beta_2 - mJ_2^*}{\eta_2 + \mu + \gamma + mJ_2^*} = \frac{J_2^*}{A_2^*} = \frac{\alpha + \gamma + mA_2^*}{\eta_2 - mA_2^*}$$

i.e.

$$(\alpha + \gamma + mA_2^*)(\eta_2 + \mu + \gamma + mJ_2^*) = (\eta_2 - mA_2^*)(\beta_2 - mJ_2^*).$$

Hence

$$\det(\mathbf{D}_2) > 0.$$

In consequence, $\mathbf{D}_2$ *has negative eigenvalues.*

The above arguments show that the matrix $\mathbf{D}$ *has negative eigenvalues. Therefore, the susceptible extinction equilibrium is locally asymptotically stable.*

5. GLOBAL STABILITY

In this section, we present global stability results for the equilibria of the system (**??**). As in the previous section, all our conditions are in terms of the biologically meaningful threshold values defined in the section 3.

Our first result provides conditions for the disease to die out.

Theorem 3 *If* $R_1 > (\eta_1 + \mu)(+\alpha)/(\eta_1\alpha)$ *and* $R_2 \leq 1 - R_0$, *then* $\limsup_{t\to\infty} N(t) > 0$ *and the disease-free equilibrium* $(J_1^*, A_1^*, 0, 0)$ *is a global attractor in* R_+^4.

Proof 3 *We show now that if* $R_1 > (\eta_1 + \mu)(+\alpha)/(\eta_1\alpha)$, *then the trivial solution of the given system is a repeller in the sense that* $\limsup_{t\to\infty} N(t) > 0$. *We will prove this by contradiction. Assume that* $\limsup_{t\to\infty} N(t) = 0$. *Let* $x = A_1 + rJ_1$, *with* $r \in (0, 1)$ *to be be chosen later. The derivative of* x *along a solution of the system* (**??**) *is*

$$x' \geq \left(\frac{\eta_1}{r} - (\eta_1 + \mu)\right) rJ_1 + (r\alpha R_1 - -\alpha) A_1 - mxN.$$

Since $R_1 > (\eta_1 + \mu)(+\alpha)/(\eta_1\alpha)$, *there exists an* $0 < \epsilon < 1$ *such that*

$$\eta_1\alpha R_1 > (\eta_1 + \mu + \epsilon)(+\alpha + \epsilon).$$

Then, we can choose r *such that*

$$\frac{+\alpha + \epsilon}{\alpha R_1} < r < \frac{\eta_1}{\eta_1 + \mu + \epsilon} < 1.$$

We thus have

$$x' > \epsilon x - mxN = x(\epsilon - mN).$$

If $\limsup_{t\to\infty} N(t) = 0$, *then* $x \to \infty$, *but this is a contradiction since* $x = A_1 + rJ_1 < N$. *Therefore,* $\limsup_{t\to\infty} N(t) > 0$.

We now proceed to show that the disease-free is a global attractor in R^4_+. *Consider the following potential Liapunov function*

$$V(X) = \varrho_2 J_2 + A_2, \qquad X = (J_1, A_1, J_2, A_2) \in R^4_+,$$

with $\varrho_2 > 0$ *to be chosen later. The derivative of* V *along a solution of the system (??) is*

$$\begin{aligned}\dot{V}(X) &= (\eta_2+\mu+\gamma)\left(\frac{\eta_2}{\eta_2+\mu+\gamma} - \varrho_2\right) J_2 \\ &\quad +(\varrho_2\xi\beta_2 - (\alpha+\gamma))A_2 + \frac{A_1A_2}{A} - mVN.\end{aligned}$$

We choose $\varrho_2 = \dfrac{\eta_2}{\eta_2+\mu+\gamma} < 1$. *Then*

$$\begin{aligned}\dot{V}(X) &\le (\alpha+\gamma-)\left(\frac{\xi\beta_2\eta_2}{(\eta_2+\mu+\gamma)(\alpha+\gamma-)} - 1\right) A_2 \\ &\quad -mVN.\end{aligned}$$

Notice that he hypothesis $R_2 \le 1 - R_0$ *is equivalent to* $\dfrac{\xi\beta_2\eta_2}{(\eta_2+\mu+\gamma)(\alpha+\gamma-)} < 1$. *Hence* $\dot{V}(X) \le 0$ *for* $X \in R^4_+$. *This shows that* V *is indeed a Liapunov function with our selection of* ϱ_2. *We will now apply the Lyapunov-LaSalle's invariance principle. Clearly,*

$$\begin{aligned}E &= \{X \in R^4_+ : \dot{V}(X) = 0\} \\ &= \{(J_1, A_1, 0, 0) : J_1 \ge 0, A_1 \ge 0\}\end{aligned}$$

is an invariant set. As a consequence, the largest invariant set M *in* E *is* E. *Hence all positive solutions of system (??) tend to* E.

In the following, we will apply the Bendixson-Dulac criterion to show that positive solutions of the limiting system

$$\begin{cases} J_1' &= \beta_1 A_1 - (\eta_1+\mu)J_1 - mJ_1(J_1+A_1) \\ &\equiv F(J_1, A_1), \\ A_1' &= \eta_1 J_1 - \alpha A_1 - mA_1(J_1+A_1) \\ &\equiv G(J_1, A_1), \end{cases} \tag{2}$$

will tend to the positive equilibrium $E_1 = (J_1^*, A_1^*)$. *An application of the limiting equation theory (Thieme 2003) allows us to conclude that positive solutions of the system (??) tend to the disease-free equilibrium.*

Let $\rho(J_1, A_1) = \dfrac{1}{J_1 A_1}$. *We have*

$$\begin{aligned}\frac{\partial(\rho F)}{\partial J_1} + \frac{\partial(\rho G)}{\partial A_1} &= \left(-\frac{\beta_1}{J_1^2} - \frac{m}{A_1}\right) + \left(-\frac{\eta_1}{A_1^2} - \frac{m}{J_1}\right) \\ &< 0\end{aligned}$$

for all $(J_1, A_1) \in (0,\infty)^2$. *By the Bendixson-Dulac criterium, we see that there are no nontrivial periodic orbits in* $(0,\infty)^2$. *The solutions of system (??) are bounded and there are only two equilibrium points, the trivial one and the positive one. Recall that the trivial equilibrium is a repeller for system (??) therefore the trivial equilibrium of the system (??)is also a repeller. In addition the proof of Theorem 4.1 shows that the positive steady state of the system (??) is locally asymptotically stable. These statement together show that the* ω*-limit set of a bounded positive solution of system (??) is the only positive equilibrium* (J_1^*, A_1^*). *Therefore the disease-free equilibrium* $(J_1^*, A_1^*, 0, 0)$ *is a global attractor of the system (??).*

The conditions for the case when infected subpopulation eventually win the competition with the susceptible subpopulation eventually disappear are given by the following theorem. The proof of this theorem resembles the previous one.

Theorem 4 *If* $\xi = 1$, $R_1 \le 1$ *and* $R_0 > \dfrac{\mu+\eta_2+\gamma}{\eta_2}$, *then* $\limsup_{t\to\infty} N(t) > 0$ *and the susceptible extinction equilibrium* $(0, 0, J_2^*, A_2^*)$ *is a global attractor in* R^4_+.

Proof 4 *We show first that if* $R_2 > \dfrac{\mu+\eta_2+\gamma}{\eta_2}$, *then* $\limsup_{t\to\infty} N(t) > 0$. *Again, we will prove this by contradiction. Assume that* $\limsup_{t\to\infty} N(t) = 0$. *Consider* $y = A_2 + sJ_2$, *with* $s \in (0,1)$ *to be chosen later. The derivative of* y *along a solution of the system (??) is*

$$\begin{aligned}y' &\ge \left(\frac{\eta_2}{s} - (\eta_2+\mu+\gamma)\right) sJ_2 + (\alpha+\gamma)(sR_0 - 1)A_2 \\ &\quad -myN.\end{aligned}$$

Since $R_0 > \dfrac{\eta_2+\mu+\gamma}{\eta_2}$, *then* $R_0 > \dfrac{(\eta_2+\mu+\gamma)(\alpha+\gamma)}{\eta_2(\alpha+\gamma)}$, *in consequence, there exists an* $0 < \tilde{\epsilon} < 1$ *such that*

$$\eta_2(\alpha+\gamma)R_2 > (\eta_2+\mu+\gamma+\tilde{\epsilon})(\alpha+\gamma+\tilde{\epsilon}).$$

We choose s *such that*

$$\frac{\alpha+\gamma+\tilde{\epsilon}}{(\alpha+\gamma)R_2} < s < \frac{\eta_2}{\eta_2+\mu+\gamma+\tilde{\epsilon}} < 1\ .$$

Then

$$y' > \tilde{\epsilon}y - myN = y(\tilde{\epsilon} - mN)\ .$$

If $\limsup_{t\to\infty} N(t) = 0$ *then* $y \to \infty$, *but this is a contradiction since* $y = A_2 + sJ_2 < N$ *is bounded. Hence,* $\limsup_{t\to\infty} N(t) > 0$.

We now proceed to show that the susceptible extinction is a global attractor in R^4_+ *for the system (??). Consider the following potential Liapunov function*

$$V(X) = \varrho_1 J_1 + A_1\,, \qquad X = (J_1, A_1, J_2, A_2)\,,$$

with $\varrho_1 > 0$ *to be chosen later. The derivative of* V *along a solution of the system (??) is*

$$\begin{aligned}\dot{V}(X) &= (\eta_1+\mu)\left(\frac{\eta_1}{\eta_1+\mu} - \varrho_1\right) J_1 + (\varrho_1\beta_1 - \alpha)A_1 \\ &\quad -\frac{CA_1A_2}{A} - mVN.\end{aligned}$$

We choose $0 < \varrho_1 = \dfrac{\eta_1}{\eta_1+\mu} < 1$. *Then*

$$\dot{V}(X) = \alpha(R_1-1)A_1 - \frac{\nu A_1 A_2}{A} - mVN.$$

Since $R_1 \leq 1$, *we see that* $\dot{V}(X) \leq 0$ *for* $X \in R^4_+$. *Similar to the proof of the previous theorem, an application of the Liapunov-LaSalle's invariance principle shows that positive solutions tend to*

$$\begin{aligned} E &= \{X \in R^4_+ : \dot{V}(X) = 0\} \\ &= \{(0,0,J_2,A_2) : J_2 \geq 0, A_2 \geq 0\} \end{aligned}$$

as $t \to \infty$. *Likewise, through the standard application of Bendixson-Dulac criterion and phase plane analysis, we can show that positive solutions of the limiting system*

$$\begin{cases} J_2' = \beta_2 A_2 - (\eta_2+\mu+\gamma)J_2 - mJ_2(J_2+A_2) \\ \quad\equiv F(J_2,A_2), \\ A_2' = \eta_2 J_2 - (\alpha+\gamma)A_2 - mA_2(J_2+A_2) \\ \quad\equiv G(J_2,A_2), \end{cases}$$

tend to the positive equilibrium (J_2^*, A_2^*). *Therefore, the susceptible extinction equilibrium* $(0,0,J_2^*,A_2^*)$ *is a global attractor of the system (**??**).*

To end this section, we present a very natural result that describes the devastating scenario of the extinction of the total population.

Theorem 5 *If* $\xi = 1$, $R_1 \leq 1$ *and* $R_0 \leq 1$, *then all solutions of the system (**??**) satisfies*

$$\lim_{t\to\infty}(J_1(t), A_1(t), J_2(t), A_2(t)) = (0,0,0,0).$$

Proof 5 *Consider the following Liapunov function in* R^4_+,

$$V(X) = \varrho_1 J_1 + A_1 + \varrho_2 J_2 + A_2\,, \qquad X = (J_1, A_1, J_2, A_2),$$

with $\varrho_1 = \dfrac{\eta_1}{\eta_1+\mu} < 1$ *and* $\varrho_2 = \dfrac{\eta_2}{\eta_2+\mu+\gamma} < 1$. *The derivative of* V *along a solution of the system (**??**) is*

$$\begin{aligned} \dot{V}(X) &= (\varrho_1\beta_1 - \alpha)A_1 + (\varrho_2\beta_2 - (\alpha+\gamma))A_2 - mVN \\ &= \alpha(R_1-1)A_1 + (\alpha+\gamma)(R_0-1)A_2 - mVN. \end{aligned}$$

Since $R_1 \leq 1$ *and* $R_0 \leq 1$, *we have that* $\dot{V}(X) \leq 0$ *for* $X \in R^4_+$. *Clearly, the zero set for* $\dot{V}(X)$ *is simply*

$$E = \{X \in R^4_+ : \dot{V}(X) = 0\} = \{(0,0,0,0)\}$$

*which is an invariant set. By the LaSalle Liapunov theorem, we conclude that all positive solutions of system (**??**) tend to* E *as* $t \to \infty$.

6. RESULTS AND CONCLUSIONS

The local stability result Theorem 4.1 for the disease-free equilibrium suggests that in a disease without recovery, if the susceptible reproduction ratio is larger than 1, then a unique disease free equilibrium is feasible. If in addition, the sum of the infectious contact number and the infective reproduction ratio is less than one and the susceptible reproduction ratio is greater than the inverse of the product of the probability of surviving from the susceptible juvenile stage and the probability of dying during the susceptible adult stage, then our global stability result Theorem 5.1 states that the infected classes will eventually disappear and the susceptible classes approach a positive constant value.

In a disease without recovery with any initial population size, if the infected reproduction ratio is greater than one, then a unique susceptible extinction equilibrium is feasible. Moreover, if the susceptible reproduction ratio is less than one, 100% of the babies from infected pregnant women are infected and the infected reproduction ratio is greater the inverse of the probability to survive the infected juvenile stage, then our global stability result Theorem 5.2 states that the susceptible classes will eventually disappear and the infected classes approach a positive constant value.

Theorem 5.3 states that in a disease without recovery, if the susceptible and infected reproduction ratio is less than or equal to one, then the total population eventually disappear. This is intuitive as it simply states that if both the susceptible and infected can not generate enough growth to replacing themselves, then they are doomed.

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Tabela 1: Table 1: Terms

Term	Meaning
$\beta_1 A_1$	recruitment of susceptible juveniles from the birth due to susceptible adults
$(1-\xi)\beta_2 A_2$	recruitment of susceptible juveniles from the birth due to infected adults
$\eta_1 J_1$	reduction of susceptible juveniles by maturation
μJ_1	baseline death process of the susceptible juveniles
$mJ_1 N$	additional mortality of susceptible juveniles due to crowdedness
$\eta_1 J_1$	additional mortality due to crowdedness
$\eta_1 J_1$	recruitment of susceptible adults from the maturing susceptible juveniles
$\upsilon A_1 A_2/A$	reduction of susceptible adults by the infection
αA_1	reduction of susceptible adults by the baseline death process
$mA_1 N$	additional mortality of susceptible adults due to crowdedness
$\xi\beta_2 A_2$	recruitment of infected juveniles from the birth due to infected adults
$\eta_2 J_2$	reduction of infected juveniles by maturation
μJ_2	baseline death process of the infected juveniles
γJ_2	additional mortality of infected juveniles due to the disease
$mJ_1 N$	additional mortality of infected juveniles due to crowdedness
$\eta_2 J_2$	recruitment of infected adults from the maturing infected juveniles
$\upsilon A_1 A_2/A$	newly infected adults
αA_2	reduction of infected adults by the baseline death process
γA_2	additional mortality due to the disease
$mA_2 N$	additional mortality of infected adults due to crowdedness