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OPTIMALITY CONDITIONS WITH RESPECT TO AN ORDERING MAP USING AN EXACT SEPARATION PRINCIPLE

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In this note, we are concerned with a multiobjective optimization problem with respect to a variable ordering map. Using a special (nonlinear) scalarization [1], together with an exact separation principle recently introduced by Zheng,Yang and Zou [10], we give necessary optimality conditions for locally weakly nondominated solutions with respect to a given ordering map. To get the results, a nonsmooth sequential Guignard constraint qualification is introduced.

1. Introduction

It is well known that the convex separation principle plays a fundamental role in many aspects of nonlinear analysis and optimization. The whole convex analysis revolves around the use of separation theorems; see Rockafellar [8]. In fact, many crucial results with their proofs are based on separation arguments which are applied to convex sets (see [7]). There is another approach initiated by Zheng,Yang and Zouthe [10], which does not involve any convex approximations and convex separation arguments. Using the Ekeland variational principle, those authors gave an exact separation result that can be applied to disjoint sets; which supplement the extremal principle [4, 5].

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Let X be an Asplund space and consider the following multi-objective programming problem

$$(P):\begin{cases} \text{Min } F(x) = (F_1(x), \ldots, F_n(x)) \\ \text{Subject to :} \quad h_j(x) \le 0, \ j = 1, \ldots, p \end{cases}$$

where $F_i : X \to \mathbb{R}$ and $h_j : X \to \mathbb{R}$ are lower semicontinuous functions. To define an ordering cone, let $l : \mathbb{R}^n \to \mathbb{R}^n$ be a given Lipschitz continuous map and suppose that $\mathbb{R}^n$ is equipped with a variable ordering structure defined by the following cone-valued map $D : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that

$$D(y) := C(l(y)) = \{v \in \mathbb{R}^n : \|v\| \le l(y)(v)\}, \qquad \forall y \in \mathbb{R}^n$$

is a Bishop-Phelps cone (see [2]). These cones are often used as second-order cones. Local Lipschitz continuity of the function l is needed in the proof of Theorem 3.4 below. Remark that the images of D cover a wide range of different cones; however, in order to represent all nontrivial convex closed pointed cones as Bishop-Phelps cones, one might need to replace the norm $\|.\|$, in the definition of $D(y)$, by other different equivalent norms; for more details, see [1].

Let C be the set of all feasible solutions defined by

$$C = \left\{x \in X : \ h_j(x) \le 0, \ j = 1, \ldots, p\right\}.$$

The point $\bar{x} \in C$ is said to be a locally weakly nondominated solution of the problem (P) with respect to the ordering cone valued map D [1, 9] if there is no $x \in C$ such that

$$F(\bar{x}) - F(x) \in int\, D(F(x)),$$

where

$$D(F(x)) = \bigcup_{y \in F(x)} D(y).$$

For all the following, we assume that $\|l(y)\| > 1$, for all $y \in F(C)$. Under this condition, Eichfelder and Ha [1] have proved that the interior of $D(y)$ is nonempty and the interior is the set

$$int\, D(y) = \{v \in \mathbb{R}^n : \|v\| < l(y)(v)\}, \qquad \forall y \in F(C). \tag{1}$$

Such problem has been discussed by several authors at various levels of generality. Using a special (nonlinear) scalarization [1], together with an exact separation principle [10], under a nonsmooth sequential Guignard constraint qualification, we investigate necessary optimality conditions for locally weakly nondominated solutions with respect to the given ordering map D [1, 9]. The obtained results are given in terms of Fréchet subdifferentials.

Throughout this work, we use standard notations. We denote by X^* the topological dual of X with the canonical dual pairing $\langle\cdot,\cdot\rangle$; $\|(x,y)\| := \|x\|+\|y\|$ is the l_1-norm of (x,y); $\mathbb{B}_X$ and $\mathbb{B}_{X^*}$ stand for the closed unit balls in the space and dual space in question; and w^* denotes the weak* topology on the dual space. For a multifunction $F : X \rightrightarrows X^*$, the expressions

$\limsup\limits_{x\to\bar{x}} F(x) := \left\{x^* \in X^* \mid \exists x_k \to \bar{x},\ \exists x_k^* \overset{w^*}{\to} x^* : x_k^* \in F(x_k)\ \forall k \in \mathbb{N}\right\}$

and

$\liminf\limits_{x\to\bar{x}} F(x) := \left\{x^* \in X^* \mid \forall x_k \to \bar{x},\ \exists x_k^* \overset{w^*}{\to} x^* : x_k^* \in F(x_k)\ \forall k \in \mathbb{N}\right\}$

signify, respectively, the *sequential* Painlevé-Kuratowski upper/outer and lower/inner limits in the norm topology in X and the weak* topology in X^*; $\mathbb{N} := \{1,2,\ldots\}$.

The rest of the paper is organized in this way : Section 2 contains basic definitions and preliminary material from nonsmooth variational analysis. Section 3 addresses main results (optimality conditions).

2. Preliminaries

In this section, we give some definitions, notations and results, which will be used in the sequel. For a subset D of X, the sets *int D*, *cl D*, *cl conv D* and D° stand for the topological interior of D, the closure of D, the closed convex hull of D and the negative polar cone of D, repectively. The contingent cone $K(D,x)$ to D at $x \in cl\ D$ is defined by

$$K(D,x) = \{v \in X : \exists t_n \downarrow 0 \text{ and } \exists v_n \to v \text{ such that } x + t_n v_n \in D\}.$$

A function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be locally Lipschitzian around $\bar{x} \in dom f$ if there exist a neighbourhood V of $\bar{x}$ and $k > 0$ such that

$$| f(x) - f(y) | \leq k\|x-y\| \qquad \forall x,y \in V.$$

The following definitions are crucials for our investigation.

Definition 2.1. [4] Let $\Omega \subset X$ be locally closed around $\bar{x} \in \Omega$. Then the *Fréchet normal cone* $\widehat{N}(\bar{x};\Omega)$ and the *Mordukhovich normal cone* $N(\bar{x};\Omega)$ to Ω at $\bar{x}$ are defined by

$$\widehat{N}(\bar{x};\Omega) := \left\{x^* \in X^* :\ \limsup_{x \overset{\Omega}{\to} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\right\}, \tag{2}$$

$$N(\bar{x};\Omega) := \limsup_{x \overset{\Omega}{\to} \bar{x}} \widehat{N}(x;\Omega), \tag{3}$$

where $x \overset{\Omega}{\to} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$.

Definition 2.2. [4]Let φ be a lower semicontinuous function around $\bar{x}$.
1. The *Fréchet subdifferential* of φ at $\bar{x}$ is

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ x^* \in X^* : \liminf_{x \xrightarrow{\Omega} \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

2. The *Mordukhovich subdifferential* of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) := \limsup_{x \xrightarrow{\Omega} \bar{x}} \widehat{\partial}\varphi(x), \tag{4}$$

where $x \xrightarrow{\varphi} \bar{x}$ means that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$.

One clearly has

$$\widehat{N}(\bar{x};\Omega) = \widehat{\partial}\delta(\bar{x};\Omega), \quad N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega),$$

where $\delta(\cdot;\Omega)$ is the indicator function of Ω.

Remark 2.3. [6]1. For any closed set $\Omega \subset X$ and $\bar{x} \in \Omega$ one has

$$N_c(\bar{x};\Omega) = cl\ conv\ N(\bar{x};\Omega) \tag{5}$$

and for any Lipschitz continuous function $\varphi\colon X \to \overline{\mathbb{R}}$ around $\bar{x}$, one has

$$\partial_c\varphi(\bar{x}) = cl\ conv\ \partial\varphi(\bar{x}) \tag{6}$$

where $N_c(\bar{x};\Omega)$ and $\partial_c\varphi(\bar{x})$ denote respectively the Clarke's normal cone and the Clarke's subdifferential.
2. The *Fréchet normal cone* $\widehat{N}(\bar{x};\Omega)$ is always convex while the *Mordukhovich normal cone* $N(\bar{x};\Omega)$ is nonconvex in general.

As for the extremal principle, the following exact separation theorem can be considered as a generalization of the convex separation theorem to nonconvex sets and used as a powerful tool for deducing optimality conditions in nonconvex optimization. In the separation theorem below, it is supposed that the intersection between the sets is empty and each set is considered near its own point; which is not the case in the extremal principle.

Theorem 2.4. *[10]Let X be an Asplund space and $A,\ A_1,\ \ldots,\ A_n$ be nonempty closed (not necessarily convex) subsets of X such that A is compact and $A \cap \left(\bigcap_{i=1}^{n} A_i \right) = \varnothing$. Let $1 \leq p,\ q \leq +\infty$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for any $\varepsilon \in]0,\ +\infty[$ and $\rho \in]0,\ 1[$ there exist $a \in A,\ a_i \in A_i$ and $a_i^* \in X^*,\ i = 1,\ \ldots,\ n$, such that the following statements hold:

1.

$$\left(\sum_{i=1}^{n}\|a_i-a\|^p\right)^{\frac{1}{p}} \leq \gamma_p\left(A_1,\ ...,\ A_n,\ A\right)+\varepsilon.$$

2.

$$\begin{cases} a_i^* \in \widehat{N}\left(A_i,\ a_i\right), & i=1,\ ...,\ n, \\ -\sum\limits_{i=1}^{n} a_i^* \in \widehat{N}\left(A,\ a\right) \text{ and } \left(\sum\limits_{i=1}^{n}\|a_i^*\|^q\right)^{\frac{1}{q}}=1. \end{cases}$$

3.

$$\rho\left(\sum_{i=1}^{n}\|a_i-a\|^p\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n}\left\langle a_i^*,\ a-a_i\right\rangle.$$

Here, $\gamma_p\left(A_1,\ ...,\ A_n,\ A\right)$ denotes the (p-weighted) non-intersect index of $A_1,\ ...,\ A_n,\ A$ defined by

$$\gamma_p\left(A_1,\ ...,\ A_n,\ A\right)=\inf\left\{\left(\sum_{i=1}^{n}\|a_i-a\|^p\right)^{\frac{1}{p}} : a_i\in A_i,\ i=1,\ ...,\ n\right\}.$$

3. Necessary optimality conditions

In this section, we maintain the notations given in the previous section and we give necessary optimality conditions for the multiobjective optimization problem (P). The following result has been proved by Eichfelder and Ha [1]; for the convinience of the reader, we allow ourselves to give a proof.

Proposition 3.1. *[1, Theorem 3.7] $\bar{x}$ is a local weak nondominated solution of problem (P) with respect to the ordering map D if and only if $\bar{x}$ is a local minimiser of the functional φ defined over C by*

$$\varphi(x) := \psi(F(x)) \text{ for all } x\in C$$

where

$$\psi(y) := l(y)^t\left(y-F(\bar{x})\right)+\|y-F(\bar{x})\| \text{ for all } y.$$

Proof. $\bar{x}$ is a local weak nondominated solution of the problem (P) with respect to the ordering map D, if and only if there exists a neighborhood V of $\bar{x}$ such that for all $x\in V\cap C$

$$F(\bar{x})-F(x)\notin int\, D(F(x)).$$

$$\Longleftrightarrow \qquad \psi(F(x))=l(F(x))(F(x)-F(\bar{x}))+\|F(x)-F(\bar{x})\|\geq 0.$$

$$\iff \qquad (\psi \circ F)(x) \geq (\psi \circ F)(\bar{x}) = 0,$$

since $\psi(F(\bar{x})) = 0$, one has equivalently

$$\psi(F(x)) \geq \psi(F(\bar{x})).$$

Thus, one deduces that $\bar{x}$ is a minimiser of the functional φ over $C \cap V$.

□

Remark 3.2. [1] If the functions $l : \mathbb{R}^n \to \mathbb{R}^n$ and $F : X \to \mathbb{R}^n$ are Lipschitz continuous then, the function $\varphi : X \to \mathbb{R}$ is Lipschitz continuous on the set C, too.

Definition 3.3. We say that the nonsmooth sequential Guignard constraint qualification holds at $\bar{x} \in C$ if for every sequences $\{w_k\} \subset C$, such that

$$\lim_{k \to +\infty} \|w_k - \bar{x}\| = 0 \tag{7}$$

one has

$$[K(C, w_k)]^\circ \subseteq \left[T^{Lin}(w_k)\right]^\circ,$$

where

$$\Delta(w_k) = \left\{ \begin{array}{c} \alpha^* \in \mathbb{R}^p_+ : \\ \|\alpha^*\| \leq 1 \text{ and } \sum\limits_{j=1}^{p} \alpha_j^* h_j(w_k) = 0 \end{array} \right\}$$

and

$$T^{Lin}(w_k) = \left\{ \begin{array}{c} d \in X : \\ \forall \alpha^* \in \Delta(w_k),\ \forall \pi_j^* \in \widehat{\partial} h_j(w_k),\ \left\langle \sum\limits_{j=1}^{p} \alpha_j^* \pi_j^*, d \right\rangle \leq 0 \end{array} \right\}.$$

Theorem 3.4. *Assume that C is bounded and that F is locally Lipschitz continuous at $\bar{x}$, where the nonsmooth sequential Guignard constraint qualification holds. Suppose that $\bar{x}$ is a locally weakly nondominated solution of the problem (P), with respect to the ordering map D. Then, there exist sequences $\lambda_k = \left(\lambda_k^1, \ldots, \lambda_k^n\right) \in \left(-\mathbb{R}^n_+\right)^\circ \setminus \{0\}$, $\{v_k\} \subset X$, $\{\omega_k\} \subset C$ and $a_k^* \in \{l(F(v_k))\} + \mathbb{B}_{\mathbb{R}^n}$ such that*

$$\lim_{k \to +\infty} F(v_k) = F(\bar{x}) \qquad , \qquad \lim_{k \to +\infty} \|v_k - w_k\| = 0$$

$$0 \in \widehat{\partial} \langle a_k^*, F \rangle (v_k) + clconv \left\{ \bigcup_{i=1}^{p} \alpha_j^* \widehat{\partial} h_j(w_k) \text{ such that } \alpha^* \in \Delta(w_k) \right\}.$$

Proof. Since $\bar{x}$ is a locally weakly nondominated solution of the problem (P) with respect to the ordering map D, it is also a local minimiser of the functional $\varphi = \psi \circ F$ over C. Since φ is lower semicontinuous and since C is compact, one deduces that $epi(\varphi)$ is a closed subset of $X \times \mathbb{R}$ and that $\varphi(\bar{x}) > -\infty$. For each $k \in \mathbb{N}$, let

$$A = C \times \left\{ \varphi(\bar{x}) - \frac{1}{k+1} \right\}.$$

In this case, A is a compact subset of $X \times \mathbb{R}$, $A \cap epi(\varphi) = \varnothing$ and

$$\gamma_1 (epi(\varphi), A) = d(epi(\varphi), A) \leq \frac{2}{k+1}.$$

Applying Theorem 2.4, for each fixed $k \in \mathbb{N}$, there exist $w_k \in C$, $v_k \in X$, $(v_k, \alpha_k) \in epi(\varphi)$ and $\left(v_k^*, \beta_k\right) \in X \times \mathbb{R}$ such that

$$\|(v_k^*, -\beta_k)\|_\infty = 1, \left\| \left(w_k, \varphi(\bar{x}) - \frac{1}{k+1} \right) - (v_k, \alpha_k) \right\| < \frac{2}{k+1} \tag{8}$$

and

$$(v_k^*, -\beta_k) \in \widehat{N}(epi(\varphi), (v_k, \alpha_k)) \cap -\widehat{N}\left(A, \left(w_k, \varphi(\bar{x}) - \frac{1}{k+1}\right)\right). \tag{9}$$

- From (8), one gets that (v_k, α_k) is not an interior point of $epi(\varphi)$; consequently,

  $$\alpha_k = \varphi(v_k).$$

  Then, using (8), one obtains the following inequalities :

  $$\|w_k - v_k\| \leq \frac{2}{k+1} \text{ and } |\varphi(v_k) - \varphi(\bar{x})| \leq \frac{3}{k+1}.$$

  This implies that

  $$\varphi(v_k) > -\infty, \lim_{k \to +\infty} \varphi(v_k) = \varphi(\bar{x}), \lim_{k \to +\infty} \|v_k - w_k\| = 0. \tag{10}$$

- Since $(v_k, \alpha_k) \in epi(\varphi)$ and $\left(v_k^*, -\beta_k\right) \in \widehat{N}(epi(\varphi), (v_k, \alpha_k))$, by a result of [3], one has

  $$\beta_k \geq 0.$$

  By (9), the equality in (8) and the Lipschitz continuity of φ, implies that $\beta_k \neq 0$. Setting $\pi_k^* = \dfrac{v_k^*}{\beta_k}$, from (9), one has

  $$(\pi_k^*, -1) \in \widehat{N}(epi(\varphi), (v_k, \alpha_k)) \cap -\widehat{N}\left(A, \left(w_k, \varphi(\bar{x}) - \frac{1}{k+1}\right)\right). \tag{11}$$

Thus,

$$\pi_k^* \in \widehat{\partial}\varphi(v_k)$$

and

$$(\pi_k^*,\ -1) \in -\widehat{N}\left(A,\ \left(w_k,\ \varphi(\bar{x}) - \frac{1}{k+1}\right)\right) = -\widehat{N}(C,w_k) \times \mathbb{R}$$

That is,

$$\pi_k^* \in \widehat{\partial}\varphi(v_k) \cap -\widehat{N}(C,w_k).$$

Consequently,

$$0 \in \widehat{\partial}\varphi(v_k) + \widehat{N}(C,w_k). \tag{12}$$

Then, there exists $a_k^* \in \{l(F(v_k))\} + \mathbb{B}_{\mathbb{R}^n}$ such that

$$0 \in \widehat{\partial}\langle a_k^*, F\rangle(v_k) + \widehat{N}(C,w_k).$$

- According to [4, Corollary 1.11], one has

$$\widehat{N}(C,w_k) = \{\alpha^* \in X^* : \langle \alpha^*, d\rangle \leq 0 \text{ whenever } d \in K(C,w_k)\}.$$

Since the nonsmooth sequential Guignard constraint qualification holds at $\bar{x}$, one deduces that

$$\widehat{N}(C,w_k) \subseteq \left\{\alpha^* \in X^* : \langle \alpha^*, d\rangle \leq 0 \text{ whenever } d \in T^{Lin}(w_k)\right\}.$$

Finally,

$$0 \in \widehat{\partial}\langle a_k^*, F\rangle(v_k) + clconv\left\{\bigcup_{i=1}^{p} \alpha_j^* \widehat{\partial} h_j(w_k) \text{ such that } \alpha^* \in \Delta(w_k)\right\}.$$

□

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