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Free A*-algebras

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Abstract

In this paper we studied the free A*-algebras, the sub A*-algebra generated by a subset and its characterization, an A*-algebra freely generated by a subset and introduced the concept of A*-field of sets. Also we established some theorems on making A*-field of sets into A*-algebras.

Keywords: A*-algebra, Free A*-algebra, A* field of sets

INTRODUCTION

In 1994, P. Koteswara Rao introduced the concept of A*-algebras and studied their equivalence with Adas, C – algebras, Stone type representation and introduced the concept of A*- clones and the if-then-else structure over A*- algebras and ideals of A*-algebras. In this paper we introduced the concept of free A*-algebras analogous to the free Boolean algebras.

PRELIMINARIES

Definition: For any non-empty set X, a class *M* of subsets of X which is closed under finite union of sets, finite intersection of sets, complementation of sets is called a field of sets.

Definition: A Boolean algebra is an algebra $(B,\vee, \wedge, (-)', 0, 1)$ with two binary operations, one unary operation(called complementation), and two nullary operations which satisfies:

(1) $(B,\vee, \wedge)$ is a distributive lattice.
(2) $x \wedge 0 = 0, \quad x \vee 1 = 1$
(3) $x \wedge x' = 0, x \vee x' = 1$

Definition: An indexed set $\{B_t\}_{t\in T}$ of subalgebras of a Boolean algebra B is said to be independent if $a_1\wedge a_2\wedge\ldots\wedge a_n \neq 0$ for every finite sequence of non zero elements a_i choosen from subalgebras B_t with different indices.

Definition: Let $\{B_t\}_{t\in T}$ be an indexed set of Boolean algebras. By a Boolean product of $\{B_t\}_{t\in T}$ we mean any pair $\{\{i_t\}_{t\in T}, B\}$ such that

a) B is a Boolean algebra.
b) For every $t\in T$, $i_t : B_t \to B$ is an isomorphism.
c) The indexed set $\{i_t(B_t)\}_{t\in T}$ of subalgebras of B is independent.
d) The union of all subalgebras $\{i_t(B_t)\}_{t\in T}$ generates B.

Definition: Suppose $\{X_t\}_{t\in T}$ is an indexed set of non-empty sets. Let X be the Cartesian product of X_t's. Let M_t be the field of subsets of X_t. For every set $A\subseteq X_t$, let A* be the set of all points $x\in X$ whose t^{th} coordinate $x_t\in A$ and let M_t^* be the field composed of all sets A* where $A\in M_t$. The field *M* of subsets of X generated by all the classes $\{M_t^*\}_{t\in T}$ is called the field product of $\{M_t\}_{t\in T}$.

Definition: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A* - algebra if it satisfies :

For $a, b, c\in A$

i. $a_\pi \vee (a_\pi)^{\sim} = 1$, $(a_\pi)_\pi = a_\pi$, where $a\vee b = (a^{\sim}\wedge b^{\sim})^{\sim}$.
ii. $a_\pi\vee b_\pi = b_\pi\vee a_\pi$
iii. $(a_\pi\vee b_\pi)\vee c_\pi = a_\pi\vee (b_\pi \vee c_\pi)$
iv. $(a_\pi \wedge b_\pi)\vee (a_\pi \wedge (b_\pi)^{\sim}) = a_\pi$
v. $(a \wedge b)_\pi = a_\pi\wedge b_\pi$, $(a \wedge b)^{\#} = a^{\#} \vee b^{\#}$, where $a^{\#} = (a_\pi \vee a^{\sim}_\pi)^{\sim}$
vi. $a^{\sim}_\pi = (a_\pi\vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$
vii. $(a*b)_\pi = a_\pi$, $(a*b)^{\#} = (a_\pi)^{\sim}\wedge(b^{\sim}_\pi)^{\sim}$
viii. $a = b$ if and only if $a_\pi = b_\pi$, $a^{\#} = b^{\#}$. We write 0 for $1^{\sim}$, 2 for $0*1$.

Example: 3 $=\{0,1,2\}$ with the operations defined below is an A* - algebra

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$*$	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x_π	0	1	0
$x^{\#}$	0	0	1

Definition : Let $(A_1, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ and $(A_2, \wedge, (-)^{\sim}, (-)_\pi, *, 1)$ be two A*-algebras. A Mapping $f: A_1\to A_2$ is called an A* - homomorphism if for all

$a, b\in A_1$

I. $f(a\wedge b) = f(a)\wedge f(b)$
II. $f(a\vee b) = f(a)\vee f(b)$
III. $f(a * b) = f(a) * f(b)$
IV. $f(a_\pi) = (f(a))_\pi$
V. $f(a^{\sim}) = (f(a))^{\sim}$

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VI. f(1) = 1

VII. f(0) = 0.

Definition: Let C be a non-empty set. By a free A*-algebra on C we shall mean an A*-algebra F together with a mapping f: C→F such that for every A*-algebra A and every mapping g: C→A there is a unique A*-homomorphism h: F→A such that hog = f.
Now we state some routine theorems without proof.

Theorem: If (F,f) is a free A*-algebra on a non-empty set C then f is injective and Im f generates F.

Theorem: Let (F,f) be a free A*-algebra on a nom empty set C. Then (F^{l}, f^{l}) is also a free A*-algebra on C iff there is a unique A*-isomorphism j: F→ F^{l} such that jof = f^{l}.

Theorem: Let (B, ∧, (-)′, 0, 1) be a Boolean algebra. The *A*(B) = {(a, b) /a, b ∈ B, a ∧ b = 0 } becomes an A*- algebra with the following A*- algebraic operations:
For a = (x_π, $x^\#$), b = (y_π, $y^\#$) ∈ *A*(B)

I. a ∧ b = ($x_\pi y_\pi$, $x_\pi y^\# + x^\# y_\pi + x^\# y^\#$)
II. a ∨ b = ($x_\pi y_\pi + x_\pi y^\# + x^\# y_\pi$, $x^\# y^\#$)
III. a ∗ b = (x_π, $((x_\pi)' y^\#)$)
IV. $a^{\sim}$ = ($x^\#$, x_π)
V. a_π = (x_π, $(x_\pi)'$)
VI. 1 = (1, 0), 0 = (0, 1), 2 = (0, 0).

Theorem : If A is an A*- algebra then

I. *B*(*A*(B)) ≅ B.
II. *A*(*B*(A)) ≅ A.

Theorem: Let A_1,A_2 be A*- algebras and B_1,B_2 be Boolean algebras then

I. A_1 ≅ A_2 iff *B*(A_1) ≅ *B*(A_2).
II. B_1 ≅ B_2 iff *A*(B_1) ≅ *A*(B_2).

MAIN RESULTS

Theorem: For every non-empty set C, there exists a free A*-algebra on C.

Proof: Let *F* = {θ / θ: C→3 = {0, 1, 2}, θ(c) = 0 for all most all c∈C }
Define ∨, ∧, $(-)^{\sim}$,$(-)_\pi$, ⋆, 0,1, 2 as follows:
Let θ, ξ ∈ *F*. For all c∈C, define
(θ ∨ ξ)(c) = (θ)(c) ∨ (ξ)(c)
(θ ∧ ξ)(c) = (θ)(c) ∧ (ξ)(c)
(θ ⋆ ξ)(c) = (θ)(c) ⋆ (ξ)(c)
$\theta^{\sim}(c) = (\theta(c))^{\sim}$
0(c) = 0
1(c) = 1
2(c) = 2
Then *F* is an A*-algebra.
Now define f : C→ *F* as follows:
For any c∈C, f(c) : C → **3** by f(c)(d) = 1 if d = c
= 0 if d ≠ c

Claim: (***F***, f) is a free A*-algebra on C.
Let A be any A*-algebra and g : C → A be a mapping.
Define h : ***F*** → A as follows:
Let θ ∈ ***F***

$$h(\theta) = \vee_{c\in C} \theta(c)\, g(c)$$

Where θ(c).g(c) = $(\theta(c).g(c))_\pi \star (\theta(c).g(c))^\#$
$(\theta(c).g(c))_\pi = (\theta(c) \wedge g(c))_\pi \vee (\theta(c)^\# \wedge g(c)^\#)$
$(\theta(c).g(c))^\# = (\theta(c)_\pi \wedge g(c)^\#) \vee (\theta(c)^\# \wedge g(c)_\pi)$
Clearly h is an A*- homomorphism
$h(f(c)) = \vee_{t\in C} (f(c))(t).\, g(t)$ = g(c)
Therefore hog = g.
Uniqueness of h:
Let θ ∈ ***F***, t∈ C
θ (t) = θ (t).1

$$= \vee_{c\in C} \theta(c).[f(c)](t)$$
$$= [\vee_{c\in C} \theta(c).f(c)](t)$$
$$Therefore\ \theta = \vee_{c\in C} \theta(c) f(c)$$

Suppose $\overline{h}: F \to A$ is another A*- homomorphism such that $\overline{h} o f = g$

$$\overline{h}(\theta) = \overline{h}(\vee_{c\in C} \theta(c) f(c))$$
$$= \vee_{c\in C} \theta(c) \overline{h}(f(c))$$
$$= \vee_{c\in C} \theta(c) g(c)$$
$$= h(\theta)$$

Therefore $\overline{h} = h$

Remerk: The free A*-algebra *F* constructed as in the above theorem is called the free A*-algebra on C.

Definition: Suppose A is an A*-algebra and $C \subseteq A$. Then A is said to be freely generated by C if

(i) <C> = A
(ii) Every mapping f : C → A^{l}, A^{l} is another A*-algebra, can be extending uniquely to a homomorphism h : A→ A^{l} .

Theorem: If A is an A*-algebra and B = B(A), $C \subseteq A.$ then A is freely generated by C if and only if B is freely generated by C_π, where C_π = {a_π / a ∈ C}.

Proof: Suppose A is an A*-algebra, B = B(A) and $C \subseteq A.$ .Assume that A is freely generated by C.
We have to prove that B is freely generated by C_π = {a_π / a ∈ C}.
Let g : C → ***B´*** is a mapping where ***B´*** is another Boolean algebra.

Define f : C → *A*(***B´***) by f(a) = $(g(a)_\pi, g(a)_\pi^{\sim})$.

Since C generates A, f can be extended to a unique homomorphism

h :A → *A*(*B´*).
h_π : B →B(*A*(*B´*)) is also a unique homomorphism of Boolean

algebras.
Since $B(A(B')) \cong B'$, let $I : B(A(B'))$ is an isomorphism such that $I(a, a^{\sim}) = a, \forall (a, a^{\sim}) \in B(A(B'))$.

Then $I\,h_\pi : B \to B'$ is a homomorphism and $I\,h_\pi$ is a unique extension of g.
Therefore C_π generates B freely.
Conversely assume that C_π generates B freely.
We have to prove that A is freely generated by C.
Suppose $f : C \to A'$ is another mapping where A' is another A*-algebra.
Define $f_\pi : C_\pi \to B(A')$ by $f_\pi(a) = (f(b))_\pi$
Since C_π generates B freely $\exists$ a unique homomorphism $g : B \to B(A')$
Such that $f_\pi(a) = g(a)$ $\forall a \in C_\pi$.
Define $h : A \to A'$ by $h(a) = (g(a_\pi) * g(a_\pi))$
Then h is a homomorphism. Since g is unique, his also unique.
We now show that h = f on C.

Let $a \in C \Rightarrow a_\pi \in C_\pi$

$$h(a) = (g(a_\pi) * g(a_\pi)^{\sim}) = f_\pi(a_\pi) * (f_\pi(a_\pi)^{\sim})$$
$$= [f(a)_\pi * ((f_\pi(a_\pi)^{\sim}]$$
$$= f(a)$$

Therefore h is a unique extension of f.

Therefore C generates A freely.

Definition: Let X be a non-empty set. A class $F^* = \{(A_1, A_2) / A_1, A_2 \subseteq X, A_1 \cap A_2 = \phi\}$ is called an A*-field of subsets of X if

(i) $(X, \phi) \in F^*$
(ii) $(A_1, A_2) \in F^* \Rightarrow (A_2, A_1) \in F^*$
(iii) $(A_1, A_2), (B_1, B_2) \in F^* \Rightarrow (A_1B_1, A_1B_2 + A_2B_1 + A_2B_2)$
(iv) $(A_1, A_2), (B_1, B_2) \in F^* \Rightarrow (A_1, A_1^C B_2) \in F^*$

Juxtaposition and addition stand for intersection and union of sets

From the above definition immediately we have the following theorem.

Theorem: Let X be a non-empty set. A class $F^* = \{(A_1, A_2) / A_1, A_2 \subseteq X, A1 \cap A2 = \phi\}$. Then F^* is an A*-algebra with the following operations:

(i) $1 = (X, \phi)$, $0 = (\phi, X)$, $2 = (\phi, \phi)$
(ii) $(A_1, A_2)_\pi = (A_2, A_1^C)$
(iii) $(A_1, A_2)^C = (A_2, A_1)$
(iv) $(A_1, A_2) * (B_1, B_2) = (A_1, A_1^C B_2)$
(v) $(A_1, A_2) \wedge (B_1, B_2) = (A_1B_1, A_1B_2 + A_2B_1 + A_2B_2)$
(vi) $(A_1, A_2) \vee B_1, B_2) = (A_1B_1 + A_1B_2 + A_2B_1, A_2B_2)$

Proof: It is routine to verify the axioms in 1.6.

Theorem: Let F^* be an A*- field of subsets of a non-empty set X and $F = \{A / (A, A^C) \in F^*\}$

Then F is a field of subsets of X and $B(F^*) \cong F$.

Proof: Suppose F^* is an A*- field of subsets of X.

We have to prove that $F = \{A / (A, A^C) \in F^*\}$ is a field of subsets of X.

Let $A, B \in F$

Consider $(A, A^C) \wedge (B, B^C) = (AB, AB^C + A^CB + A^CB^C) \in F^*$
$= (AB, AB^C + A^C(B + B^C)) \in F^*$
$= (AB, AB^C + A^C) \in F^*$ $(AB, (A + A^C).(B^C + A^C)) \in F^*$
$= (AB, A^C + B^C) \in F^*$
$= (AB, (AB)^C) \in F^*$
$= AB \in F$

Let $A \in F$. Then $(A, A^C) \in F^* \Rightarrow (A^C, A) \in F^*$
$\Rightarrow (A^C, (A^C)^C) \in F^*$
$\Rightarrow A^C \in F$

Therefore F is a field of subsets of X and clearly $F \cong B(F^*)$

Theorem: Let F is a field of subsets of a non-empty set X. Then $A(F) = \{(A, B) / A, B \in F, A \cap B = \phi\}$ is an A*- field of subsets of X.

Proof: It is routine to verify the axioms in 2.5.
We now prove the following theorem.

Theorem: Every A*- algebra A is isomorphic to an A*- field of subsets F^* of a Stone space.

Proff: Let A be an A*- algebra.
Let B = B(A). Then there exists a Stone space H such that B $\cong$ F, where F is the field of clopen subsets of H.
By the known result A(B) $\cong$ $A(F)$.
But A $\cong$ A(B). Thus A $\cong$ $A(F)$.

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