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Recursive method for inversion of lower triangular matrix

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Abstract
Algorithm for finding by recursion, the inverse of a lower triangular matrix of order n is developed, using its last row and the sub matrix obtained by deleting the last row and last column. In these algorithms the hitherto using double suffix notation for the entries of amatrix is replaced by single suffix notation. The necessary results are established.

Keywords: Lower triangular matrix, Recursive method, Single suffix

INTRODUCTION

The algebra of lower triangular matrices of order n can be identified with the space $R^{n(n+1)/2.}$ This identification enables one to switch over to single suffix notation from the double suffix notation for the entries of a matrix. As the underlying sets are bijective vector multiplication on $R^{n(n+1)/2.}$ can be suitably defined using matrix multiplication, so that these are isomorphic as algebras. This observation leads to simplification of computational procedure for inversion. In this paper I discussthese aspects and present algorithm for recursive method.

Notation

For any positive integer n, let S_n=n(n+1)/2. We denote the vectors in R^{Sn} by $(a^1,a^2,\ldots..a^n)$ where ai=(a_{l+1}, a_{l+2}, a_{l+i}) and i=S_{i-1}. R^{Sn} is a vector space with component wise addition and scalar multiplication. Before defining multiplication between vectors in R^{Sn} we introduce the following notation.

If $A_n = (a^1, a^2, \ldots, a^n)$ we write $A_{n-1} = (a^1, a^2, \ldots, a^{n-1})$
$a^n = (a, a_{S_n})$ Where $a = (a_{N+1}, a_{N+2}, \ldots, a_{N+n-1})$ and $N = S_{n-1}$
When n=1, $A_1, B_1 \in R^1$. We write $A_1, B_1 = (a_1, b_1)$, where $A_1 = (a_1), B_1 = (b_1)$.
If $b^n \in R^n$ we write $b^n = (b_1, b_2, \ldots, b_{n-1}, b_n) = (b^{n-1}, b_n)$.
If $A_n = (A_{n-1}, a, a_{S_n}) \in R^{S_n}$, we define
$b^n A_n = (b^{n-1}, b_n)(A_{n-1}, a, a_{S_n}) = (b^{n-1}A_{n-1} + b_n a, b_n a_{S_n})$

This gives us the definition of b^nA_n inductively for all n.
Let $A_n = (A_{n-1}, a, a_{S_n})$ and $B_n = (B_{n-1}, b, b_{S_n})$.

If n=1, $A_1 = (a_1),\ B_1 = (b_1),\ A_1B_1 = (a_1b_1)$.
If A_{n-1}, B_{n-1} are defined, we define
$A_nB_n = (A_{n-1}B_{n-1}, aB_{n-1} + a_{S_n}b, a_{S_n}b_{S_n})$

Theorem

R^{Sn} is an algebra with identity $I_n = (I_{n-1}, 0, 1)$ and $I_1 = (1)$
Proof when n=1, $R^{Sn}=R^1$ and $A_1=(a_1)$
∴ we identify A_1 with a_1

Vector addition coincides with addition in R , the scalar multiplication and vector multiplication coincide with multiplication in R. thus R^1 is an algebra with identity I_{n-1}.

$A_n = (A_{n-1}, a, a_{S_n})$ and $B_n = (B_{n-1}, b, b_{S_n}) \in R^{S_n}$
$A_n + B_n = (A_{n-1} + B_{n-1}, a + b, a_{S_n} + b_{S_n})$,
$A_nB_n = (A_{n-1}B_{n-1}, aB_{n-1} + a_{S_n}b, a_{S_n}b_{S_n})$ and
$\alpha A_n = (\alpha A_{n-1}, \alpha a, \alpha a_{S_n})$, $\forall \alpha \in R$.

By induction hypothesis R^{Sn-1} is algebra with identity
$I_n = (I_{n-1}, 0, 1)$

Theorem

A Є R^{Sn} is invertibleiff $\exists\, B \in R^{S_n}$ such that Ab=I_n.
Proof It is enough to show that $AB = I_n \Leftrightarrow BA = I_n$. When n=1, S_n=1and in this case the statement is true.
Assume for n-1. Let
$A = (A_1, a, a_{S_n}) \in R^{S_n}, B = (B_1, b, b_{S_n}) \in R^{S_n}$
$AB = (A_1B_1, aB_1 + a_{S_n}b, a_{S_n}b_{S_n})$,
$BA = (B_1A_1, bA_1 + b_{S_n}a, b_{S_n}a_{S_n})$
Assume that $AB = I_n = (I_{n-1}, 0, 1)$. Then $A_1B_1 = I_{n-1}$, $aB_1 + a_{S_n}b = 0$, $a_{S_n}b_{S_n} = 1$.
By induction hypothesis $A_1B_1 = I_{n-1}$. Hence it is enough to prove that $bA_1 + b_{S_n}a = 0$.

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Since

$aB_1 + a_{S_n} b = 0$

$aB_1 = -a_{S_n} b \Rightarrow aB_1A_1 = -a_{S_n} bA_1 \Rightarrow aI_{n-1} = -a_{S_n} bA_1$

$b_{S_n} a = -b_{S_n} a_{S_n} bA_1 = -bA_1 \Rightarrow b_{S_n} a + bA_1 = 0$.

Theorem

$A = (A_1, a, a_{S_n})$ is invertible iff $a_{S_1}, a_{S_2}, ..., a_{S_n} \neq 0$. In this case $A^{-1} = (A_1^{-1}, x, a_{S_n}^{-1})$ where $x = -a_{S_n}^{-1} a A_1^{-1}$.

Proof we prove by induction on n.

If n=1, $S_n = 1$, $R^{S_n} = R^1 = R$ and in this case the statement is clearly true. Assume the validity of the statement for n-1. Let $A = (A_1, a, a_{S_n})$ be any vector in R^{S_n}.

Clearly $a_{S_1}, a_{S_2}, ..., a_{S_n} \neq 0$ if $a_{S_1}, a_{S_2}, ..., a_{S_{n-1}} \neq 0 \neq a_{S_n}$.

By induction hypothesis $A_1 \in R^{S_{n-1}}$ is invertible iff $a_{S_1}, a_{S_2}, ..., a_{S_{n-1}} \neq 0$.

Now we assume that $a_{S_1}, a_{S_2}, ..., a_{S_n} \neq 0$.

$(A_1, a, a_{S_n})(A_1^{-1}, x, a_{S_n}^{-1}) =$

$(A_1A_1^{-1}, aA_1^{-1} + a_{S_n} x, a_{S_n} a_{S_n}^{-1}) = (I_{n-1}, 0, 1) = I_n$.

Hence A is invertible.

Conversely assume that A is invertible and let $A^{-1} = B = (B_1, b, b_{S_n})$ then

$AB = (A_1, a, a_{S_n})(B_1, b, b_{S_n}) = (A_1B_1, aB_1 + a_{S_n} b, a_{S_n} b_{S_n})$

$\Rightarrow A_1B_1 = I_{n-1}$, $aB_1 + a_{S_1} b = 0, a_{S_n} b_{S_n} = 1 \Rightarrow A_1$ is invertible and $a_{S_n} \neq 0$

$\Rightarrow a_{S_1}, a_{S_2}, ..., a_{S_{n-1}} \neq 0 \neq a_{S_n}$. $\Rightarrow a_{S_1}, a_{S_2}, ..., a_{S_n} \neq 0$.

$\varphi(A_n) = L_n$ (L for lower triangular) where L_n is the lower triangular matrix of order n, with first I entries of the i^{th} row coinciding with the corresponding entries of a^i. By partitioning L_n through bordering at the last row and last column and ignoring the zero column in the bordered matrix, we can represent $L_n = (L_{n-1}, l, a_{S_n})$ where $(l, a_{S_n}) = a^n$. This representation enables us to write $L_n = \varphi(A_n) = \varphi(A_{n-1}, l, a_{S_n})$. It is now clear that addition, scalar multiplication and vector multiplication in R^n correspond to the corresponding operations in the algebra of lower triangular matrices. Thus we have the followings

Theorem

The algebra R^{S_n} is isomorphic to L_n, the algebra of lower triangular matrices of order n.

Corollary

let $L_n = (L_{n-1}, l, l_{nn})$ be a lower triangular matrices of order n where, L_{n-1} is the lower triangular matrices of order n-1 formed by the first n-1 rows and columns of L_n and (l, l_{nn}) is the last row of L_n. L_n is nonsingular if L_{n-1} is nonsingular and $l_{nn} \neq 0$. In this case $L_n^{-1} = (L_{n-1}^{-1}, x, l_{nn}^{-1})$ where $x = -l_{nn}^{-1} l L_{n-1}^{-1}$.

Proof Follows from theorems (1.5) and (1.6).

Remark

The above identification of a lower triangular matrix of order n with a vector in R^{S_n} enables us to adopt single suffix notation for entries in triangular matrix instead of the present double suffix notation. The corollary (1.7) yields a recursive method for finding the inverse of a nonsingular lower triangular matrix. We present below an algorithm for this recursive method.

Algorithm Recursive algorithm for inversion of a lower triangular matrix of order n.

Write $S_i = i(i+1)/2, \quad 0 \le i \le n$

$l^i = (l_{I+1}, l_{I+2}, .., l_{I+i-1}, l_{S_i})$, and $I = S_{i-1}$

$L_1 = (l^1) = (l_1), x^1 = (x_1) = (l_1^{-1}), L_1^{-1} = (x^1)$

Let $2 \le i \le n$ $x_{I+j} = -l_{S_i}^{-1} \sum_{k=j}^{n-1} l_{I+k} x_{K+j}$, where $I = S_{i-1}$, $K = S_{k-1}$, $1 \le j \le i-1$

$x_{S_i} = l_{S_i}^{-1}$, $x^i = (x_{I+1}, x_{I+2}, .., x_{I+i-1}, x_{S_i})$ where $I = S_{i-1}$

$L_i^{-1} = (L_{i-1}, x^i)$

Example Let $L_4 = \begin{bmatrix} 1 & & & \\ 2 & 4 & & \\ 1 & 0 & 3 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$

Step 1 $L_1 = (1), L_1^{-1} = (1)$

Step 2 $L_2 = \begin{bmatrix} 1 & \\ 2 & 4 \end{bmatrix}$,

$l^2 = (l_2, l_3) = (2 \quad 4)$, $x^2 = (x_2 \quad x_3)$ where

$x_2 = -l_3^{-1} l_2 x_1 = -0.5, \quad x_3 = l_{S_2}^{-1} = l_3^{-1} = 0.25$,

$L_2^{-1} = (L_1^{-1}, x^2) = \begin{bmatrix} 1 & \\ -0.5 & 0.25 \end{bmatrix}$.

Step 3 $L_3 = \begin{bmatrix} 1 & & \\ 2 & 4 & \\ 1 & 0 & 3 \end{bmatrix}$

$l^3 = (1 \quad 0 \quad 4)$, $x^3 = (x_4 \quad x_5 \quad x_6)$ where

$x_4 = -0.25, \; x_5 = 0, \; x_6 = 0.25$

$L_3^{-1} = (L_2^{-1} \quad x^3) = \begin{bmatrix} 1 & & \\ -0.5 & 0.25 & \\ -0.25 & 0 & 0.25 \end{bmatrix}$.

Step 4 $L_4 = \begin{bmatrix} 1 & & & \\ 2 & 4 & & \\ 1 & 0 & 3 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$

$l^4 = (0 \quad 0 \quad 2 \quad 1)$, $x^4 = (x_7 \quad x_8 \quad x_9 \quad x_{10})$

where $x_7 = 0.5 \quad x_8 = 0 \quad x_9 = -0.5 \quad x_{10} = 1$

$$L_4^{-1} = (L_3^{-1}, x^4) = \begin{bmatrix} 1 & & & \\ -0.5 & 0.25 & & \\ -0.25 & 0 & 0.25 & \\ 0.5 & 0 & -0.5 & 1 \end{bmatrix}.$$

CONCLUSION

The formulae for finding the inverse of the lower triangular matrices are obtained in vector form and necessary algorithms are developed. This facilitates adoption of a single suffix notation for matrix computations, there by save the computer memory and computational time.

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