Journal of Experimental Sciences 2012, 3(2): 58-63
ISSN: 2218-1768
Available Online: http://jexpsciences.com/
journal of experimental sciences

On the stability of a four species: a prey-predator-host- commensal-competition-syn eco-system-I (fully washed out state)

N. Shanker[1] **and K. Lakshmi Narayan**[2]

[1]Department. of Mathematics, C.M.R. College of Engineering & Technology, Hyderabad, Andhra Pradesh - 501 401, India.
[2]Department of Mathematics, SLC's Institute of Engineering & Technology, Hyderabad-501 512, India.

Abstract

This paper deals with an investigation on a four Species Syn-Ecological System (Fully Washed out State). The System comprises of a prey (S_1) , a predator (S_2) that survives upon S_1, two hosts S_3 and S_4 for which S_1, S_2 are commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 and S_4 are competitors. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of one of the sixteen equilibrium points: the fully washed out state is established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearised equations for the perturbations over the equilibrium point are analyzed to establish the criteria for stability and the trajectories illustrated.

Keywords: Commensal, Eco-system, Equilibrium points, Host, Competition, Prey, Predator, Stability.

INTRODUCTION

Population sizes of species are affected by ecological interactions such as competition, predation and parasitism. Mathematical modeling of ecosystems was initiated in 1925 by Lotka [10] and by Volterra [17]. The general concepts of modeling have been presented in the treatises of Meyer [11], Kushing [7] and Kapur [5, 6]. K. Lakshminarayan and N.Ch. Pattabhi Ramacharyulu [8] studied the two species prey-predator ecological models incorporating a partial cover for the prey and alternate food for the predator. These authors have also analysed a prey-predator model with alternative food for the predator, harvesting of both the species [9]. The study on competitive eco-systems of two and three species with limited and unlimited resources was done by N.C. Srinivas [16]. R. Archana Reddy [1, 2] and B. Bhaskara Rama Sharma [3] investigated on interacting species with harvesting of both the species at constant rate and competitive eco-systems with time delay, employing analytical and numerical techniques. Further study on the stability of a Host – a flourishing commensal species pair with limited resources was done by N. Phani Kumar, N. Seshagiri Rao and N.Ch. Pattabhi Ramacharyulu [12]. The stability analysis of a four species eco-system with the interaction between S_3 and S_4 is neutralism was considered by B. Hari Prasad and N.Ch. Pattabhi Ramacharyulu [4]. Following this N. Shanker, K. Lakshminarayan and N.Ch. Pattabhi Ramacharyulu studied stability analysis of a four species eco-system with the interaction between S_3 and S_4 being mutual [13, 14,15,16].

The present investigation is on an analytical study of a four species (S_1, S_2, S_3, S_4) Prey-Predator-Host-Commensal-Competition-Syn Eco-System. Fig.1 shows a Schematic Sketch of the system under investigation. In all sixteen equilibrium points are identified based on model equations and the stability analysis is carried out only for the fully washed out state.

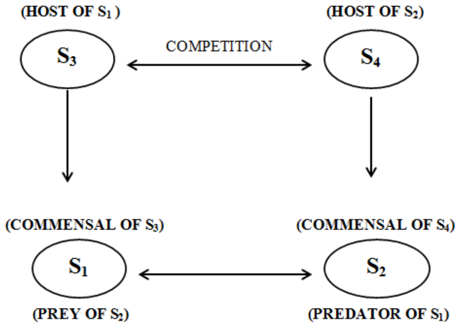

Fig. 1 Schematic Sketch of the Syn Eco - System Under Investigation

NOTATION ADOPTED

N_1 (t): The population of the prey species (S_1)
N_2 (t) : The population of the predator species (S_2)
N_3 (t) :The population of the host species (S_3) of the prey (S_1)
N_4 (t) : The population of the host (S_4) of the predator (S_2)
T : Time instant
a_1, a_2, a_3, a_4 : Natural growth rates of S_1, S_2, S_3, S_4
$a_{11}, a_{22}, a_{33}, a_{44}$: Self inhibition coefficients of S_1, S_2, S_3, S_4
a_{12}, a_{21} : Interaction (prey-predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13} : Coefficient of commensalism of S_3 towards S_1

Received: Dec 14, 2011; Revised: Jan 22, 2012; Accepted: Feb 15, 2012.

*Corresponding Author

N. Shanker
Department. of Mathematics, C.M.R. College of Engineering & Technology,
Hyderabad, Andhra Pradesh - 501 401, India.

Tel: +91-9490373327
Email: shankermaths@yahoo.co.in

a_{24} : Coefficient of commensalism of S_4 towards S_2

a_{34} : Coefficient of competition of S_4 towards S_3

a_{43} : Coefficient of competition of S_3 towards S_4

$K_i = \frac{a_i}{a_{ii}}$: *Carrying* capacity of S_i, i=1,2,3,4

Further the variables N_1, N_2, N_3 and N_4 are non-negative and the model parameters

$a_1, a_2, a_3, a_4, a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{21}, a_{13}, a_{24}, a_{34}, a_{43}$ are assumed to be non-negative constants.

BASIC MODEL EQUATIONS

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \tag{3.1}$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 + a_{24}N_2N_4 \tag{3.2}$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 - a_{34}N_3N_4 \tag{3.3}$$

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 - a_{43}N_3N_4 \tag{3.4}$$

EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

$$\frac{\mathrm{dN_i}}{\mathrm{dt}} = 0, \qquad \mathrm{i} = 1,2,3,4 \tag{4.1}$$

are given in the following table.

Table I. Equilibrium states

S.No.	Equilibrium states	Equilibrium point
1*	Fully washed out state	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
2	Only the prey S_1 survives	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
3	Only the predator S_2 survives	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$
4	Only the host (S_3) of S_1 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
5	Only the host (S_4) of S_2 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
6	Prey (S_1) and the predator (S_2) survives	$\overline{N_1} = \frac{a_1a_{22} - a_2a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_2} = \frac{a_2a_{11} + a_1a_{21}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0$
7	Predator (S_2) and the host (S_4) of S_2 washed out	$\overline{N_1} = \frac{a_1a_{33} + a_3a_{13}}{a_{11}a_{33}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
8	Predator (S_2) and the host (S_3) of S_1 washed out	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
9	Prey (S_1) and the host (S_4) of S_2 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
10	Prey (S_1) and the host(S_3) of S_1 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2a_{44} + a_4a_{24}}{a_{22}a_{44}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
11	Prey (S_1) and the predator (S_2) washed out	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{\alpha_2}{\alpha_1}, \overline{N_4} = \frac{\alpha_3}{\alpha_1}$ where $\alpha_1 = a_{33}a_{44} - a_{34}a_{43}$ $\alpha_2 = a_3a_{44} - a_4a_{34}$ $\alpha_3 = a_4a_{33} - a_3a_{43}$

12	Only the host (S_4) of S_2 washed out	$\overline{N_1}=\frac{\beta_2}{\beta_1},\overline{N_2}=\frac{\beta_3}{\beta_1},\overline{N_3}=\frac{a_3}{a_{33}},\overline{N_4}=0$ $where\ \ \beta_1=a_{33}(a_{11}a_{22}+a_{12}a_{21})$ $\beta_2=a_1a_{22}a_{33}+a_3a_{13}a_{22}-a_2a_{12}a_{33}$ $\beta_3=a_2a_{11}a_{33}+a_1a_{21}a_{33}+a_3a_{13}a_{21}$
13	Only the host(S_3) of S_1 washed out	$\overline{N_1}=\frac{\theta_2}{\theta_1},\overline{N_2}=\frac{\theta_3}{\theta_1},\overline{N_3}=0,\overline{N_4}=\frac{a_4}{a_{44}}$ $where\ \ \theta_1=a_{44}(a_{11}a_{22}+a_{12}a_{21})$ $\theta_2=a_1a_{22}a_{44}-a_2a_{12}a_{44}-a_4a_{12}a_{24}$ $\theta_3=a_2a_{11}a_{44}+a_4a_{11}a_{24}+a_1a_{21}a_{44}$
14	Only the Predator (S_2)washed out	$\overline{N_1}=\frac{\psi}{a_{11}\alpha_1},\overline{N_2}=0,\overline{N_3}=\frac{\alpha_2}{\alpha_1},\overline{N_4}=\frac{\alpha_3}{\alpha_1}$ $where\ \ \psi=a_1\alpha_1+a_{13}\alpha_2$
15	Only the prey (S_1) washed out	$\overline{N_1}=0,\overline{N_2}=\frac{\delta}{a_{22}\alpha_1},\overline{N_3}=\frac{\alpha_2}{\alpha_1},\overline{N_4}=\frac{\alpha_3}{\alpha_1}$ where $\delta=a_2\alpha_1-a_3a_{24}a_{43}+a_4a_{24}a_{33}$
16	The co-existent state (or) Normal steady state	$\overline{N_1}=\frac{\sigma_2}{\sigma_1},\overline{N_2}=\frac{\sigma_3}{\sigma_1},\overline{N_3}=\frac{\alpha_2}{\alpha_1},\overline{N_4}=\frac{\alpha_3}{\alpha_1}$ $where\ \ \sigma_1=(a_{11}a_{22}+a_{12}a_{21})\alpha_1$ $\sigma_2=(a_1a_{22}-a_2a_{12})\alpha_1+a_3(a_{12}a_{24}a_{43}+a_{13}a_{22}a_{44})$ $-\ a_4(a_{12}a_{24}a_{33}+a_{13}a_{22}a_{34})$ $\sigma_3=(a_1a_{21}+a_2a_{11})\alpha_1+a_3(a_{13}a_{21}a_{44}-a_{11}a_{24}a_{43})$ $+a_4(a_{11}a_{24}a_{33}-a_{13}a_{21}a_{34})$

The present paper deals with the stability of fully washed out state (marked *) of the above table only. The stability of the other Equilibrium states will be presented in the forthcoming communications.

Stability of the fully washed out equilibrium state (Sl.No. 1 in the above table)

To discuss the stability of equilibrium point

$$\overline{N_1}=0,\overline{N_2}=0,\overline{N_3}=0,\overline{N_4}=0$$

Let us consider small deviations $u_1(t),\ u_2(t),\ u_3(t),\ u_4(t)$ from the steady state i.e.,

$$N_i(t)=\overline{N}_i+u_i(t)\ ,\qquad i=1,2,3,4 \tag{5.1}$$

Where $u_i(t)$ is a small perturbation in the species S_i.

Substituting (5.1) in (3.1), (3.2), (3.3), (3.4) and neglecting products and higher *powers of* $u_1,\ u_2,\ u_3,\ u_4$ we get

$$\frac{du_i}{dt}=a_iu_i\ ,\qquad i=1,2,3,4 \tag{5.2}$$

The characteristic equation of which is

$$(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)=0 \tag{5.3}$$

whose roots $a_1,\ a_2,\ a_3,\ a_4$ *are all positive*.

Hence the Fully Washed-out State is *Unstable*.

The solutions of the equations (5.2) are

$$u_i=u_{i0}e^{a_it},\qquad i=1,2,3,4 \tag{5.4}$$

where u_{10} , u_{20} , u_{30} *and* u_{40} *are the initial values of* $u_1,\ u_2,\ u_3,\ u_4$ *respectively.*

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates $a_1,\ a_2,\ a_3,\ a_4$ and the initial values of the pertubation $u_{10}(t),\ u_{20}(t),\ u_{30}(t),\ u_{40}(t)$ *of the* species S_1, S_2, S_3, and S_4. Of these 576 situations some typical variations are illustrated in figures 2 to 9 through respective solution curves that would facilitate to make some reasonable observations and the conclusions are presented here.

Conclusions of the Perturbation Graphs

Case (i): $If\ u_{40} < u_{20} < u_{30} < u_{10}, \quad a_2 < a_4 < a_1 < a_3$

In this case predator (S$_2$) has the least natural growth rate and host (S$_4$) of the predator (S$_2$) has the least initial population strength. The host (S$_3$) of the prey (S$_1$) initially dominates over the predator (S$_2$) and also the host (S$_4$) of predator (S$_2$) till the time instant $t_{23}^* = \frac{1}{a_3 - a_2}\log(\frac{u_{20}}{u_{30}})$, $t_{13}^* = \frac{1}{a_3 - a_1}\log(\frac{u_{10}}{u_{30}})$ respectively and thereafter the dominance is reversed. Also the prey (S$_1$) initially dominates over the predator (S$_2$) and also the host (S$_4$) of predator (S$_2$) till the time instant $t_{21}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$, $t_{41}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}})$ respectively and thereafter the dominance is reversed as shown in Fig. 2.

Case (ii): $If\ u_{40} < u_{10} < u_{20} < u_{30}, \quad a_1 < a_3 < a_4 < a_2$

In this case the prey (S$_1$) has the least natural growth rate and host (S$_4$) of the predator (S$_2$) has the least initial population strength. The predator (S$_2$) dominates over the prey (S$_1$) and also over the host (S$_4$) of predator (S$_2$) initially till the time instant $t_{12}^* = \frac{1}{a_2 - a_1}\log(\frac{u_{10}}{u_{20}})$, $t_{42}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}})$ respectively and thereafter the dominance is reversed. Also the host (S$_3$) of the prey (S$_1$) dominates over the prey (S$_1$) till the time instant $t_{13}^* = \frac{1}{a1 - a2}\log(\frac{u20}{u10})$ and thereafter the dominance is reversed as shown in Fig. 3.

Case (iii): $If\ u_{10} < u_{30} < u_{40} < u_{20}, \quad a_3 < a_4 < a_2 < a_1$

In this case the host (S$_3$) of the prey (S$_1$) has the least natural growth rate and the prey (S$_1$) has the least initial population strength. The host (S$_4$) of predator (S$_2$) initially dominates over the host (S$_3$) of the prey (S$_1$) till the time instant $t_{34}^* = \frac{1}{a_4 - a_3}\log(\frac{u_{30}}{u_{40}})$ and thereafter the dominance is reversed. Also the predator (S$_2$) initially dominates over the host (S$_3$) of the prey (S$_1$) and also over the host (S$_4$) of the predator (S$_2$) till the time instant $t_{32}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}})$ and $t_{42}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}})$ and thereafter the dominance is reversed as shown in Fig. 4.

Case (iv): $If\ u_{10} < u_{20} < u_{40} < u_{30}, \quad a_2 < a_1 < a_4 < a_3$

In this case the predator (S$_2$) has the least natural growth rate and the prey (S$_1$) has the least initial population strength. The host (S$_4$) of the predator (S$_2$) initially dominates over the predator (S$_2$) and also over the prey (S$_1$) till the time instant $t_{24}^* = \frac{1}{a_4 - a_2}\log(\frac{u_{20}}{u_{40}})$, $t_{14}^* = \frac{1}{a_4 - a_1}\log(\frac{u_{10}}{u_{40}})$ respectively and thereafter the dominance is reversed. Also the host (S$_3$) of the prey (S$_1$) dominates over the predator (S$_2$), prey (S$_1$) and host (S$_4$) of the predator (S$_2$) till the time instant $t_{23}^* = \frac{1}{a_3 - a_2}\log(\frac{u_{20}}{u_{30}})$, $t_{13}^* = \frac{1}{a_3 - a_1}\log(\frac{u_{10}}{u_{30}})$ and $t_{43}^* = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}})$ respectively and thereafter the dominance is reversed as shown in Fig. 5.

Case (v): $If\ u_{30} < u_{40} < u_{10} < u_{20}, \quad a_4 < a_2 < a_3 < a_1$

In this case the host (S$_4$) of the predator (S$_2$) has the least natural growth rate and the host (S$_3$) of the prey (S$_1$) has the least initial population strength. The prey (S$_1$) initially dominates over the host (S$_4$) of the predator (S$_2$) and the host (S$_3$) of the prey (S$_1$) till the time instant $t_{41}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}})$, $t_{31}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}})$ respectively and thereafter the dominance is reversed. Also the predator (S$_2$) dominates over the host (S$_4$) of the predator (S$_2$) till the time instant $t_{42}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}})$ and thereafter the dominance is reversed as shown in Fig. 6.

Case (vi): $If\ u_{30} < u_{10} < u_{20} < u_{40}, \quad a_4 < a_1 < a_2 < a_3$

In this case the host (S$_4$) of the predator (S$_2$) has the least natural growth rate and the host (S$_3$) of the prey (S$_1$) has the least initial population strength. The predator (S$_2$) initially dominates over the prey (S$_1$) till the time instant $t_{12}^* = \frac{1}{a_2 - a_1}\log(\frac{u_{10}}{u_{20}})$ and thereafter the dominance is reversed as shown in Fig.7.

Case (vii): $If\ u_{20} < u_{30} < u_{10} < u_{40}, \quad a_3 < a_2 < a_1 < a_4$

In this case the host (S$_3$) of the prey (S$_1$) has the least natural growth rate and the predator (S$_2$) has the least initial population strength. The prey (S$_1$) initially dominates over host (S$_3$) of the prey (S$_1$) and also the predator (S$_2$) till the time instant $t_{31}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}})$, $t_{21}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$ respectively and thereafter the dominance is reversed. Also the host (S$_4$) of the predator (S$_2$) dominates over the host (S$_3$) of the prey (S$_1$), the predator (S$_2$) and the prey (S$_1$) till the time instant $t_{34}^* = \frac{1}{a_4 - a_3}\log(\frac{u_{30}}{u_{40}})$, $t_{24}^* = \frac{1}{a_4 - a_2}\log(\frac{u_{20}}{u_{40}})$ and $t_{14}^* = \frac{1}{a_4 - a_1}\log(\frac{u_{10}}{u_{40}})$ respectively and thereafter the dominance is reversed as shown in Fig. 8.

Case (viii): $If\ u_{20} < u_{40} < u_{30} < u_{10}, \quad a_1 < a_4 < a_3 < a_2$

In this case prey (S$_1$) has the least natural growth rate and the highest initial population strength. And the predator (S_2) has the highest natural growth rate and the least initial population strength. The host (S_3) of the prey (S_1) initially dominates over the host (S_4) of the predator (S_2) till the time instant $t_{43}^{*} = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}})$ and thereafter the dominance is reversed as shown in Fig. 9.

Trajectories of Perturbations

The trajectories in $u_1 - u_2,\ u_1 - u_3,\ u_1 - u_4,\ u_2 - u_3,\ u_2 - u_4,\ u_3 - u_4$ *planes are*

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1}, \left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_1},$$

$$\left(\frac{u_1}{u_{10}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_1},$$

$$\left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2}, \quad \left(\frac{u_2}{u_{20}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_2} \ \text{and}$$

$$\left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3}$$ respectively.

GRAPHS OF THE PERTURBATION

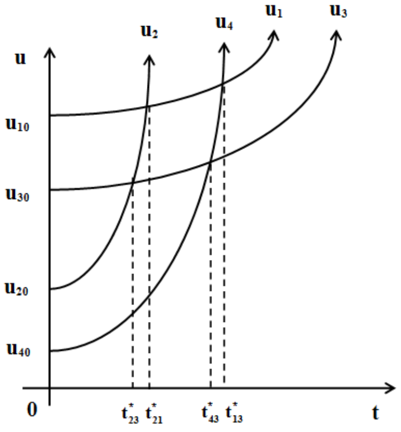

Fig. 2

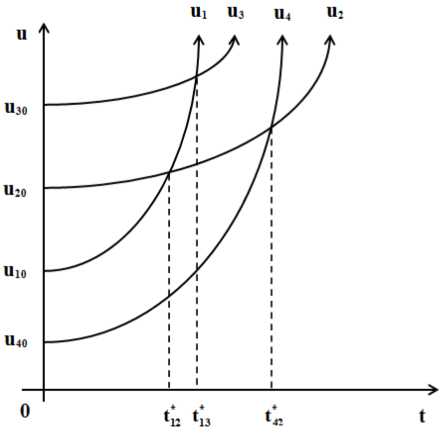

Fig. 3

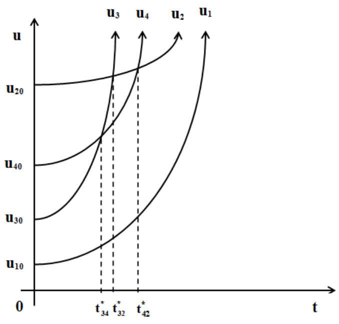

Fig. 4

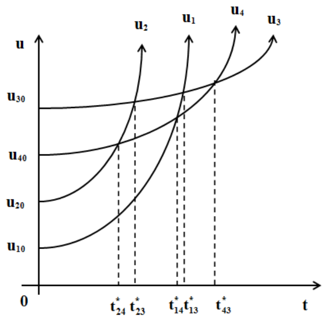

Fig. 5

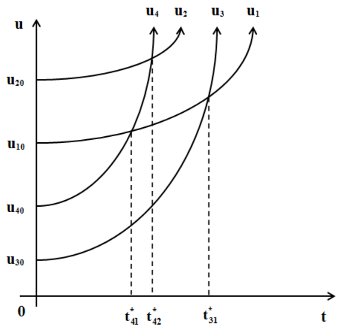

Fig. 6

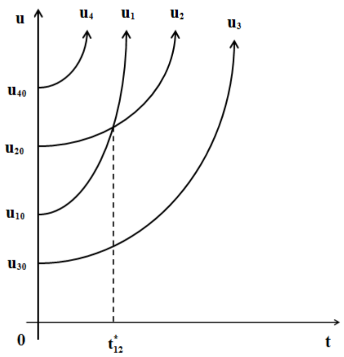

Fig. 7

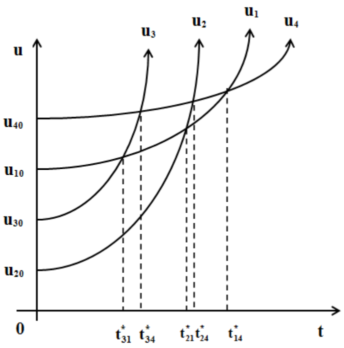

Fig. 8

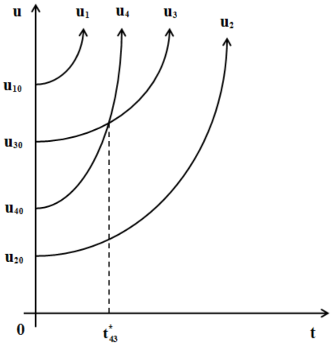

Fig. 9

ACKNOWLEDGEMENT

I would like to express deep sense of gratitude to my guide Prof. N. Ch. Pattabhi Ramachryulu garu for his invaluable guidance and support in my research work without whom my academic ambitions would not have been fulfilled.

REFERENCES

[1] Archana Reddy R., 2009, “On the stability of some Mathematical Models in Biosciences – Interacting Species,” Ph.D. thesis, J.N.T.U.

[2] Archana Reddy R., Pattabhi Ramacharyulu N. Ch., and Krishna Gandhi B., 2007, “A stability analysis of two competitive interacting species with harvesting of both the species at a constant rate,” *International journal of scientific computing* (1), p. 57-68.

[3] Bhaskara Rama Sharma B., 2009, “Some Mathematical Models in Competitive Eco- Systems,” Ph.D. thesis, Dravidian University.

[4] Hari Prasad B. and Pattabhi Ramacharyulu. N. Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Syn Eco-System-VI (Predator Washed out states),” *International eJournal of Mathematics and Engineering*, Volume 1, Issue III, p. 398-410.

[5] Kapur J.N., 1985, “Mathematical Modelling in Biology and Medicine,” affiliated *east west*.

[6] Kapur J.N., 1985, “Mathematical Modelling,” *Wiley Easter*.

[7] Kushing J.M., 1977, “Integro-Differential Equations and Delay Models in Population Dynamics, Lecture Notes in Bio-Mathematics,” *Springer Verlag*, 20.

[8] Lakshmi Narayan K., 2005, “A Mathematical Study of a Prey-Predator Ecological Model with a Partial Cover for the Prey and Alternate Food for the Predator,” Ph.D. thesis, J.N.T.U.

[9] Lakshmi Narayan K., and Pattabhi Ramacharyulu N. Ch., 2008, “A Prey-Predator model with cover for prey and alternate food for the predator, harvesting of both species,” Int. *J. Open Problems Compt. Math*., Vol.1¸ no.1.

[10] Lotka A.J., 1925, “Elements of Physical Biology,” *Williams & Wilking Baltimore*.

[11] Meyer W.J., 1985, “Concepts of Mathematical Modelling,” Mc. *Grawhill*.

[12] Phani Kumar N., Seshagiri Rao N., and Pattabhi Ramacharyulu N.Ch., 2009, “On the Stability of a Host – A Flourishing Commensal Species pair with Limited Resources,” *International Journal of Logic Based Intelligent Systems*, Vol.3, No.1.

[13] Shanker N., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System-I(Fully washed out state),” *International eJournal of Mathematics and Engineering*, Vol.1, Issue 3, p. 304-316.

[14] Shanker N., Lakshmi Narayan K., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System-II(Prey and Predator Washed out states),” *International Journal of Mathematical Sciences and Engineering Applications,* Vol. 4, No. V, p. 409-427.

[15] Shanker N., Lakshmi Narayan K., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System- III(Predator Washed out states),” *Global Journal of Pure and Applied Mathematics* Volume 3, Number 3(2011), p.245-255.

[16] Shanker N., Lakshmi Narayan K., and Pattabhi Ramacharyulu N.Ch., “On the stability of a four species: A Prey-Predator-Host-Commensal-Mutual-Syn Eco-System- IV(Prey Washed out state),” *International eJournal of Mathematics and Engineering*, Vol.2, Issue 3, p. 1082-1090.

[17] Srinivas N. C., 1991, “Some Mathematical Aspects of Modellingin Bio-medical Sciences,” Ph.D thesis, Kakatiya University.

[18] Volterra V., 1931, “Leconssen La Theorie Mathematique De La Leitte Pou Lavie,” *Gauthier-Villars*, Paris.