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Global stability of a commensal- host ecological model with limited resources and both are harvesting at a constant rate

N. Phani Kumar

Faculty in Mathematics, Departmental Humanities & Sciences, Malla Reddy Engineering College, Secunderabad, India

Abstract
In this paper we establish the global stability of a commensalism-host ecological model surviving with limited resources and both are harvesting at a constant rate, by constructing a suitable Liapunov's function in case of co-existent equilibrium state.
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Keywords: Equilibrium state, Liapunov's function, Global stability.

INTRODUCTION

The present investigation is mainly devoted to establish the global stability of the co-existent equilibrium state of the commensal-host model by employing a properly constructed Liapunov's function.

Phani Kumar and N.Ch.Pattabhi Ramacharyulu etc. [9] examined the local stability of a commensal -host ecological model surviving with limited resources and both are harvesting at a constant rate, on the quasi-linear basic balancing equations. The present authors has been also discussed the local stability analysis for a host-commensal eco-system in their earlier work [6, 7, 8, 11].

LIAPUNOV'S STABILITY ANALYSIS

Many approaches are available for the stability analysis of linear,time-invarient systems. However for non-linear systems and/or time-varying systems, stability analysis may be extremely difficult or impossible. Liapunov Stability analysis is one method that may be applied for non-linear systems. In 1892 A.M. Liapunov introduced the direct method to study the global stability of equilibrium states in case of linear and non-linear systems. His method is based on the chief characteristic of constructing a scalar function called Liapunov's function. That is by using the direct method of Liapunov,we can determine the stability of a system without solving the state. This is quite advantageous because solving non-linear and/or time-invarying state equation is very difficult. To day this method is widely recognized as an efficient tool in theory of control systems, dynamical systems, systems with time lag, power system analysis, and time varying non-linear feed back systems, multi species ecological systems and so on.

The stability behavior of solutions of linear and weakly non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighborhood of operating point i.e., local stability.

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*Corresponding Author
N. Phani Kumar
Faculty in Mathematics, Departmental Humanities & Sciences, Malla Reddy Engineering College, Secunderabad, India

Tel: +91-9866102086
Email: phanikumar_nandanaranam@yahoo.com

Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. The stability behavior of a physical system is discussed by several authors like Kapoor [2], Lotka [3], Ogata [4] Bhaskara Rama Sarma and N.Ch.PattabhiRamacharyulu [1], Lakshminarayan and N.Ch.PattabhiRamacharyulu [5], Phanikumar and N.Ch .Pattabhi Ramacharyulu[10]etc

If the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable. This idea was generalized by Liapunov to study stability problems in a broader context.

STABILITY BY LIAPUNOV'S DIRECT METHOD

Consider an autonomous system

$$\left.\begin{aligned}\frac{dx}{dt} &= F(x,y)\\ \frac{dy}{dt} &= G(x,y)\end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point taken as (0, 0). Consider a function E(x, y) possessing continuous partial derivatives along the path of (1). This path is represented by C= [(x (t), y (t)] in the parametric form. E(x, y) can be regarded as a function of't' along C with rate of change

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}F(x,y) + \frac{\partial E}{\partial y}G(x,y) \qquad (2)$$

BASIC EQUATIONS OF THE MODEL

The basic equations for the growth rate of a flourishing commensal and host species with limited resources are given by

$$\left.\begin{aligned} \frac{dN_1}{dt} &= a_1N_1 - a_{11}N_1^{\,2} + a_{12}N_1N_2 - a_{11}H_1 \\ \frac{dN_2}{dt} &= a_2N_2 - a_{22}N_2^{\,2} - a_{22}H_2 \end{aligned}\right\} \tag{3}$$

THE EQULIBRIUM STATES

The system (3) under investigation has nine equilibrium states given by $\frac{dN}{dt}$=0 and these are classified into two categories A and B.

When the harvesting rates are interdependent

When $H_1 < \frac{1}{4}\left[K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right]^2;\ H_2 < \frac{K_2^{\,2}}{4}$

E₁ : $\overline{N_1} = \left(K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right) - \frac{H_1}{K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)}$; $\overline{N_2} = K_2 - \frac{H_2}{K_2}$

E₂ : $\overline{N_1} = \frac{H_1}{K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)}$; $\overline{N_2} = K_2 - \frac{H_2}{K_2}$

The above two states exists only when

$K_2^{\,2} > H_2$ and $\left[K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right]^2 > H_1$

When the harvesting rates are not interdependent

(B.1) When $H_1 > \frac{1}{4}\left[K_1 + \frac{3CK_2}{4}\right]^2;\ H_2 < \frac{K_2^{\,2}}{4}$

E₃ : $\overline{N_1} = \frac{K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)}{2}$; $\overline{N_2} = K_2 - \frac{H_2}{K_2}$,

this would exists only when $K_2^{\,2} > H_2$

(B.2) When $H_1 < \frac{1}{4}\left[K_1 + \frac{CK_2}{4}\right]^2;\ H_2 < \frac{K_2^{2}}{4}$

E₄ : $\overline{N_1} = \left(K_1 + \frac{CH_2}{K_2}\right) - \frac{H_1}{K_1 + \frac{CH_2}{K_2}}$; $\overline{N_2} = \frac{H_2}{K_2}$

this would exists only when $\left(K_1 + \frac{CH_2}{K_2}\right)^2 > H_1$

E₅ : $\overline{N_1} = \frac{H_1}{K_1 + \frac{CH_2}{K_2}}$; $\overline{N_2} = \frac{H_2}{K_2}$

E₆ : $\overline{N_1} = \frac{K_1 + \frac{CH_2}{K_2}}{2}$; $\overline{N_2} = \frac{H_2}{K_2}$

(B.3) When $H_1 < \frac{1}{4}\left[K_1 + \frac{CK_2}{2}\right]^2;\ H_2 = \frac{K_2^{2}}{4}$

E₇ : $\overline{N_1} = \left(K_1 + \frac{CK_2}{2}\right) - \frac{H_1}{K_1 + \frac{CK_2}{2}}$; $\overline{N_2} = \frac{K_2}{2}$

this would exists only when $\left(K_1 + \frac{CK_2}{2}\right)^2 > H_1$

E₈ : $\overline{N_1} = \frac{H_1}{K_1 + \frac{CK_2}{2}}$; $\overline{N_2} = \frac{K_2}{2}$

(B.4)When $H_1 = \frac{1}{4}\left[K_1 + \frac{CK_2}{2}\right]^2;\ H_2 = \frac{K_2^{2}}{4}$

E₉ : $\overline{N_1} = \frac{1}{2}\left[K_1 + \frac{CK_2}{2}\right]$; $\overline{N_2} = \frac{K_2}{2}$

where

Ki= $\frac{a_i}{a_{ii}}$, i = 1, 2 are the carrying capacities of N_i.

C= $\frac{a_{12}}{a_{11}}$ Commensal coefficient

$h_1 = a_{11} H_1$ is rate of harvest of the commensal

$h_2 = a_{22} H_2$ is rate of harvest of the host.

LOCAL STABILITY ANALYSIS

The present authors [9] discussed the local stability of the above nine equilibrium states. From which the equilibrium point

E₁ : $\overline{N_1} = \left(K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)\right) - \frac{H_1}{K_1 + C\left(K_2 - \frac{H_2}{K_2}\right)}$;

$\overline{N_2} = K_2 - \frac{H_2}{K_2}$ is stable and rest of them are unstable

LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

The linearized perturbed equations over the perturbations (u_1, u_2) of the system (3) are

$$\frac{du_1}{dt} = -a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)u_1 + Ca_{11}\overline{N_1}u_2 \tag{4}$$

$$\frac{du_2}{dt} = -a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)u_2 \qquad (5)$$

The characteristic equation is

$$\left(\lambda + a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\right)\left(\lambda + a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\right) = 0$$

i.e.,

$$\lambda^2 + \left[a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]\lambda + a_{22}a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N_2} - \frac{H_2}{K_2}\right) = 0 \qquad (6)$$

equation (6) is in the form of $\lambda^2 + p\lambda + q = 0$
where

$$p = a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right) \qquad (7)$$

$$q = a_{22}a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N_2} - \frac{H_2}{K_2}\right) \qquad (8)$$

By using

$$\left[a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]^2 > 4a_{22}a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N_2} - \frac{H_2}{K_2}\right),$$

we have p, q > 0.

$\therefore$ Liapunov's conditions are satisfied.

Define E (u_1, u_2) = ½ ($au_1^2 + 2b\,u_1u_2 + cu_2^2$)

Where
$$a = \frac{a_{22}^2\left(\overline{N_2} - \frac{H_2}{K_2}\right)^2 + a_{11}a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)}{D} \qquad (9)$$

$$b = \frac{Ca_{11}a_{22}\overline{N_1}\left(\overline{N_2} - \frac{H_2}{K_2}\right)}{D} \qquad (10)$$

$$c = \frac{a_{11}^2\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)^2 + C^2a_{11}^2\overline{N_1}^2 + a_{11}a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)}{D} \qquad (11)$$

Where

$$D = pq = \left[a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]\left[a_{22}a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N_2} - \frac{H_2}{K_2}\right)\right] \qquad (12)$$

Now

D^2 (ac-b^2) =

$$D^2\left\{\left(\frac{a_{22}^2\left(\overline{N_2} - \frac{H_2}{K_2}\right)^2 + a_{11}a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)}{D}\right)\left(\frac{a_{11}^2\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)^2 + C^2a_{11}^2\overline{N_1}^2 + a_{11}a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)}{D}\right) - \left(\frac{Ca_{11}a_{22}\overline{N_1}\left(\overline{N_2} - \frac{H_2}{K_2}\right)}{D}\right)^2\right\}$$

$\Rightarrow$ D^2 (ac – b^2)> 0 i.e., b^2-ac < 0 (13)

$\therefore$ The function E (u_1, u_2) is positive definite.

Further

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} =$$

$$(au_1 + bu_2)\left(-a_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right)u_1 + Ca_{11}\overline{N_1}u_2\right) + (bu_1 + cu_2)\left(-a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)u_2\right)$$

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} =$$

$$\left(aCa_{11}\overline{N_1} - ba_{11}\left(\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right) - ba_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\right)u_1u_2 + \left(bCa_{11}\overline{N_1} - ca_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\right)u_2^2 \qquad (14)$$

Substituting the values of a, band c from (9), (10) and (11) in (14) we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\left(\frac{D}{D}u_1^2 + \frac{D}{D}u_2^2\right)$$

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\left(u_1^2 + u_2^2\right) \qquad (15)$$

which is clearly negative definite

$\therefore$ E (u_1, u_2) is Liapunov function for the linear system.

Now we prove that E (u_1, u_2) is also a Liapunov function for the non-linear system

Define F (N_1, N_2) = $a_1N_1 - a_{11}N_1^2 + a_{12}N_1N_2 - a_{11}H_1$ (16)

G (N_1, N_2) = $a_2N_2 - a_{22}N_2^2 - a_{22}H_2$ (17)

By putting N_1 = $\overline{N_1}$ + u_1 and N_2 = $\overline{N_2}$ + u_2 in (4) and (5)

$$F(u_1, u_2) = \frac{du_1}{dt} = -a_{11}\left[\overline{N_1} - \frac{H_1}{K_1 + C\overline{N_2}}\right]u_1 + Ca_{11}\overline{N_1}u_2\, u_2 + f(u_1, u_2)$$

$$G(u_1, u_2) = \frac{du_2}{dt} = -a_{22}\left[\overline{N_2} - \frac{H_2}{K_2}\right]u_2 + g(u_1, u_2)$$

Where

f (u_1, u_2) = $-a_{11}u_1^2 + a_{12}u_1u_2$

g (u_1, u_2) = - $a_{22}u_2^2$

Now

$$\frac{\partial E}{\partial u_1}F + \frac{\partial E}{\partial u_2}G = -(u_1^2 + u_2^2) + (au_1 + bu_2)f(u_1, u_2) + (bu_1 + cu_2)g(u_1, u_2) \qquad (18)$$

By introducing polar co-ordinates,

$$\frac{\partial E}{\partial u_1}F + \frac{\partial E}{\partial u_2}G = -r^2 + r[(a\cos\theta + b\sin\theta)f(u_1, u_2) + (b\cos\theta + c\sin\theta)g(u_1, u_2) \qquad (19)$$

Let us denote largest of the numbers | a|, |b|, |c| by K, our assumptions imply that

| f (u_1, u_2)| < $\frac{r}{6K}$ and | g (u_1, u_2)| < $\frac{r}{6K}$,for all satisfying small r > 0.
So

$$\frac{\partial E}{\partial u_1} F + \frac{\partial E}{\partial u_2} G < -r^2 + \frac{4Kr^2}{6K} = - \frac{r^2}{3} < 0$$

Thus E (u_1, u_2) is a positive definite function with the property that

$$\frac{\partial E}{\partial u_1} F + \frac{\partial E}{\partial u_2} G$$ is negative definite.

$\therefore$ The equilibrium point is asymptotically stable.

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