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CHARACTERIZATION OF PERFECT ROMAN DOMINATION EDGE CRITICAL TREES

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A perfect Roman dominating function on a graph $G = (V,E)$ is a function $f : V \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a perfect Roman dominating function f is the sum of the weights of the vertices. The perfect Roman domination number of G, denoted by $\gamma_R^p(G)$, is the minimum weight of a perfect Roman dominating function in G. In this paper, we study the graphs for which adding any new edge decreases the perfect Roman domination number. We call these graphs γ_R^p-edge critical. The purpose of this paper is to characterize the class of γ_R^p-edge critical trees.

1. Introduction

Roman domination is a variation of domination introduced by ReVelle [12, 13]. Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighboring location only if there was a second army which would stay and protect the home. Thus, there are two types of armies, stationary and traveling. A vertex with no army must have a neighboring vertex with a traveling army. Stationary armies then dominate their own vertices. A

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vertex with two armies is dominated by its stationary army, and its open neighborhood is dominated by the traveling army. The concept of Roman domination can be formulated in terms of graphs. In this paper, we continue the study of a variant of Roman dominating functions, namely, perfect Roman dominating functions introduced in [7]. We first present some necessary definitions and notations. For notation and graph theory terminology not given here, we follow [6]. We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and the *degree* of v, denoted by $\deg_G(v)$, is the cardinality of its open neighborhood. A *leaf* of a tree T is a vertex of degree one, while a support vertex of T is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. In this paper, we denote the set of all support vertices of T by $S(T)$ and the set of leaves by $L(T)$. We denote $\ell(T) = |L(T)|$ and $s(T) = |S(T)|$. We also denote by $L(x)$ the set of leaves adjacent to a support vertex x, and denote $\ell_x = |L(x)|$. If T is a rooted tree, then for any vertex v we denote by T_v the sub-rooted tree rooted at v. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V \setminus S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A *perfect dominating set* is a set $S \subseteq V$ such that for all $v \in V$, $|N[v] \cap S| = 1$. The minimum size of a perfect dominating set for a graph G is the *perfect domination number* of G, denoted by $\gamma_P(G)$. Perfect dominating sets and several variations on perfect domination have received much attention in the literature; for example, see some discussion in [6] or the survey in [11].

For a graph G, let $f : V(G) \to \{0,1,2\}$ be a function, and let (V_0,V_1,V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0,1,2$. There is a one-by-one correspondence between the functions $f : V(G) \to \{0,1,2\}$ and the ordered partitions (V_0,V_1,V_2) of $V(G)$. So, we will write $f = (V_0,V_1,V_2)$. A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (or briefly, RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. Roman dominating functions with several further conditions have been studied, for example, among other types, see [1–3, 10].

Recently, Henning, Klostermeyer and MacGillivray [7] introduced the concept of perfect Roman domination in graphs. As defined in [7], an RDF $f = (V_0,V_1,V_2)$ is called a *perfect Roman dominating function* (or just PRDF) if very vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The perfect Roman domination number $\gamma_R^P(G)$ is the minimum weight of a

PRDF. Note that $\gamma_R^P(G)$ is defined for any graph G, since $(\emptyset, V(G), \emptyset)$ is an PRDF for G. We refer to a $\gamma_R^P(G)$-function as a PRDF of G with minimum weight.

For many graph parameters, criticality is a fundamental question. The concept of criticality with respect to various operations on graphs has been studied for several domination parameters. Much has been written about graphs, where a parameter increases or decreases whenever an edge or vertex is removed or added. This concept has been considered for several domination parameters such as domination, 2-rainbow domination and Roman domination, by several authors and the concept is now well studied in domination theory. For references on the criticality concept on various domination parameters see, for example [4, 5, 8, 9]. In this paper, we consider this concept for perfect Roman domination number.

Our aim is to study the graphs for which adding any new edge decreases the perfect Roman domination number. We say that G is perfect Roman domination edge critical, or just γ_R^P-edge critical, if for any $e \in E(\bar{G})$, we get $\gamma_R^P(G+e) < \gamma_R^P(G)$, where $\bar{G}$ is the complement of G. The purpose of this paper is to give a descriptive characterization of the class of γ_R^P-edge critical trees.

2. Main Results

We first present some properties of the γ_R^P-edge critical graphs.

Lemma 2.1. *For every edge $e = xy$ in a graph $\bar{G}$, we get $\gamma_R^P(G) - 1 \leq \gamma_R^P(G+e)$.*

Proof. Let $e = xy \in E(\bar{G})$ and $f = (V_0, V_1, V_2)$ be a $\gamma_R^p(G+e)$-function. If $V_2 \cap \{x,y\} = \emptyset$ or $\{x,y\} \subseteq V_2 \cup V_1$, then f is a *PRDF* of graph G, as desired. Thus we may assume that $x \in V_2$ and $y \in V_0$. Now we define function g by $g(y) = 1$ and $g(u) = f(u)$, if $u \in V - \{y\}$. Then function g is a *PRDF* of graph G, and therefore $\gamma_R^P(G) \leq \gamma_R^P(G+e) + 1$. □

The next corollary is immediate from Lemma 2.1.

Corollary 2.2. *For any edge $e \in E(\bar{G})$ in a γ_R^P-edge critical graph G, we have $\gamma_R^P(G+e) + 1 = \gamma_R^P(G)$.*

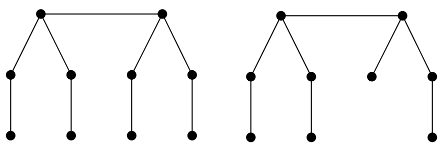
Next, we give a characterization of γ_R^P-edge critical graphs.

Theorem 2.3. *A graph G is γ_R^P-edge critical if and only if for any two non-adjacent vertices u, v, there exists a $\gamma_R^P(G)$-function $f = (V_0, V_1, V_2)$ such that $\{f(u), f(v)\} = \{1,2\}$ and also if for $x \in \{u,v\}$, $f(x) = 1$, then for any vertex $y \in N(x)$, $f(y) \neq 2$.*

Proof. Let G be a graph and $e = uv \in E(\bar{G})$. First, suppose that there exists a γ_R^P-function $f = (V_0, V_1, V_2)$ such that $\{f(u), f(v)\} = \{1,2\}$ and also if for $x \in \{u,v\}$, $f(x) = 1$, then for any vertex $y \in N(x)$, $f(y) \neq 2$. Suppose that $f(u) = 1$ and $f(v) = 2$. We define $g : V(G+uv) \longrightarrow \{0,1,2\}$ by $g(u) = 0$ and $g(z) = f(z)$ if $z \neq u$. Then g is a perfect Roman dominating function for $G+uv$, and so $\gamma_R^P(G+uv) \leq \gamma_R^P(G)-1$. This implies that G is $\gamma_R^P(G)$-edge critical graph.

For the converse, suppose that G is $\gamma_R^P(G)$-edge critical graph. Then by Corollary 2.2, we have $\gamma_R^P(G+uv) = \gamma_R^P(G) - 1$. Let $g = (V_0, V_1, V_2)$ be a $\gamma_R^P(G+uv)$-function. If $\{g(u), g(v)\} \neq \{0,2\}$, then g is a perfect Roman dominating function for G, which implies that $\gamma_R^P(G) \leq \gamma_R^P(G+uv) = \gamma_R^P(G) - 1$, a contradiction. Thus, $\{g(u), g(v)\} = \{0,2\}$. Let $g(u) = 0$, then for any vertex $w \in N_{G+uv}(u) - \{v\}$, $g(w) \neq 2$, since g is a perfect Roman dominating function. We define $h : V(G) \longrightarrow \{0,1,2\}$ by $h(u) = 1$ and $h(z) = g(z)$ if $z \neq u$. Then h is a perfect Roman dominating function for G with weight $\gamma_R^P(G+uv)+1$ and also any vertex $w \in N_G(u)$, $h(w) \neq 2$. On other hand, since $\gamma_R^P(G) = \gamma_R^P(G+uv)+1$, it follows that h is a $\gamma_R^P(G)$-function, and the result follows. □

Next, we give a characterization of the class of γ_R^P-edge critical trees. Let T_1 be a tree obtained from two path P_5 by joining central vertices which is depicted in Fig. 1(a), and T_2 be a tree obtained from a path P_5 with central vertex u and a path P_4 with support vertex v by joining u to v illustrated in Fig. 1(b).



$(a)\ T_1$ $(b)\ T_2$

Figure 1: The trees T_1 and T_2.

Theorem 2.4. *A tree T is γ_R^P-edge critical if and only if $T \in \{T_1, T_2\}$.*

Proof. Let T be a γ_R^P-edge critical tree. If $diam(T) \in \{2,3\}$, T is star or a double star. It is straightforward to see that T is not γ_R^P-edge critical. Thus we assume that $diam(T) \geq 4$. We root T at a leaf x_0 of a diametrical path $x_0 x_1 \ldots x_d$ from x_0 to a leaf x_d farthest from x_0. Without loss of generality, we may assume that for $i \in \{2, d-2\}$, $\deg(x_{i-1}) \geq \deg(u)$, where u is any child support vertex of x_i. We proceed with the following claims:

Claim 1. T has no strong support vertex with exactly one adjacent non-leaf vertex and degree at least 4.

Proof Assume, towards a contradiction, that u is a strong support vertex of T with degree at least 4 such that $\deg(u) = \ell_u + 1$. Let w, z be two leaves adjacent to u and x be non-leaf neighbors of u. It follows from Theorem 2.3, that there exists a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(w), f(z)\} = \{1, 2\}$ and $f(u) \neq 2$. Clearly, for each leaf v adjacent to u, $f(v) \geq 1$, $f(u) = 0$ and since f is a $\gamma_R^P(T)$-function, $(N(u) - \{w, z\}) \cap V_2 = \emptyset$. Then, $g : V(T) \longrightarrow \{0, 1, 2\}$ defined by $g(u) = 2$, $g(v) = 0$ if $v \in L(u)$, $g(x) = \max\{f(x), 1\}$ and $g(v) = f(v)$ if $v \notin L(u) \cup \{u, x\}$, is a perfect Roman dominating function for T with weight less than $\gamma_R^P(T)$, a contradiction.

By Claim 1, we get $\max\{\deg(x_1), \deg(x_{d-1})\} \leq 3$.

Claim 2. T has no two strong support vertices u and v such that $|N(u) \cup N(v)| = \ell_u + \ell_v + 1$.

Proof Assume that u and v are two strong support vertex of tree T such that $|N(u) \cup N(v)| = \ell_u + \ell_v + 1$ and $N(u) \cap N(v) = \{w\}$. By Claim 1, $\ell_u = \ell_v = 2$. Let $L(u) = \{u_1, u_2\}$ and $L(v) = \{v_1, v_2\}$. It follows from Theorem 2.3, that there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(u_1), f(v_1)\} = \{1, 2\}$. Without loss of generality, we may assume that $u_1 \in V_1$ and $v_1 \in V_2$. Then, $f(u) \neq 2, f(v_2) = 1$ and $f(v) = 0$. Since f is a $\gamma_R^P(T)$-function, $f(w) \neq 2$ and so $f(u) + f(u_2) \geq 2$. Then, $g : V(T) \longrightarrow \{0, 1, 2\}$ defined by $g(u) = 2$, $g(v) = 2$, $g(u_1) = g(u_2) = g(v_1) = g(v_2) = 0$, $g(w) = max\{1, f(w)\}$ and $g(z) = f(z)$ if $z \notin N[u] \cup N[v]$, is a perfect Roman dominating function for T with weight less than $\gamma_R^P(T)$, a contradiction.

Claim 2, implying that if x_1 be a strong support vertex, then each vertex of $N(x_2) - \{x_3\}$ is a weak support vertex or a leaf.

We now assume that $diam(T) = 4$. Without loss of generality, we may assume that $\deg(x_1) \geq \deg(x_3)$. By Claim 1, $\deg(x_3) \leq \deg(x_1) \leq 3$. We first assume that $\deg(x_1) = 3$. Then Claim 2, implying that every neighbor of x_2 is a leaf or a weak support vertex and so $\deg(x_3) = 2$. Let $L(x_1) = \{x_0, x_1'\}$. It follows from Theorem 2.3, that there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_1')\} = \{1, 2\}$. Without loss of generality, we may assume that $x_0 \in V_1$ and $x_1' \in V_2$. Then $f(x_1) = 0, f(x_2) \neq 2$ and $f(x_3) + f(x_4) = 2$. Then, $g : V(G) \longrightarrow \{0, 1, 2\}$ defined by $g(x_1) = 2$, $g(x_1') = g(x_0) = g(x_2) = 0$ and $g(z) = 1$ if $z \notin \{x_0, x_1, x_2, x_1'\}$, is a perfect Roman dominating function for T with weight less than $\gamma_R^P(T)$, a contradiction. Thus, we assume that $\deg(x_3) = \deg(x_1) = 2$ and also any neighbor of x_2 is a leaf or weak support vertex. If $\deg(x_2) = 2$, then $T = P_5$ and clearly T is not γ_R^P-edge critical. Hence, we assume that $\deg(x_2) \geq 3$. It follows from Theorem 2.3, that there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_4)\} = \{1, 2\}$. Without loss of generality, we may assume that $x_0 \in V_1$ and $x_4 \in V_2$. Then $f(x_3) = 0, f(x_2) \neq 2$ and $f(x_1) = 1$. Hence if $u \in L(x_2)$, then $f(u) = 1$ and also if $v \in N(x_2)$ is a support vertex with leaf neighbors v', then $f(v)+f(v') = 2$. Therefore, $w(f) = 2\deg(x_2) - \ell_{x_2} + f(x_2) \geq 2\deg(x_2) - \ell_{x_2}$. function $g = (N(x_2), L(T) - L(x_2), \{x_2\})$ is a perfect Roman dominating function for T with weight $2+\deg(x_2) - \ell_{x_2}$ and so $w(g) < \gamma_R^P(T)$, a contradiction. Thus, we may assume that $diam(T) \geq 5$.

Claim 3. $diam(T) = 5$.

Proof Assume that $diam(T) \geq 6$. Then $x_2 \notin N[x_{d-2}]$ and so by Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_2), f(x_{d-2})\} = \{1, 2\}$. Without loss of generality, we may assume that $x_2 \in V_1$ and $x_{d-2} \in V_2$. By Theorem 2.3, for any $u \in N(x_2)$, $f(u) \neq 2$ and so $f(x_3) \neq 2$. Since $f(x_2) = 1$, we have $f(N[x_1]) \geq 3$. Then, reassigning to each neighbor of x_1 the weight 0 and to x_1 the weight 2 produces a PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Hence $diam(T) = 5$.

Then Claim 1, implying that $\deg(x_1) \leq 3$ and $\deg(x_4) \leq 3$. Without loss of generality, we may assume that $\deg(x_4) \leq \deg(x_1)$. We consider the following cases:

Case 1. $\deg(x_1) = \deg(x_4) = 3$.
Suppose that $N(x_1) = \{x_0, x_1', x_2\}$ and also $N(x_4) = \{x_5, x_4', x_3\}$. We first assume that $5 \leq max\{\deg(x_2), \deg(x_3)\}$. Without loss of generality, we assume that $\deg(x_2) \geq 5$. By Claim 2, each neighbor of x_2 other of x_1 and x_3 is a leaf or a weak support vertex. Let K_i be the set of weak support neighbors of x_i for $i \in \{2, 3\}$. Then $\ell_{x_2} + |K_2| \geq 3$. It follows from Theorem 2.3, that there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $f(x_0) = 1$, $f(x_1') = 2$. Then clearly $f(x_1) = 0$, $f(x_2) \neq 2$, for $u \in L(x_2)$, $f(u) \geq 1$ and for any vertex $v \in K_2$, $f(v) + f(v') = 2$, where v' is leaf neighbor of v. Hence $\gamma_R^P(T) = w(f) \geq w(f|_{T_{x_3}}) + \ell_{x_2} + 2|K_2| + 3$. We define function $g : V(T) \longrightarrow \{0, 1, 2\}$ defined by $g(x_2) = g(x_1) = 2$, $g(x_0) = g(x_1') = 0$, for each vertex $u \in L(x_2)$, $g(u) = 0$, for $u \in K_2$, $g(u) = 0$ and $g(u') = 1$, $g(x_3) = \max\{1, f(x_3)\}$ and for $w \in V(T_{x_3}) - \{x_3\}$, $g(w) = f(w)$. Then g is a perfect Roman dominating function with weight at most $5 + |K_2| + w(f|_{T_{x_3}})$ and so $w(g) \leq 5 + |K_2| + w(f|_{T_{x_3}}) < w(f|_{T_{x_3}}) + \ell_{x_2} + 2|K_2| + 3 \leq \gamma_R^P(T)$, a contradiction. Thus, we assume that $max\{\deg(x_2), \deg(x_3)\} \leq 4$.

Then by Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_1), f(x_4)\} = \{1, 2\}$. Without loss of generality, we may assume that $x_1 \in V_2$ and $x_4 \in V_1$. Clearly, $f(x_0) = f(x_1') = 0$, $f(x_5) = f(x_4') = 1$ and we can assume that $f(x_3) = 0$ and for each $v \in L(x_3)$, $f(v) = 1$. Let $N(x_3) \cap V_2 = \{u\}$. If $u \in K_3$ and $L(u) = \{u'\}$, then $f(u') = 0$. Then, reassigning to x_5 and x_4' the weight 0, to x_4 the weight 2 and to each $u \in K_3$ and u' the weight 1, and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Thus, we assume that $u \notin K_3$ and so $u = x_2$. Then for each vertex $v \in L(x_2)$, $f(v) = 0$ and for each vertex $w \in K_2$, $f(w) = 0$ and $f(w') = 1$, that $\{w'\} = L(w)$. Hence $w(f) = \ell_{x_3} + 2|K_3| + |K_2| + 7$. Then, reassigning to x_2 the weight 0, to x_4 the weight 2, to x_5 and x'_4 the weight 0, to each $u \in K_2$ the weight 1, to each $u \in L(x_2)$ the weight 1 and leaving all other weights unchanged produces a new PRDF g with weight $\ell_{x_3} + 2|K_3| + 2|K_2| + \ell_{x_2} + 4$ and so $w(g) = \ell_{x_3} + 2|K_3| + 2|K_2| + \ell_{x_2} + 4 < \ell_{x_3} + 2|K_3| + |K_2| + 7 = w(f) = \gamma_R^P(T)$, a contradiction.

Case 2. $\deg(x_1) = 3$ and $\deg(x_4) = 2$.
Without loss of generality, we may assume that any child of x_3 is a weak support or a leaf. We first assume that $\deg(x_2) \geq 4$. Then any neighbors of x_2 other of x_3 is a weak support or a leaf. By Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_1), f(x_3)\} = \{1, 2\}$. We first assume that $x_1 \in V_2$ and $x_3 \in V_1$ and so by theorem 2.3, $f(x_2) \neq 2$. Clearly $f(x_0) = f(x'_1) = 0$, $f(x_5) + f(x_4) = 2$ and we can assume that for each $v \in L(x_3)$, $f(v) = 1$. Then reassigning to x_5 and x_3 the weight 0, to x_4 the weight 2 and to each $u \in K_3$ and $u' \in N(u) - \{x_3\}$ the weight 1, and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Thus, we assume that $f(x_1) = 1, f(x_3) = 2$ and $f(x_2) \neq 2$. Let $T' = T - T_{x_3}$. Then, reassigning to x_2 the weight 2, to $u \in N(x_2) - \{x_3\}$ the weight 0 and to each other vertex of tree T' the weight 1, and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Thus, we assume that $\deg(x_2) \leq 3$.

Assume that $\deg(x_2) = 3$ and $N(x_2) = \{x_1, x_3, x'_2\}$. Then Claim 2 implying that x'_2 is a leaf or a weak support vertex. We first assume that x_2 is a support vertex. By Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_2), f(x_4)\} = \{1, 2\}$. We first assume that $f(x_2) = 2$ and $f(x_4) = 1$ and so $f(x_3) \neq 2$. Clearly, $f(x'_2) = 0$, $f(x_5) = 1$ and $f(x_0) + f(x_1) + f(x'_1) = 2$. We can assume that $f(x_0) = f(x'_1) = 0$ and $f(x_1) = 2$. If $f(x_3) = 1$, then reassigning to x_2 the weight 0, reassigning to x'_2 the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Thus we may assume that$f(x_3) = 0$. Then for $u \in N(x_3) - \{x_2\}$, $f(u) \neq 2$, since $f(x_2) = 2$ and f is a $\gamma_R^P(T)$-function. Then reassigning to x_2 and x_5 the weight 0, reassigning to x'_2 the weight 1, reassigning to x_4 the weight 2 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Thus we assume that $f(x_2) = 1$ and $f(x_4) = 2$ and so $f(x_1) \neq 2$. Then $f(x_1) + f(x'_1) + f(x_0) = 3$ and so reassigning to x_1 the weight 2, reassigning to x'_1 and x_0 the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Thus we assume that x'_2 is a weak support vertex. Let x''_2 is leaf adjacent to x'_2. Then by Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_4)\} = \{1, 2\}$. We first assume that $f(x_0) = 2$ and $f(x_4) = 1$. Clearly

$f(x_5) = f(x_1') = 1$, $f(x_1) = 0$, $f(x_3) \neq 2$, and so we can assume that $f(x_2') = 2$ and $f(x_2'') = f(x_2) = 0$. Then, reassigning to x_1 the weight 2, reassigning to x_1' and x_0 the weight 0, reassigning to x_2' and x_2'' the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Hence, we assume that $f(x_0) = 1$ and $f(x_4) = 2$. Clearly, $f(x_5) = 0$, $f(x_3) \neq 2$, $f(x_1) \neq 2$. Then, $f(x_0) + f(x_1) + f(x_1') = 3$ and so as before we can assume that $f(x_2) = 0$ and $f(x_2') = 2$ and $f(x_2'') = 0$. Then, reassigning to x_1 the weight 2, reassigning to x_1' and x_0 the weight 0, reassigning to x_2' and x_2'' the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction.

Case 3. $\deg(x_1) = 2$ and $\deg(x_4) = 2$.
Assume that $\deg(x_2) = 2$. Then by Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_2)\} = \{1, 2\}$. We first assume that $f(x_0) = 2$ and $f(x_2) = 1$. Clearly $f(x_1) = 0$ and $f(x_3) \neq 2$. Then, reassigning to x_1 the weight 2, reassigning to x_0 and x_2 the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Hence, we assume that $f(x_0) = 1$ and $f(x_2) = 2$. If $f(x_3) \neq 2$, then reassigning to x_1 the weight 2, reassigning to x_0 and x_2 the weight 0 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Now assume that $f(x_3) = 2$. Then, reassigning to x_1 and x_0 the weight 1, reassigning to x_2 the weight 0, and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Hence, $\deg(x_2) \geq 3$ and similarly $\deg(x_3) \geq 3$.

Assume that $\deg(x_2) \geq 4$. We can assume that any vertex in $N(x_2) - \{x_3\}$ is a leaf or a weak support vertices. We first assume that x_2 is a strong support vertex. Let $\{x_2', x_2''\} \subseteq L(x_2)$. It follows from Theorem 2.3, that there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $f(x_2') = 1$, $f(x_2'') = 2$. Then clearly $f(x_2) = 0$, $f(x_3) \neq 2$. Also, we can assume that for $u \in N(x_2) - (L(x_2) \cup \{x_3\})$, $f(u) + f(u') = 2$, where u' is leaf neighbor of u. Then, reassigning to x_2 the weight 2, reassigning to $u \in N(x_2) - \{x_3\}$ the weight 0, reassigning to each leaf of tree $T - T_{x_3}$ the weight 1, reassigning to vertex x_3 the weight $\max\{1, f(x_3)\}$ and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Next, assume that x_2 is not a strong support vertex. Then, there is weak support vertex $x_2' \in N(x_2) - \{x_1, x_3\}$. Let x_2'' is leaf adjacent to x_2'. Then, by Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x_0), f(x_2'')\} = \{1, 2\}$. Without loss of generality, we may assume that $x_0 \in V_2$ and $x_2'' \in V_1$. Clearly, $f(x_1) = 0$, $f(x_2) \neq 2$. Hence, for each leaf u adjacent to x_2, $f(u) \geq 1$ and for any support vertex $u \in N(x_2) - \{x_3\}$, $f(u) + f(u') = 2$, where u' is the leaf adjacent to u. If $f(x_3) = 0$, then for each leaf v adjacent to x_3,$f(v) \geq 1$ and for any support vertex $v \in N(x_3) - \{x_2\}$, $f(v) + f(v') =$

2, where v' is the leaf adjacent to v. Hence $w(f) \geq 2|K_2| + 2|K_3| + \ell_{x_2} + \ell_{x_3}$. Then, function $g = (N(x_2), V(T) - N[x_2], x_2)$ is a perfect Roman dominating function with weight $\ell_{x_3} + 2|K_3| + |K_2| + 2$ and so $w(g) = \ell_{x_3} + 2|K_3| + |K_2| + 2 < 2|K_2| + 2|K_3| + \ell_{x_2} + \ell_{x_3} = w(f) = \gamma_R^P(T)$, a contradiction. Next, assume that $f(x_3) \neq 0$. Let $T' = T - T_{x_3}$. Then reassigning to x_2 the weight 2, reassigning to $u \in N(x_2) - \{x_3\}$ the weight 0, reassigning to each leaf of tree T' the weight 1 and leaving all other weights unchanged produces a new PRDF with weight less than $\gamma_R^P(T)$, a contradiction. Hence, $\deg(x_2) = 3$ and similarly $\deg(x_3) = 3$. Let $N(x_2) = \{x'_2, x_1, x_3\}$ and $N(x_3) = \{x'_3, x_4, x_2\}$. We first assume that x'_2 is a leaf. If x_3 be a support vertex, then by Theorem 2.3, there is a $\gamma_R^P(T)$-function $f = (V_0, V_1, V_2)$ such that $\{f(x'_2), f(x'_3)\} = \{1, 2\}$. Without loss of generality, we may assume that $f(x'_2) = 2$ and $f(x'_3) = 1$. Clearly, $f(x_2) = 0$, $f(x_3) \neq 2$, $f(x_0) + f(x_1) = 2$, $f(x_4) + f(x_5) = 2$ and so $w(f) \geq 7 > \gamma_R^P(T) = 6$, a contradiction. Hence, x'_3 is a weak support vertex and so $T = T_1$. Now, assume that x'_2 is a support vertex. Then $T \in \{T_1, T_2\}$. The converse part is obvious. $\square$

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