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Some threshold results and global stability of a host-mortal commensal ecological model with immigrated host species

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Abstract

The paper consists some threshold results on a Mathematical Model of a Host-Mortal Commensal Ecological Model with host is being harvested (Immigrated) at a constant rate by the identification of threshold regions through illustrations. Further, the global stability of this model with limited resources is established by Liapunov's stability criteria. It is elicited by constructing a suitable Liapunov's function for evaluating the global stability of the model in the case of co-existence equilibrium state. A pair of non-linear system of ordinary differential equations characterizes the model and the two equilibrium points are identified. Some threshold results are derived to establish the stability of the coexistent equilibrium state.

Keywords: Mortal Commensal-Immigrated Host Model, Equilibrium Points,Normal Steady State, Stability,Threshold Diagrams, Liapunov's function for global stability.

INTRODUCTION

In the field of Mathematical modeling apart from other branches of it Ecology occupies an important and prominent place. The word ecology is derived from Greek. It means "The interdisciplinary scientific study of interactions between organisms and their environment". It also studies the ecosystems. Ecosystems give us the information of the network about the relations among the organisms at different scales. Ecology can refer to any form of bio-diversity. Hence ecologists are capable of conducting research from the smallest bacteria to the global atmospheric gases. Ecology is a modern discipline of science, which came into existence in 19th century. It blossomed from natural science. Ecology is not synonymous to environment, environmentalism or environment science. The main scientific disciplines of physiology, evolution, genetics and behaviour are closely related to ecology. In short we can say that ecology is the study of living beings such as animals and plants in relation to their habits and habitats. It mainly deals with the evolutionary biology, which explains us about how the living beings are regulated in nature. Mathematical Modeling of ecosystems was initiated by Lotka [9] and Volterra [16]. The general concepts of Modeling have been presented in the treatises of Mayer [10], Kushing [6], Paul Colinvaux [11], Freedman [3], Kapur [4, 5] and several others. The Ecological Symbiosis can be broadly classified as Prey-Predator, Competition, Mutualism and Commensalism and so on. N.C.Srinivas [15] studied the Competitive Ecosystems of two species and three species with limited and unlimited recourses. Lakshminarayan and Pattabhi Ramacharyulu [7, 8] investigated Prey-Predator Ecological Models with a partial cover for the Prey and alternative food for the Predator and Prey-Predator Model with cover for Prey and alternate food for the Predator and time delay. The stability analysis of Competitive species were carried out by Archana Reddy, Pattabhi Ramacharyulu and Gandhi [1]; by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [2], while the Mutualism between two species was examined by Ravindra Reddy [13]. Recently Phanikumar, Seshagiri Rao and Pattabhi Ramacharyuly [12] studied on the stability of a Host- A flourishing Commensal species pair with limited resources. Seshagiri Rao, Phanikumar and Pattabhi Ramacharyuly [14] investigated on the stability of a Host- A decaying Commensal species pair with limited resources.

The present investigation is on an analytical study of some threshold results of two species Commensalism model where the host species is harvested (Immigrated) at a constant rate and also focuses on the establishment of the global stability of the co-existent equilibrium state of a Mathematical Model of Commensalism pair with limited resources and commensal species has Mortality rate by employing a property constructed by Liapunov's function with Liapunov's criteria for global stability.

Brief History of Liapunov's Stability Analysis

A.M. Liapunov initiated a meritorious method in 1892 to examine the global stability of equilibrium points in the case of linear and non-linear systems. This method yields stability information directly without solving the differential equations involved in the system. Hence it is also called Liapunov's direct method to detect the criteria for global stability. His method is based on the chief charactestic of constructing a scalar function called Liapunov's function. The stability behaviour of solutions of linear and non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighborhood of operating point i.e., local stability. Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. Hence, the stability behaviour of a physical system is curbed by these limitations. In this connection

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several authors like Lotka [15], Kapur [13], Pattabhi Ramacharyulu [1-11], Lakshminarayan [14] and Bhaska Raramasarma [12] etc. applied this method in various situations for global stability.

An outline sketch of Liapunov's Method for Global stability
Consider an autonomous system

$$\frac{dx}{dt} = F_1(x, y) \quad \text{and} \quad \frac{dy}{dt} = F_2(x, y) \tag{1}$$

Assume that this system has an isolated initial point taken as (0, 0). Consider a function E(x, y) possessing continuous partial derivatives along with the path of (1). This path is represented by C = [(x (t), y (t)] in the parametric form. E(x, y) can be regarded as a function of 't' along C with rate of change.

If the total energy of physical system has a local minimum at a certain equilibrium point then the point is said to be stable .Liapunov generalized this principle by constructing a function E (N1, N2) whose rate of change is given by

$$\frac{dE}{dt} = \frac{\partial E}{\partial N_1}\frac{\partial N_1}{\partial t} + \frac{\partial E}{\partial N_2}\frac{\partial N_2}{\partial t} \tag{2}$$

corresponding to the system.

(ii) Theorem (A): If there exists a Liapunov's function E (x,y) for the system (1), then the critical point (0,0) is stable. Further, if this function has additional property that the function (2) is negative definite, then the critical point (0,0) is asymptotically stable. (3)

The following theorem assists to ascertain the definiteness of a Liapunov's function.

(iii) Theorem (B): The function $E(x, y) = ax^2 + bxy + cy^2$ is positive definite if $a > 0$ and $b^2 - 4ac < 0$ and negative definite if $a < 0$, b2 – 4ac<0. (4)

Notations adopted

$N_1(t)$: The population of the Commensal species (S_1).
$N_2(t)$: The population of the Host species (S_2).
d_1 : The natural death rate of the Commensal species (S_1).
a_{11} : The rate of decrease of the commensal due to insufficient food.
a_2 : The natural growth rate of the Host Species.
a_{22} : The rate of decrease of the Host due to insufficient food.
$e_1 (= d_1 / a_{11})$: The Mortal coefficient of S_1.
$c (= a_{12} / a_{11})$:The coefficient of commensal.
$k_2 (= a_2 / a_{22})$: The carrying capacity of S_2.
$H_2 \left(= \frac{h_2}{a_{22}}\right)$: The constant harvesting (Immigration) rate of the Host Species.

The state variables $N_1(t)$ and $N_2(t)$ as well as all the model parameters d_1,a_2,a_{11},a_{22} ,a_{12}, H_2 are assumed to be non-negative constants.

Construction of Basic Equations of the Model

The equation for the growth rate of the Commensal species (S_1) with limited resources is given as

$$\frac{dN_1}{dt} = a_{11}N_1[-e_1 - N_1 + cN_2] \tag{5}$$

The equation for the growth rate of the Host species (S_2) with limited resources is given as

$$\frac{dN_2}{dt} = a_{22}\left[k_2 N_2 - N_2^{\,2} + H_2\right] \tag{6}$$

EQUILIBRIUM STATES

The system under investigation has the following two equilibrium points given by $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$. These are obtain as follows

E_1: Commensal washed out Equilibrium State

$$\overline{N_1} = 0 \,; \quad \overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2} \tag{7}$$

E_2: The Co-existent State

$$\overline{N_1} = c\left(\frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2}\right) - e_1 \,;$$

$$\overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2} \tag{8}$$

This state may also be called as the "Normal steady state" and this state exists only when $e_1 < c\left(\frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2}\right)$.

Threshold Diagrams (or) Phase- plane diagram

Gause's Priciple of competitive exclusion: "Two species cannot indefinitely co-exist in the same locality if there have identical ecological requirements".

In consonance with the above principle, we intend to derive threshold diagrams according to Gause's principle of competitive exclusion of this model.

The conditions $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$ imply that neither N_1 nor N_2 changes its density. When we impose these conditions the basic equations give rise to four straight lines N_1=0, $N_1 - cN_2 + e_1 = 0$, $N_2 = \frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2}$ and $N_2 = \frac{k_2 - \sqrt{k_2^{\,2} + 4H_2}}{2}$. At the points where $\frac{dN_1}{dt} = 0$; $\frac{dN_2}{dt} = 0$, the resulting straight lines divide the phase plane in to four regions in the first quadrant $N_1 \geq 0$, $N_2 \geq 0$.

Signs of N_1 and N_2 in the specific regions

Region I: The perturbations in both the species N_1 and N_2 flourish with time t.
Region II: The perturbations in commensal species N_1 declines and the perturbations in the host species N_2 flourishes with time t.
Region III: The perturbations in both the species N_1 and N_2 decline with time t.
Region IV: The perturbations in commensal species N_1 flourishes and the perturbations in host species N_2 declines with time t.

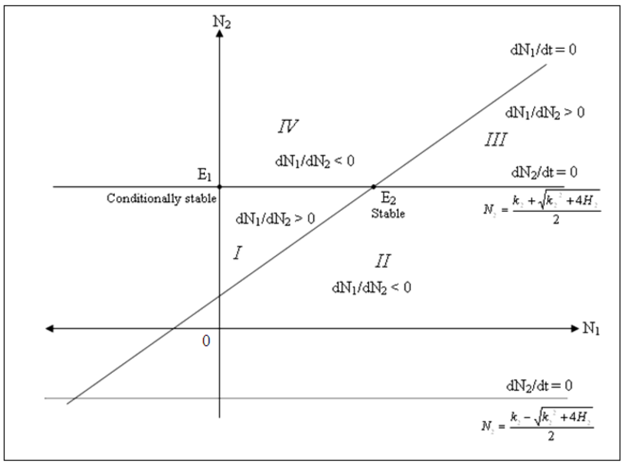

Fig 1.

For deriving threshold diagrams to this case, we need the following Lemmas.

Lemma(I) : The solution which starts in region I at time t = t_0 of $N_1(t)$ and $N_2(t)$ will remain in this region for all future time $t \geq t_0$ and ultimately tends towards equilibrium solution

$N_1(t) = \overline{N_1}$, $N_2(t) = \overline{N_2}$

Lemma (II) : The solution which starts in region II at time t = t_0 of $N_1(t)$ and $N_2(t)$ will remain in this region for all future time $t \geq t_0$ and ultimately approaches the equilibrium solution

$N_1(t) = \overline{N_1}$, $N_2(t) = \overline{N_2}$.

Lemma (III) : The solution which starts in region - III at time t = t_0 of $N_1(t)$ and $N_2(t)$ will remain in this region for all future time $t \geq t_0$ and ultimately reaches the equilibrium solution

$N_1(t) = \overline{N_1}$, $N_2(t) = \overline{N_2}$.

Lemma (IV) : The solution of $N_1(t)$ and $N_2(t)$ which starts in the region IV for all future time $t \geq t_0$ and ultimately approaches the equilibrium solution

$N_1(t) = \overline{N_1}$, $N_2(t) = \overline{N_2}$.

By the concept of Threshold diagrams and above Lemmas, we can conclude that every solution $N_1(t)$, $N_2(t)$ of (1) and (2) starts in region I, II, III and IV at time $t = t_0$ and remains there for all future time and finally they approach equilibrium solution

$N_1(t) = \overline{N_1}$, $N_2(t) = \overline{N_2}$ as 't' approaches infinity.

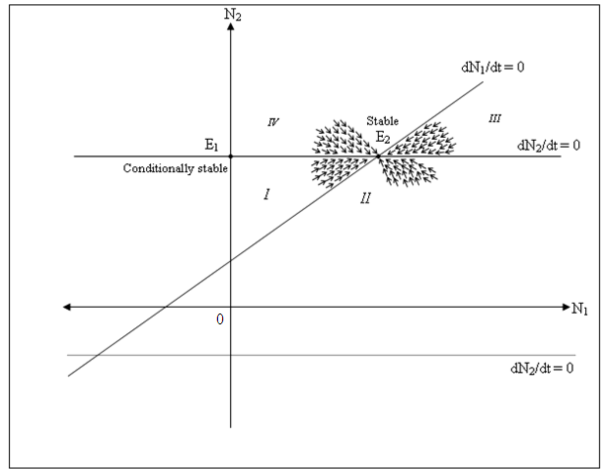

Fig 2.

CONCLUSION

By observing all four regions as shown in the above figure, all the solutions, which start in respective regions, finally approach the equilibrium solution E_2 (e_1,e_2). It concludes that the state corresponding to the equilibrium point is stable.

Liapunov's Function for Global Stability

We have observed that the equilibrium point E_2 is locally stable. We now examine the global stability of the dynamical system (5) and (6). We have already noted that this system has a unique, stable non-trivial co-existent equilibrium state at

$$\overline{N_1} = c\left(\frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2}\right) - e_1 \quad ;$$

$$\overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} + 4H_2}}{2} .$$

Basic Equations

$$\frac{dN_1}{dt} = a_{11}N_1\left[-e_1 - N_1 + cN_2\right] \qquad (9)$$

$$\frac{dN_2}{dt} = a_{22}\left[k_2N_2 - N_2^{\,2} + H_2\right] \qquad (10)$$

The linearized perturbed equations over the perturbations (u_1, u_2) are

$$\frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 \qquad (11)$$

$$\frac{du_2}{dt} = -2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]u_2 \qquad (12)$$

The corresponding characteristic equation is

$$\left(\lambda + a_{11}\overline{N_1}\right)\left(\lambda + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) = 0 \qquad (13)$$

$$\lambda^2 + \left[a_{11}\overline{N_1} + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right]\lambda + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right] = 0 \qquad (14)$$

Equation (14) is of the form $\lambda^2 + p\lambda + q = 0$
where

$$p = a_{11}\overline{N_1} + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right] > 0 \qquad (15)$$

$$q = 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right] > 0 \qquad (16)$$

$\therefore$ The conditions for the existence of Liapunov's function are satisfied.
Now define

$$E\left(u_1, u_2\right) = \frac{1}{2}\left(Au_1^{\,2} + 2Bu_1u_2 + Cu_2^{\,2}\right) \qquad (17)$$

where

$$A = \frac{\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \qquad (18)$$

$$B = \frac{2ca_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \qquad (19)$$

$$C = \frac{a_{11}^{\,2}\overline{N_1}^{\,2} + \left(ca_{11}\overline{N_1}\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \qquad (20)$$

$$D = pq > 0 \qquad (21)$$

From the equations (15) and (16) it is clear that D>0.

Also

$$D^2\left(AC - B^2\right) = \left\{\left(\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)\left(a_{11}^{\,2}\overline{N_1}^{\,2} + \left(ca_{11}\overline{N_1}\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) - \left(2ca_{11}a_{22}\overline{N_1}\right)^2\left[\overline{N_2} - \frac{k_2}{2}\right]^2\right\} \qquad (22)$$

$\therefore$ The function E (u_1, u_2) at (17) is positive definite.
Further

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = \left(Au_1 + Bu_2\right)\left(-a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2\right) + \left(Bu_1 + Cu_2\right)\left(-2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]u_2\right) \qquad (23)$$

Substituting the values of A, B and C from (18) (19) and (20) in (23) we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\frac{1}{D}\left[a_{11}\overline{N_1}\left[\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right]\right]u_1^2$$

$$+\frac{1}{D}\left[ca_{11}\overline{N_1}\left\{\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right\} - a_{11}c\overline{N_1}\left(2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) - 4ca_{11}a_{22}^{\,2}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]^2\right]u_1u_2$$

$$-\frac{1}{D}\left\{ca_{11}\overline{N_1}\left(2ca_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) - 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\left[\left(a_{11}\overline{N_1}\right)^2 + \left(ca_{11}\overline{N_1}\right)^2 + \left(2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)\right]\right\}u_2^{\,2}$$

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\frac{1}{D}\left[Du_1^2 + Du_2^2\right] \qquad (24)$$

$$= -\left(u_1^2 + u_2^2\right) \qquad (25)$$

$$\therefore \frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\left(u_1^2 + u_2^2\right) \qquad (26)$$

which is clearly negative definite. So E (u_1, u_2) is a Liapunov's function for the linear system.

Next we prove that E (u_1, u_2) is also a Liapunov's function for the non-linear system.

Let f_1 and f_2 be two functions of N_1 and N_2 defined by

$$f_1\left(N_1, N_2\right) = a_{11}N_1\left[-e_1 - N_1 + cN_2\right] \qquad (27)$$

$$f_2\left(N_1, N_2\right) = a_{22}\left(k_2N_2 - N_2^{\,2} + H_2\right) \qquad (28)$$

Now, we have to show that $\frac{\partial E}{\partial u_1}f_1 + \frac{\partial E}{\partial u_2}f_2$ is negative definite.

By putting $N_1 = \overline{N_1} + u_1$; $N_2 = \overline{N_2} + u_2$ in (9) and (10), we get

$$f_1\left(u_1, u_2\right) = \frac{du_1}{dt} = a_{11}\left(\overline{N_1} + u_1\right)\left[-e_1 - \left(\overline{N_1} + u_1\right) + c\left(\overline{N_2} + u_2\right)\right]$$

$$= -a_{11}e_1\overline{N_1} - a_{11}\overline{N_1}^2 - a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}\,\overline{N_2} + ca_{11}\overline{N_1}u_2 - e_1a_{11}u_1 - a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_2}u_1 + ca_{11}u_1u_2 - a_{11}u_1^2$$

$$= a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 + u_1a_{11}\left(-e_1 - \overline{N_1} + c\overline{N_2}\right) - a_{11}u_1^2 + ca_{11}u_1u_2$$

$$\Rightarrow f_1\left(u_1, u_2\right) = \frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 + F\left(u_1, u_2\right) \qquad (29)$$

where $F\left(u_1, u_2\right) = -a_{11}u_1^{\,2} + ca_{11}u_1u_2$ (30)
Also

$$f_2\left(u_1, u_2\right) = \frac{du_2}{dt} = a_{22}\left(k_2\left(\overline{N_2} + u_2\right) - \left(\overline{N_2} + u_2\right)^2 + H_2\right)$$

$$\Rightarrow f_2\left(u_1, u_2\right) = \frac{du_2}{dt} = -2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]u_2 + G\left(u_1, u_2\right) \qquad (31)$$

where $G\left(u_1, u_2\right) = -a_{22}u_2^{\,2}$ (32)

From (17)

$$\frac{\partial E}{\partial u_1} = Au_1 + Bu_2 \qquad (33)$$

$$\frac{\partial E}{\partial u_2} = Bu_1 + Cu_2 \qquad (34)$$

Now

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = \left(Au_1 + Bu_2\right)\left(-a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 + F(u_1,u_2)\right)$$

$$+ \left(Bu_1 + Cu_2\right)\left(-2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]u_2 + G(u_1,u_2)\right) \qquad (35)$$

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -(u_1^2 + u_2^2) + \left(Au_1 + Bu_2\right)F(u_1,u_2) + \left(Bu_1 + Cu_2\right)G(u_1,u_2) \qquad (36)$$

Introducing polar co-ordinates $u_1 = r\cos\theta$, $u_2 = r\sin\theta$, the equation (6.28) can be written as

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -r^2 + r[(A\cos\theta + B\sin\theta)F(u_1,u_2) + (B\cos\theta + C\sin\theta)G(u_1,u_2)] \qquad (37)$$

Let us denote the largest of the numbers $|A|$, $|B|$ and $|C|$ by K. Our assumptions imply that $|F(u_1,u_2)| < \frac{r}{6K}$ and $|G(u_1,u_2)| < \frac{r}{6K}$ for all sufficiently small r > 0.

So,

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 < -r^2 + \frac{4Kr^2}{6K} = -\frac{r^2}{3} < 0 \qquad (38)$$

Thus E(u_1,u_2) is a positive definite function with the condition that $\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite.

$\therefore$ The equilibrium state E_2 is "globally stable".

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