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Almost η-duals of some sequence spaces

K. B. Gupta[1*] and Ashfaque A. Ansari[2]

[1]Department of Mathematics, St. Andrew's College, Gorakhpur-273001, India.
[2]Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur – 273009, India.

Abstract
Ansari and Shukla [1] have generalized the notion of α-duals and developed the concept of almost η–duals by using the concept of absolutely almost convergence. P Chandra and B.C. Tripathy [8] have introduced the concept of η-duals. We have introduced the concept of almost η-duals and determined almost η-duals of some sequence spaces.

Keywords: Almost η-dual, c_0-space, c-space, l_∞space, $l_\infty(p)$-space, -space, (p)-space, -space. Almost α-dual.

INTRODUCTION

After Lorentz [6] introduced the concept of almost convergence. Das, Kuttener and Nanda [3] have developed the concept of absolutely almost convergence of sequence space. α and β-duals were defined by Kothe-Toeplitz [4] in scalar form which was later generalized in operator version [7]. Using the concept of almost α-duals developed by Ansari & Shukla [1] and the concept of η-duals developed by P. Chandra and B.C. Tripathy [7], we have developed and determined the almost η-duals of some sequence space in this paper.

Some definitions and Relations

The idea of the dual sequence spaces was introduced by köthe and Toeplitz [4] whose main results concerned with α-duals; the α-dual of E ⊂ w [where w is the linear space of all complex sequences and E denote a set (ora space) of complex sequences] is defined as

$E^{\alpha} = \{a = (a^k) \in w : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for all } x = (x^k) \in E\}$. Let c^o, c and l^∞ be the Banach spaces of null, convergent and bounded sequences $x = (x^k)$ respectively with $\|x\|^\infty = \sup_k |x^k|$. Let D be the shift operator on l^∞ i.e. $D(x^n) = x^{n+1}$. It is proved that the α-duals of c^o, c and l^∞, respectively being denoted by , c^{α} and which are equal to l^1 (where l^1 is the space of absolutely convergent series) and the almost α-duals of c^0, c and l^∞ is $\hat{l}_r$ which is the space of absolutely almost convergent series. In our case, the almost η-duals of c^o, c and l^∞ is $\hat{l}_r$ (for $0 < r \le 1$) which is the space of r-absolutely almost summable sequences. This is a natural extension of $\hat{l}_r$.

Again, the Banach limit L [2] is a non negative linear functional on l^∞ such that L is invariant under the shift operator i.e. $L(D(x)) = L(x)$, $x \in l^\infty$ and $L(e) = 1$ where $e = (1, 1, 1, \ldots\ldots)$. Lorentz [6] has defined a sequence $x \in l^\infty$ to be almost convergent if all the Banach limits of x coincide. Let denote the set of all almost convergent sequences. Lorentz [6] proved that $\hat{c}$ $\{x = (x^k) : \lim_{k\to\infty}\left(\frac{1}{k+1}\right)\sum_{i=0}^{k} x_{n+i}$ exists uniformely in $n\}$. Das, Kuttener and Nanda [3] have introduced the concept of absolutely almost convergent series.

Let $a = \sum_{i=0}^{\infty} a_i$ be an infinite series of complex numbers and be its sequence of partial sums i.e.

$x^n = a^0 + a^1 + a^2 + \ldots\ldots\ldots + a^n$.

Define

$d^{k,n}$ as

$$d^{k,n} = d^{k,n}(x) = \frac{1}{k+1}\sum_{i=0}^{k} x_{n+i} \quad (k > 0, n \ge 0).$$

By taking $d^{0,n} = d^{0,n}(x) = x^{n-1}$

We then write for $k, n \ge 0$

$\phi^{k,n} = \phi^{k,n}(a) = d^{k+1,n} - d^{k,n}$

then $\phi^{0,n} = a^n$

and $\phi^{k,n} = \frac{1}{k(k+1)}\sum_{i=1}^{k} i \; a^{n+i} (k \ge 1)$

Then the series $a = \sum_{i=1}^{\infty} a_i$ (or the sequence $x = (x^n)$) is said to be absolutely almost convergent series, if $\sum_{k=1}^{\infty} |\phi_{k,n}|$ converges uniformely in n.

We write $\hat{l}_1$ to denote the set of all absolutely almost convergent series.

Also, the sequence $x = (x^n)$ is said to be r-absolutely almost summable sequence if $\sum_{k=1}^{\infty} |\phi_{k,n}|^r$ converges uniformely in n (where 0

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*Corresponding Author
K. B. Gupta
Department of Mathematics, St. Andrew's College, Gorakhpur-273001, India.

Tel: +91-9415500371
Email: dr.kbgupta77@gmail.com

< r ≤ 1). We write $\hat{l}_r$ to denote the set of all p-absolutely almost summable sequence.

Using the concept of absolutely almost convergent series. Ansari and Shukla [1] has introduced the concept of almost ‑duals as,

If E is a set (or a space) of sequences of complex numbers, the almost η–-dual of E is denoted by and is defined as

$$= \{a = (a^k) \in w : \sum_{k=1}^{\infty}\frac{1}{k(k+1)}\left|\sum_{i=1}^{k} i.a_{n+i}x_{n+i}\right|$$

< ∞ , $(x^k) \in$ E uniformely in n}.

Using the concept of almost η–-duals introduced by Ansari & Shukla [1] and the concept of η-duals developed by P. Chandra and Tripathy [8], we introduced the new concept of almost η-dual. Thus if E is a set or (a space) of sequence of complex numbers and $0 < r \le 1$; then the almost η-dual of E denoted by $E^{\hat{\eta}}$, is defined as

$$E^{\hat{\eta}} = \{a = (a^k) : \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} ia_{n+i}x_{n+i}\right|^r < \infty, \quad (x^k) \in$$

E uniformely in n}.

Theorem : 1. The almost ‑duals of Null, convergent and bounded sequence are $c_0^{\hat{\eta}} = c^{\hat{\eta}} = l_\infty^{\hat{\eta}} = \hat{l}_r ; \forall \ 0 < r \le 1$.

Where

$$\hat{l}_r = \{a = (a^k) : \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} ia_{n+i}\right|^r < \infty, \text{ uniformely for}$$

each n, where $0 < r \le 1$}

Proof : Since $c^0 \subset c \subset l^\infty$

$\Rightarrow l_\infty^{\hat{\eta}} \subset c^{\hat{\eta}} \subset c_0^{\hat{\eta}}$.

Therefore, we show that

(i) $l_\infty^{\hat{\eta}} = \hat{l}_r$

and (ii) It is sufficient to show that $c_0^{\hat{\eta}} \subset \hat{l}_r$, so that the theorem is complete.

(i) Let $(x^k) \in l^\infty$ and $(a^k) \in \hat{l}_r$ i.e. (a^k) be a sequence of complex numbers such that $\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i}\right|^r < \infty$ uniformely in n.

Let $y^n = \sum_{k=1}^{n} a_k x_k$, $d^{k,n} = \frac{1}{k+1}\sum_{i=0}^{k} y_{n+i}$ $(\boldsymbol{k} > 0, n \ge 0)$

and $\phi^{k,n} = \frac{1}{k(k+1)}\sum_{i=1}^{k} i a_{n+i} x_{n+i}$ $(k \ge 1)$

Then,

$$\sum_{k=1}^{\infty}|\phi_{k.n}|^r = \sum_{k=1}^{\infty}\left|\frac{1}{k(k+1)}\sum_{i=1}^{k} a_{n+i} x_{n+i}\right|^r$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i} x_{n+i}\right|^r$$

$$\le \sup_{1\le i\le\infty}|x_{n+i}|^r \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i}\right|^r$$

$$= M^r \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i}\right|^r$$

(where $M = \sup_{i\ge1}|x_{n+i}| < \infty$)

< ∞ for all n.

Thus,

$\sum_{k=1}^{\infty}|\phi_{k,n}|^r < \infty$ for all n.

Hence $(a^k) \in l_\infty^{\hat{\eta}}$

$\Rightarrow \hat{l}_r \subset l_\infty^{\hat{\eta}}$

conversely, suppose that (a^k) be a sequence of complex numbers such that $(a^k) \in l_\infty^{\hat{\eta}}$ but $(a^k) \notin \hat{l}_r$ i.e. $\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i}\right|^r = \infty$ for some n.

Define a sequence $x = (x^k)$ where $x^k = \text{sgn } a^k$, for every k, Then. $(x^k) \in l^\infty$, putting $x^{n+i} = \text{sgn } a^{n+i}$

We get

$$\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i} x_{n+i}\right|^r$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^r|k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i} \,\text{sgn}\, a_{n+i}\right|^r$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} |a_{n+i}|\right|^r = \infty$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i.a_{n+i} x_{n+i}\right|^r = \infty \text{ where } (x^k) \in l^\infty$$

which is a contradiction that $(a^k) \in l_\infty^{\hat{\eta}}$

$\Rightarrow l_\infty^{\hat{\eta}} \subset \hat{l}_r$

Thus,

$l_\infty^{\hat{\eta}} \subset \hat{l}_r$.

(ii)Now, it is sufficient to show that

$c_0^{\hat{\eta}} \subset \hat{l}_r$.

Let (a^k) be a sequence of complex numbers such that

$(a^k) \in c_0^{\hat{\eta}}$ but $(a^k) \notin \hat{l}_r$

i.e. $\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i a_{n+i}\right|^r = \infty$ for some n.

$$\text{sinc}\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i.a_{n+i}\right|^r \le \sum_{k=1}^{\infty}\left|\sum_{i=1}^{k} a_{n+i}\right|^r = \left|\sum_{i=1}^{\infty} a_{n+i}\right|^r \left\{\because \frac{i}{k(k+1)} < 1\right\}$$

$\Rightarrow \left|\sum_{i=1}^{\infty} a_{n+i}\right|^r = \infty$, for some n.

$\Rightarrow \left|\sum_{i=1}^{\infty} a_{n+i}\right| = \infty$, for some n.

$\Rightarrow \left|\sum_{i=1}^{\infty} a_{n+i}\right| \le \sum_{i=1}^{\infty}|a_{n+i}| = \infty$.

Define $x = (x^k)$ such that $x^k = 0$, .

$= \frac{\text{sgn } a_{n+i}}{i}$,

Then $(x^k) \in c^0$. But

$$\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}\,x_{n+i}\right|^r = \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k}|\,a_{n=i}\,|\right|^r.$$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)}\sum_{i=1}^{k}|\,a_{n+i}\,|^r\right)^r$$

$$= \sum_{k=1}^{\infty}\left(\left(\frac{1}{k}-\frac{1}{k+1}\right)s_k\right)^r \quad \text{where } s^k = \sum_{i=1}^{k}|\,a_{n+i}\,|$$

since $\sum_{i=1}^{\infty}|\,a_{n+i}\,|$ diverges therefore, $\sum_{k=1}^{\infty}\frac{s_k}{k}$ and

$\sum_{k=1}^{\infty}\frac{s_k}{k+1}$ also diverges

$\Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)s_k$ diverges.

$\Rightarrow \sum_{k=1}^{\infty}\left[\left(\frac{1}{k}-\frac{1}{k+1}\right)s_k\right]^r$ diverges. where $0 \le r < 1$.

Therefore the series

$\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}\,x_{n+i}\right|^r = \sum_{k=1}^{\infty}\left|\left(\frac{1}{k}-\frac{1}{k+1}\right)s_k\right|^r$ does not converse.

which is contradiction to the fact that $(a^k) \in c_0^{\hat{\eta}}$.

Therefore, $c_0^{\hat{\eta}} \subset \hat{l}_r$

Hence $c_0^{\hat{\eta}} \subset \hat{l}_r$

This completes that proof.

ALMOST η-DUALS OF GENERALIZED BOUNDED SEQUENCE

Let l^∞ be the space of bounded complex sequence and $p = (p^k)$ denote a strically positive numbers. Lascarides and Maddox [5] have defined the sequence space.

$$l^\infty(p) = \{x = (x^k) : \sup_{k\ge1}|\,x_k\,|^{p_k} < \infty\}$$

In this section, we have determined almost -dual of generalized bounded sequence of scalars.

Theorem : 2. Let $p^k > 0$, for every k, then

$[l_\infty(p)]^{\hat{\eta}} = \hat{l}_r$ (p), where

$\hat{l}_r$ (p) =

$$\bigcap_{N=2}^{\infty}\left\{a=(a_k):\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}N^{\frac{1}{p_{n+i}}}\right|^r < \infty \,\text{uniformly in } n\right\}$$

Proof : Let $a = (a^k) \in \hat{l}_r$ (p) and $x = (x^k) \in l^\infty$ (p) we choose an integer $N >$ max $\left\{1, \sup_{k\ge1}|\,x_k\,|^{p_k}\right\}$

If $\sup|\,x^k\,|^{pk} \le 1$

$\Rightarrow N > 1 \ge \sup|\,x^k\,|^{pk}$

but if $\sup|\,x^k\,|^{p^k} > 1$

$\Rightarrow N > \sup_{k\ge1}|\,x_k\,|^{p_k}$

In both above cases

$\sup_{1\le i\le\infty}|\,x_{n+i}\,|^{p_{n+i}} < N$

$\Rightarrow \sup_{1\le i\le\infty}|\,x_{n+i}\,| < N^{\frac{1}{p_{n+i}}}$

$\Rightarrow \sup_{1\le i\le\infty}|\,x_{n+i}\,|^r < N^{\frac{r}{p_{n+i}}}$

Then,

$$\sum_{k=1}^{\infty}|\,\phi_{k,n}\,|^r = \sum_{k=1}^{\infty}\left|\frac{1}{k(k+1)}\sum_{i=1}^{k} i\,a_{n+i}\,x_{n+i}\right|^r$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i.a_{n+i}\,x_{n+i}\right|^r$$

$$\le \sup_{1\le i\le\infty}|\,x_{n+i}\,|^r\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}\right|^r$$

$$< \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}N^{\frac{1}{p_{n+i}}}\right|^r$$

$< \infty$ uniformely in n.

Thus $\sum_{k=1}^{\infty}|\,\phi_{k,n}\,|^r < \infty$

$\Rightarrow a = (a^k) \in [l_\infty(p)]^{\hat{\eta}}$

Therefore $\hat{l}_r(p) \subset [l_\infty(p)]^{\hat{\eta}}$

conversely, let $a = (a^k) \in [l_\infty(p)]^{\hat{\eta}}$ but $(a^k) \notin \hat{l}_r(p)$

$\Rightarrow \exists$ an integer $N > 1$ such that

$$\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}N^{\frac{1}{p_{n+i}}}\right|^r = \infty$$

choose $x^k = N^{\frac{1}{p_k}}$ sqn a^k. we have $x \in l^\infty(p)$ but

$$\sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k} i\,a_{n+i}\,x_{n+i}\right|^r$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^r(k+1)^r}\left|\sum_{i=1}^{k}|\,a_{n+i}\,|\,N^{\frac{1}{p_{n+i}}}\right|^r = \infty$$

which is contradiction that $(a^k) \in [l_\infty(p)]^{\hat{\eta}}$

Hence $[l_\infty(p)]^{\eta} \subset \hat{l}_r(p)$

Therefore, $[l_\infty(p)]^{\hat{\eta}} = \hat{l}_r(p)$.

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