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Applicability of STWS Technique in Solving Linear System of Stiff Delay Differential Equations with Constant Delays

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Abstract

This paper presents the Single Term Walsh Series (STWS) technique to determine the numerical solution to stiff linear systems of delay differential equations (DDEs) with single and multiple constant delays. The applicability of this technique is demonstrated with examples of stiff delay systems. The discrete solutions obtained using the STWS technique is compared with their corresponding exact solutions.

Key Words: WF, BPF, STWS, DDE, Stiff, delay

Introduction

Stiff systems of ordinary differential equations (ODEs) are very important special case of the systems occur in initial value problems. Stiff problems arise in many areas such as chemical kinetics, nuclear reactor theory, control theory, biochemistry, climatology, electronics, fluid dynamics, etc. While solving stiff problems numerically by a given method with assigned tolerance, a step size is restricted by stability requirements rather than by the accuracy demands. This behaviour is usually observed in problems that have some components that decay much more rapidly than other components. Due to this behaviour, ODEs have been divided into stiff and non-stiff problems. Models with either all fast or all slow changing variables are non-stiff problems. Models with both fast and slow changing variables are stiff problems. The problem of stiffness also occurs in DDEs.

Roth [1] has proposed difference methods for solving stiff DDEs. Staay [2] has introduced composite integration-interpolation methods for the solution of stiff DDEs. Bocharov et al.[3] have considered the application of linear multistep methods (LMMs) for the numerical solution of stiff DDEs with several constant delays, which are used in mathematical modelling of immune response. El-Safty and Hussien [4] have obtained the numerical solution of stiff DDEs in the form of Chebyshev series. Guglielmi and Hairer [5] have implemented Radau IIA methods for stiff DDEs. Huang et al. [6] have discussed the error behaviour of general linear methods for stiff DDEs. Bellen and Zennaro [7] have dealt the numerical treatment of DDEs (including stiff problems) in different fields of science and engineering. Zhu and Xiao [8] have discussed parallel two-step ROW-methods (PTSROW methods) for the numerical solution to stiff DDEs and analyzed the stability behaviours of these methods.

The roles of block pulse function (BPF), Walsh function (WF) and Walsh series (WS) have been important in solving delay problems. Chen and Shih [9] have presented the WS method to solve single delay systems. Rao and Srinivasan [10] have applied BPF for synthesis of systems with time delays. Rao and Palanisamy [11] have analysed time delay systems using WFs. Hwang and Shih [12] have studied the optimal control of delay systems via BPF. Palanisamy et al. [13, 14] have presented the STWS technique to discuss the effect of time delay on system performance and for the optimal control of linear time-varying delay systems.

This paper presents the STWS technique for the numerical solution of stiff linear systems of DDEs with single and multiple constant delays. The applicability of this technique is demonstrated with examples of delay systems. The discrete solutions obtained using this technique is compared with their corresponding exact solutions.

STWS Technique to Systems with Single Delay

Consider the system of equations with single delay of the form

$$\dot{x}(t) = Ax(t) + Lx(t-\tau) + Bu(t),$$
$$x(t) = \phi(t), \quad t \in [-\tau, 0), \qquad (1)$$

where x is a state vector, $\dot{x}$ is a rate vector, u is an input vector, A, L, and B are nxn, nxn, nx1 matrices respectively.

The given functions are expanded as STWS in the normalized interval s $\in$ [0, 1) which corresponds to t$\in$ [0, 1/m) by defining t = s/m, m being any integer. In the normalized interval, Eqn. (1) becomes

$$\dot{x}(s) = \frac{A}{m}\, x(s) + \frac{L}{m}\, x(s-N) + \frac{B}{m}\, u(s) \quad (2)$$

with $N = \tau m$. Expressing Eqn. (2) in STWS at the k[th] interval as

$$\begin{aligned} \dot{x}(s) &= T(k)\, \psi_0(s) \\ x(s) &= S(k)\psi_0(s) \\ x(s-N) &= S(k-N)\psi_0(s) \\ u(s) &= H(k)\psi_0(s) \end{aligned}$$

the following set of recursive equations with the operational matrix for integration $E = \frac{1}{2}$ has been obtained:

$$\begin{aligned} T(k) &= \left[I - \frac{A}{2m} \right]^{-1} G(k) \\ S(k) &= \frac{1}{2}T(k) + x(k-1) \\ x(k) &= T(k) + x(k-1) \end{aligned} \quad (3)$$

where

$$G(k) = \frac{A}{m}\, x(k-1) + \frac{L}{m}\, S(k-N) + \frac{B}{m}\, H(k)$$

for k = 1, 2, 3, ... Using the recursive relations given by Eqn. (3), piecewise constant solutions of $\dot{x}(t)$ and x(t) can be evaluated for any length of time.

STWS Technique to Systems with Multiple Delays

Consider a system with multiple delays

$$\begin{aligned} &\dot{x} = Ax + \sum_{i=1}^{n} L_i x(t-\tau_i) + Bu(t-\tau_u), \quad \text{where} \\ &\tau_1 < \tau_2 < \tau_3 ... < \tau_n, \\ &x(t) = \varphi_i(t), \ -\tau_i \le t < 0, \\ &u(t) = \varphi_u(t), \ -\tau_u \le t < 0, \ \ x(t_0) = x(0). \end{aligned} \quad (4)$$

The given functions are expanded as STWS in the normalized interval s ∈ [0, 1) which corresponds to t ∈ [0, 1/m) by defining t = s/m, m being any integer. In the normalized interval, Eqn. (4) becomes

$$\begin{aligned} \dot{x}(s) &= \frac{A}{m} x(s) + \sum_{i=1}^{n} \frac{L_i}{m} x(s-(N_i+\alpha_i)) \\ &\quad + \frac{1}{m} Bu(s-(N_u+\alpha_u)), \end{aligned} \quad (5)$$

where $\tau_i = (N_i + \alpha_i)/m$ and $\tau_u = (N_u + \alpha_u)/m.$

Here N's represent the integer parts of the delays and $\alpha's$ represent the fractional parts.

Expressing Eqn. (5) in STWS, the following set of recursive equations with the operational matrix for integration $E = \frac{1}{2}$ has been obtained:

$$\begin{aligned} T(k) &= \left[I - \frac{A}{2m} \right]^{-1} G(k) \\ S(k) &= \frac{1}{2}T(k) + x(k-1) \\ x(k) &= T(k) + x(k-1) \end{aligned} \quad (6)$$

where

$$\begin{aligned} G(k) &= \frac{A}{m}\, x(k-1) \\ &\quad + \sum_{i=1}^{n} \frac{L_i}{m} \left[(1-\alpha_i)S(k-N_i) + \alpha_i S(k-N_i-1) \right] \\ &\quad + \frac{B}{m}\left[(1-\alpha_u)H(k-N_u) + \alpha_u H(k-N_u-1) \right], \end{aligned}$$

for k = 1, 2, 3, . . . Using the recursive relations given by Eqn. (6), piecewise constant solutions of $\dot{x}(t)$ and x(t) can be evaluated for any length of time.

Stability of computations in the Walsh series approach to delay differntial equations [12]

Consider the function f(t), its Walsh series representation $f^* = F^T \psi(t)$ and the consequent error all in L_2 -space over the interval $I : t \in [0,1)$. The representation error e(t) is given by e(t) = f(t) – f*(t) and F^T is chosen to minimize ||f – f*||. Under this condition the residual error have an upper bound such that

$$\|e\| \le \frac{1}{2^{k+1}\sqrt{3}} \sup_I \left(\frac{df}{dt} \right).$$

If the error in representing $\int_0^1 \psi(t)dt$ in terms of $\psi(t)$ is $\gamma^{(t)} = \int_0^t \psi(t)dt - E\psi(t)$, then

$$\|\gamma\| \le \frac{1}{2^{k+1}\sqrt{3}} \sup_I (f).$$

In view of this, k should be chosen suitably to remove this error totally.

Consider the case of delay differential equations. If the error in representing $\psi(t-\tau)$ in terms of $\psi(t)$ is $d(t) = F^T\psi(t-\tau) - F^T D\psi(t),$ where $\tau = (N+\alpha)/m, \ 0 < \alpha < 1,$ N = 0, 1. . . m -1 and m = 2^k, it has been shown that

$$\|d\| \le \frac{\alpha(1-\alpha)}{2^k}\left[\sup_{I\tau} - \inf_{I\tau}\right]$$

where $I\tau$ is the total interval [-τ , 1).

$\|d\|$ vanishes if ∝ = 0 or 1 and has a maximum value at ∝ = ½. In view of this, k should be chosen such that ∝ is 0 or 1 to remove this error totally. Stability of computations for the case of multiple delays is a straightforward extension of these ideas.

Numerical Examples

Example 1

Consider the stiff linear delay system with single delay [8]

$$\dot{x}(t) = Lx(t) + Mx(t-0.1) + N, \qquad 0 \le t \le 10,$$

$$x(t) = (1+e^{-t}, 1+e^{-2t})^T, \qquad -0.1 \le t \le 10,$$

where

$$L = \begin{pmatrix} -1001 & -125 \\ 8 & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 1000e^{-0.1} & 125e^{-0.2} \\ -8e^{-0.1} & -2e^{-0.2} \end{pmatrix},$$

$$N = \begin{pmatrix} 1126 - 1000e^{-0.1} - 125e^{-0.2} \\ -8 + 8e^{-0.1} + 2e^{-0.2} \end{pmatrix}.$$

The exact solution is

$$x(t) = (1+e^{-t}, 1+e^{-2t})^T.$$

For this problem, the discrete solution has been calculated using the STWS algorithm given by Eqn. (3) with m = 100 and been compared with its exact solution. These results are shown in Table 1 and the error graph of this example is shown in Fig. 1.

Example 2

Consider the stiff linear system with multiple delays [4]

$$\dot{x}_1(t) = -\frac{1}{2}x_1(t) - \frac{1}{2}x_2(t-1) + f_1(t),$$

$$\dot{x}_2(t) = -x_2(t) - \frac{1}{2}x_1(t-\frac{1}{2}) + f_2(t),$$

$$t \in [0,5],$$

with the initial functions

$x_1(t) = e^{-t/2}$ for $-\frac{1}{2} \le t \le 0,$

$x_2(t) = e^{-t}$ for $-1 \le t \le 0,$

and $f_1(t) = \frac{1}{2}e^{-(t-1)}, \qquad f_2(t) = \frac{1}{2}e^{-(t-1/2)/2}.$

The exact solution is given by

$x_1(t) = e^{-t/2}$, $x_2(t) = e^{-t}$.

The discrete solution of this system has been calculated using the STWS technique given by Eqn. (3) with m = 100. The exact and STWS solutions and the absolute errors between them are shown in Table 2. The error graph of this example is shown in Fig. 2.

Example 3

Consider the stiff delay problem [15]

$$\dot{x}(t) = -1000x + qx(t-1) + c, \quad 0 \le t \le 10$$

with q = 997/exp (3), c = 1000 – q.

The exact solution of this problem is given by

x(t) = 1+ exp(-3t).

The STWS solution of this example has been calculated using the algorithm given by Eqn. (3) with m = 100 and is compared with its exact solution. These results are shown in Table 3. The error graph of this example is shown in Fig. 3.

Example 4

Consider the stiff delay problem [15]

$$\dot{x}(t) = -1000x + qx(t-1) + c, \quad 0 \le t \le 10$$

with q = 999/e, c = 1000 – q.

The exact solution of this problem is given by

x(t) = 1+ exp(-t).

For this problem, the discrete solution has been calculated using the STWS technique given by Eqn. (3) with m = 100. The exact and STWS solutions and the absolute errors between them are shown in Table 4. The error graph of this example is shown in Fig.4.

Table 1

Time	STWS Solution		Exact Solution		Absolute Error	
t	x_1	x_2	x_1	x_2	x_1	x_2
0	2.000000	2.000000	2.000000	2.000000	0.00E+00	0.00E+00
1	1.367473	1.136489	1.367879	1.135335	4.06E-04	1.15E-03
2	1.132522	1.018456	1.135335	1.018316	2.81E-03	1.40E-04
3	1.048516	1.002491	1.049787	1.002479	1.27E-03	1.22E-05
4	1.017421	1.000338	1.018316	1.000335	8.95E-04	2.39E-06
5	1.006456	1.000045	1.006738	1.000045	2.81E-04	2.47E-07
6	1.002299	1.000007	1.002479	1.000006	1.80E-04	4.17E-07
7	1.000819	1.000001	1.000912	1.000001	9.30E-05	3.58E-07
8	1.000303	1.000000	1.000335	1.000000	3.26E-05	1.18E-07
9	1.000115	1.000000	1.000123	1.000000	8.22E-06	1.33E-08
10	1.000044	1.000000	1.000045	1.000000	1.57E-06	6.75E-09

Fig. 1

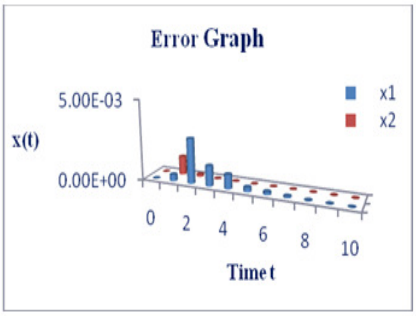


Table 2

Time t	STWS Solution x_1	STWS Solution x_2	Exact Solution x_1	Exact Solution x_2	Absolute Error x_1	Absolute Error x_2
0	1.000000	1.000000	1.000000	1.000000	0.00E+00	0.00E+00
1	0.549037	0.335213	0.606531	0.367879	5.75E-02	3.27E-02
2	0.349543	0.126543	0.367879	0.135335	1.83E-02	8.79E-03
3	0.227179	0.054349	0.22313	0.049787	4.05E-03	4.56E-03
4	0.138985	0.020556	0.135335	0.018316	3.65E-03	2.24E-03 7.74E-04

Fig. 2

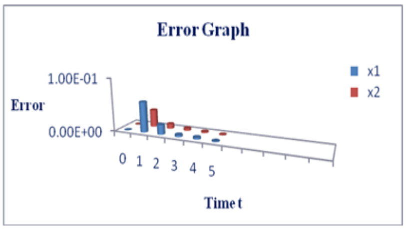


Table 3

Time t	STWS Solution	Exact Solution	Absolute Error
0	2	2	0.00E+00
1	1.049638	1.049787	1.49E-04
2	1.002464	1.002479	1.49E-05
3	1.000122	1.000123	1.11E-06
4	1.000006	1.000006	7.34E-08
5	1.000000	1.000000	4.56E-09
6	1.000000	1.000000	2.72E-10
7	1.000000	1.000000	1.58E-11
8	1.000000	1.000000	8.97E-13
9	1.000000	1.000000	5.02E-14
10	1.000000	1.000000	2.22E-15

Fig. 3

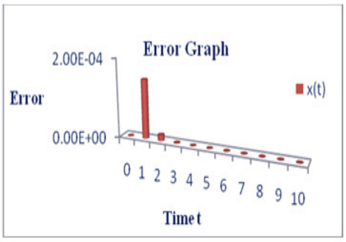
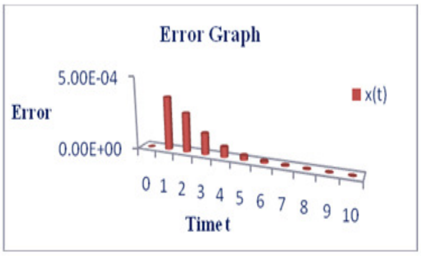


Table 4

Time t	STWS Solution	Exact Solution	Absolute Error
0	2	2	0.00E+00
1	1.367512	1.367879	3.68E-04
2	1.135065	1.135335	2.71E-04
3	1.049638	1.049787	1.49E-04
4	1.018242	1.018316	7.32E-05
5	1.006704	1.006738	3.36E-05
6	1.002464	1.002479	1.48E-05
7	1.000906	1.000912	6.36E-06
8	1.000333	1.000335	2.67E-06
9	1.000122	1.000123	1.11E-06
10	1.000045	1.000045	4.52E-07

Fig. 4



Conclusion

In this paper, the STWS technique has been presented to determine the discrete solutions for the linear delay systems with single and multiple constant delays. The stability of computations in the Walsh series approach to DDEs has been discussed. The applicability of the STWS technique has been demonstrated by considering four examples of stiff linear delay systems. The STWS solutions of these systems have been determined and compared with their corresponding exact solutions. From the numerical results, it is observed that the STWS technique is very much applicable and suitable for solving stiff linear delay systems of single as well as multiple constant delays.

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