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# DEGREE UPPER BOUNDS FOR H-BASES

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The main objective of this paper is to present upper bounds for the degree of H-bases of polynomial ideals. For this purpose, we introduce the new concept of *reduced H-bases* and show that the maximal degree of the elements of any reduced H-basis of an ideal is independent of the choice of the basis. Furthermore, we show that, given an ideal, this maximal degree is invariant after performing any linear change of variables on the ideal. These results allow us to establish explicit degree upper bounds in the case of either a zero-dimensional ideal or an ideal generated by a regular sequence.

## 1. Introduction

The concept of *Gröbner bases*, introduced by Buchberger in 1965 in his PhD thesis [9], has played an important role in the development of computational algebraic geometry. This concept is an important ingredient to study various problems in science and engineering. Furthermore, it has many applications in different areas such as optimization, coding theory/cryptography, signal and image processing, robotics, statistics and so on, see e.g. [8].

One of the main drawbacks of Gröbner bases is the fact that they are based on term orderings [14] and therefore computing a Gröbner basis of an ideal generated by a set of symmetric polynomials may break the symmetry among

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the variables. In addition, Gröbner bases are not numerically stable. Hence, investigating for alternative bases for polynomial ideals which are not tied to a term ordering (which behave numerically stable under small perturbations) is worthwhile. In this direction, several theories have been developed. One of the attractive tools for this purpose is the theory of *border bases*, which was introduced in [37]. These bases behave numerically better than Gröbner bases and preserve also the symmetry; for a discussion of their properties, see [30]. For a Gröbner-free normal form construction using linear algebra tools in the special case of vanishing ideals of points, we refer to [34]. On the other hand, as another numerically stable tool, Macaulay in 1914 in [35] introduced the concept of *H-bases* which is independent of monomial orderings and merely tied to the maximal degree (homogeneous) part of a polynomial. Indeed, he computed an H-basis for a specific example by determining syzygies among the leading parts of the polynomials. The original motivation of Macaulay was to transform systems of polynomial equations into simpler ones. Although this notion was known long before Gröbner bases, due to the lack of symbolic methods for computing H-bases, they were not developed as much as Gröbner bases. We shall note that the structure of H-bases in a constructive way was first studied in [52] and later on in [40]. The main algorithm to construct these bases in a monomial free fashion was introduced in [46] by using a reduction algorithm which is a straightforward extension of the division algorithm which uses monomial ordering. This generalized reduction provides a facility to characterize H-bases which yields a natural extension of Buchberger's algorithm to compute H-bases which will be referred to as the *H-Basis algorithm* in this paper. For a comprehensive discussion on the construction of H-bases, their properties and their applications to the solution of polynomial systems and to some numerical analysis problems such as interpolation problem, we refer to [41, 46].

In this paper, we are interested in giving upper bounds for the maximal degree of the elements of an H-basis of a given ideal. To the knowledge of the authors, this subject has not been studied in literature. For practical applications and in particular, for the implementation of algorithms in computer algebra systems, it is important to establish upper bounds for the complexity of determining an H-basis. Using Lazard's algorithm [32], a good measure to estimate such a bound for Gröbner bases, is an upper bound for the degrees of the intermediate polynomials during the Gröbner basis computation. To review some of the existing results for Gröbner bases, let $\mathcal{P}$ be the polynomial ring $\mathbb{K}[x_1,\ldots,x_n]$ where $\mathbb{K}$ is a field of characteristic zero and $\mathcal{I} \subset \mathcal{P}$ be an ideal generated by *homogeneous* polynomials of degree at most $d$ with $\dim(\mathcal{I}) = D$. The first doubly exponential upper bounds were proven by Bayer, Möller, Mora and Giusti, see [42, Chapter 38] for a comprehensive review of this topic. Based on results due to Bayer [1] and Galligo [19, 20], Möller and Mora [39] provided the upper bound $(2d)^{(2n+2)^{n+1}}$ for any Gröbner basis of $\mathcal{I}$. They also proved that this doubly exponential behavior cannot be improved. Simultaneously, Giusti [21] showed the upper bound $(2d)^{2^{n-1}}$ for the degree of the reduced Gröbner basis (w.r.t. the degree reverse lexicographic order) of $\mathcal{I}$ when the ideal is in *generic position*. Then, using a self-contained and constructive combinatorial argument, Dubé [16] proved the so far sharpest degree bound $2(d^2/2+d)^{2^{n-1}} \sim 2d^{2^n}$. Mayr and Ritscher [38], following the tracks of Dubé [16], obtained the dimension-dependent upper bound $2(1/2d^{n-D}+d)^{2^{D-1}}$ for every reduced Gröbner basis of $\mathcal{I}$. Finally, using a combination of Hermann's bound and the bound given by Dubé, Wiesinger-Widi [53] was able to give new bounds for Gröbner bases computation, see also [43, page 800].

In this article, we give first an upper bound for an H-basis in terms of the Hilbert regularity, satiety and dimension of the ideal. In addition, we discuss this bound in generic position by introducing the new notion of reduced H-bases and showing that the maximal degree of the elements of an H-basis of an ideal is independent of the choice of the basis and is stable after performing any linear change of variables on the ideal. Our study shows that in general H-bases may be reached earlier than Gröbner bases. Since in our approach, we apply Pommaret bases, let us give some historical remarks on these bases. Pommaret division was introduced by Janet [28] in order to apply Cartan test [10–12] for producing, essentially, an H-basis of minimal degree. This result and its relation with Castelnuovo-Mumford regularity was discussed in [36], see also [13].

The rest of the paper is organized as follows. Secs. 2, 3 and 4 are devoted to review some preliminaries on Gröbner bases, Pommaret bases, and H-bases, respectively. In these sections, we give some helpful properties of these bases needed throughout this paper. In Sec. 4, we state and prove an upper bound for the degrees of the elements in an H-basis of a given ideal. Finally, in Sec. 5 we analyse this bound in generic position and show that in general H-bases may be reached earlier than Gröbner bases.

## 2. Gröbner bases

In this section, we review the basic notations and some definitions related to Gröbner bases that we use in this article. Throughout this paper, we will employ the following notations. Let $\mathcal{P} = \mathbb{K}[x_1,\dots,x_n]$ be the polynomial ring over $\mathbb{K}$ where $\mathbb{K}$ is an infinite field and $n \geq 2$. We consider also the (not necessarily homogeneous) polynomials $f_1,\dots,f_k \in \mathcal{P}$ with $k \geq 2$ and the ideal $\mathcal{I} = \langle f_1,\dots,f_k\rangle$ generated by these polynomials. Assume that $f_i$ is (non-zero and) of total degree $d_i$. We number the $f_i$'s in order that $d_1 \geq d_2 \geq \cdots \geq d_k \geq 2$ and set $d = d_1$. Further, we denote by $\mathcal{R} = \mathcal{P}/\mathcal{I}$ the corresponding factor ring and by $D = \dim(\mathcal{I})$

its Krull dimension. The following convention is adopted in this paper: We shall use capital letters to designate homogeneous polynomials. Also, $\mathcal{J}$ denotes the homogeneous ideal in $\mathcal{P}$ generated by $F_1,\ldots,F_k$ and the other notations remain the same as earlier.

A power product of the variables $x_1,\ldots,x_n$ is called a *term* and $\mathcal{T}$ denotes the monoid of all terms in $\mathcal{P}$. We fix the term ordering $\prec$ on $\mathcal{P}$ given by the reverse degree lexicographic ordering with $x_n \prec \cdots \prec x_1$. The *leading term* of a polynomial $0 \neq f \in \mathcal{P}$, denoted by $\mathrm{LT}(f)$, is the greatest term (with respect to $\prec$) appearing in $f$ and its coefficient is the *leading coefficient* of $f$ and we denote it by $\mathrm{LC}(f)$. The *leading monomial* of $f$ is the product $\mathrm{LM}(f) = \mathrm{LC}(f)\mathrm{LT}(f)$. The *leading term ideal* of $\mathcal{I}$ is defined to be $\mathrm{LT}(\mathcal{I}) = \langle \mathrm{LT}(f) \mid 0 \neq f \in \mathcal{I}\rangle$. For a finite set $F = \{f_1,\ldots,f_k\} \subset \mathcal{P}$, let $\mathrm{LT}(F)$ be the set $\{\mathrm{LT}(f_1),\ldots,\mathrm{LT}(f_k)\}$. A finite subset $\{g_1,\ldots,g_m\} \subset \mathcal{I}$ of non-zero polynomials is called a *Gröbner basis* for $\mathcal{I}$ w.r.t. $\prec$ if $\mathrm{LT}(\mathcal{I}) = \langle \mathrm{LT}(G)\rangle$. In the sequel, $\deg(\mathcal{I},\prec)$ stands for the maximal degree of the elements of the reduced Gröbner basis of $\mathcal{I}$ with respect to $\prec$. We refer e.g. to [14] for more details on Gröbner bases.

If $X$ is a graded module or a homogeneous ideal and $s$ is a positive integer, we denote by $X_s$ the $\mathbb{K}$-vector space of elements of $X$ of degree $s$. To define the Hilbert regularity of a homogeneous ideal $\mathcal{J} \subset \mathcal{P}$, recall that the *Hilbert function* of $\mathcal{J}$ is defined by $\mathrm{HF}_{\mathcal{J}}(t) = \dim_{\mathbb{K}}(\mathcal{R}_t)$ where $\dim_{\mathbb{K}}(\mathcal{R}_t)$ denotes the dimension of $\mathcal{R}_t$ as a $\mathbb{K}$-vector space. From a certain degree, this function of $t$ is equal to a polynomial in $t$, called *Hilbert polynomial*, and denoted by $\mathrm{HP}_{\mathcal{J}}$ (see e.g. [14]). It is useful to remark that if $\mathcal{J}$ is zero-dimensional then $\mathrm{HP}_{\mathcal{J}}(t) = 0$ has degree $-1$. Thus, in this case, the affine variety of $\mathcal{J}$ has finitely many elements, however its projective variety is the empty set. So, the projective dimension of $\mathcal{J}$ is $-1$. Otherwise, if $D > 0$, the degree of $\mathrm{HP}_{\mathcal{J}}$ is equal to $D-1$. Note that throughout this article, for each ideal (even if the ideal is homogeneous) we use its Krull dimension which corresponds to the dimension as affine variety (and not as projective variety).

**Definition 2.1.** The *Hilbert regularity* of $\mathcal{J}$ is

$$\mathrm{hilb}(\mathcal{J}) = \min\{m \mid \forall t \geq m,\ \mathrm{HF}_{\mathcal{J}}(t) = \mathrm{HP}_{\mathcal{J}}(t)\}.$$

Recall that the *Hilbert series* of a homogeneous ideal $\mathcal{J} \subset \mathcal{P}$ is the following power series

$$\mathrm{HS}_{\mathcal{J}}(t) = \sum_{s=0}^{\infty} \mathrm{HF}_{\mathcal{J}}(s)t^s.$$

It is well-known that the Hilbert series of a homogeneous ideal may be expressed as the quotient of two polynomials.

**Proposition 2.2.** *There exists a univariate polynomial $p(t)$ so that $\mathrm{HS}_{\mathcal{J}}(t) = p(t)/(1-t)^D$ with $p(1) \neq 0$. Moreover,* $\mathrm{hilb}(\mathcal{J}) = \max\{0, \deg(p) - D + 1\}$.

For the proof of this proposition we refer to [18, Thm 7, page 130] and [7, Prop. 4.1.12]. From [14, Prop. 4, page 458] we deduce that the Hilbert function of $\mathcal{J}$ is the same as that of $\mathrm{LT}(\mathcal{J})$ and this provides an effective method to compute the Hilbert series of an ideal using Gröbner bases, see e.g. [22]. Similar to the notion of Hilbert function of a homogeneous ideal (in the projective setting), one can define the *affine* Hilbert function for an arbitrary ideal. If $X$ is a module or an ideal and $s$ is a positive integer, we denote by $X_{\leq s}$ the $\mathbb{K}$-vector space of elements of $X$ degree $\leq s$. Then, the affine Hilbert function of $\mathcal{I}$ is ${}^a\mathrm{HF}_{\mathcal{I}}(t) = \dim_{\mathbb{K}}(\mathcal{R}_{\leq t})$. The similar concepts of Hilbert polynomial, Hilbert series and Hilbert regularity may be defined for a given ideal $\mathcal{I}$. It should be noticed that $\mathrm{LT}(\mathcal{I})$ and $\mathcal{I}$ share the same affine Hilbert function. In addition, for any (not necessarily homogeneous) ideal $\mathcal{I}$, the degree of ${}^a\mathrm{HP}_{\mathcal{I}}$ is equal to $\dim(\mathcal{I})$. For example, if an ideal is zero-dimensional then its affine Hilbert polynomial is 1. For a homogeneous ideal $\mathcal{J}$ we have $\mathrm{HF}_{\mathcal{J}}(t) = {}^a\mathrm{HF}_{\mathcal{J}}(t) - {}^a\mathrm{HF}_{\mathcal{J}}(t-1)$, For more details, see [14, Chap. 9].

In all the paper, we keep the following notations concerning the homogenization of an ideal of $\mathcal{P}$. We denote by ${}^h\mathcal{P}$ the ring $\mathbb{K}[x_1,\ldots,x_{n+1}]$ where $x_{n+1}$ is a new variable. For any polynomial $f \in \mathcal{P}$, we consider its homogenization ${}^hf = x_{n+1}^{\deg(f)} f(x_1/x_{n+1},\ldots,x_n/x_{n+1}) \in {}^h\mathcal{P}$. For an ideal $\mathcal{I} \subset \mathcal{P}$, we let $\tilde{\mathcal{I}} = \langle {}^hf_1,\ldots,{}^hf_k\rangle$. Note that this notion is not well-defined, because $\tilde{\mathcal{I}}$ depends on the basis $f_1,\ldots,f_k$. In addition, the homogenization of $\mathcal{I}$ is defined as ${}^h\mathcal{I} = \langle {}^hf \mid f \in \mathcal{I}\rangle \subset {}^h\mathcal{P}$. From loc. cit., we know that ${}^a\mathrm{HF}_{\mathcal{I}}(t) = \mathrm{HF}_{{}^h\mathcal{I}}(t)$.

We end this section by giving the definition of the *degree* of a homogeneous ideal from [23, page 52]. Assume that $\mathcal{J}$ is a homogeneous ideal of dimension $D$. If $D > 0$, then the degree of $\mathcal{J}$, denoted by $\deg(\mathcal{J})$, is $(D-1)!$ times the leading coefficient of the Hilbert polynomial of $\mathcal{J}$. If $D = 0$, then it is defined to be the sum of the coefficients of $\mathrm{HS}_{\mathcal{I}}(t)$.

## 3. Pommaret bases

In this section, we give some basic properties of Pommaret bases which are used in the subsequent sections. We follow the notations fixed in Sec. 2.

Given a polynomial $f \in \mathcal{P}$ with $\mathrm{LT}(f) = x^{\alpha}$ where $\alpha = (\alpha_1,\ldots,\alpha_n)$, the *class* of $f$ is the integer $\mathrm{cls}(f) = \max\{i \mid \alpha_i \neq 0\}$. Then the *multiplicative variables* of $f$ are $\mathcal{X}_P(f) = \{x_{\mathrm{cls}(f)},\ldots,x_n\}$. The term $x^{\beta}$ is a *Pommaret divisor* of $x^{\alpha}$, and we write $x^{\beta} \mid_P x^{\alpha}$, if $x^{\beta} \mid x^{\alpha}$ and $x^{\alpha-\beta} \in \mathbb{K}[\mathcal{X}_P(x^{\beta})]$.

**Definition 3.1.** Suppose that $H \subset \mathcal{I}$ is a finite set such that no leading term of an element of $H$ is a Pommaret divisor of the leading term of another element.

Then $H$ is a *Pommaret basis* of $\mathcal{I}$, if

$$\mathrm{LT}(\mathcal{I}) = \bigoplus_{h \in H} \mathbb{K}[\mathcal{X}_P(h)] \cdot \mathrm{LT}(h) .$$

From this definition, it follows that a Pommaret basis remains a Gröbner basis of the ideal that it generates, however the converse may not be true. On the other hand, Pommaret bases do not always exist. If the base field $\mathbb{K}$ is infinite then any ideal has a Pommaret basis after a generic linear change of variables [48]. Indeed, *quasi stable position* is exactly the notion of generic position to characterize the existence of finite Pommaret bases.

**Definition 3.2.** A monomial ideal $\mathcal{J}$ is called *quasi stable*, if for any term $m \in \mathcal{J}$ and all integers $i, j, s$ with $1 \leq j < i \leq n$ and $s > 0$, if $x_i^s \mid m$ there exists an integer $t \geq 0$ such that $x_j^t m / x_i^s \in \mathcal{J}$. A homogeneous ideal is in *quasi stable position* if its leading term ideal is quasi stable.

**Proposition 3.3.** *[48] A homogeneous ideal $\mathcal{J}$ has a Pommaret basis, if and only if it is in quasi stable position.*

Let us now recall the definition of a regular sequence and the depth of an ideal. A sequence of polynomials $f_1, \ldots, f_k \in \mathcal{P}$ is called *regular* if $f_i$ is a non-zero divisor in the ring $\mathcal{P}/\langle f_1, \ldots, f_{i-1} \rangle$ for $i = 2, \ldots, k$. It is shown that the Hilbert series of a regular sequence of homogeneous polynomials $F_1, \ldots, F_k \in \mathcal{P}$ is equal to $\prod_{i=1}^{k}(1 - t^{d_i})/(1 - t^n)$ where $d_i = \deg(F_i)$, see e.g. [18, 33]. Given a homogeneous ideal $\mathcal{J}$, a sequence of homogeneous polynomials $G_1, \ldots, G_t \in \mathcal{P}$ is called *almost regular* on $\mathcal{P}/\mathcal{J}$ if $G_i$ for $i = 2, \ldots, t$ is a non-zero divisor on the ring $\mathcal{P}/(\mathcal{J} + \langle G_1, \ldots, G_{i-1} \rangle)_{\geq s}$ for some $s$ sufficiently large. In this case, we say that $G_i$ is an almost non-zero divisor on $\mathcal{P}/(\mathcal{J} + \langle G_1, \ldots, G_{i-1} \rangle)$. This is equivalent to the condition that $G_i$ is outside of all associated primes of $\mathcal{J} + \langle G_1, \ldots, G_{i-1} \rangle$ except the maximal homogeneous ideal of $\mathcal{P}$. For more details, see e.g. [31, page 290]. The *depth* of the homogeneous ideal $\mathcal{J}$ is defined to be the maximum integer $\lambda$ so that there exists a regular sequence of linear homogeneous forms $y_1, \ldots, y_\lambda$ on $\mathcal{P}/\mathcal{J}$. Furthermore, $\mathcal{P}/\mathcal{J}$ is called *Cohen-Macaulay* if $\mathrm{depth}(\mathcal{J}) = \dim(\mathcal{J})$. Note that if $\mathcal{P}/\mathcal{J}$ is Cohen-Macaulay then $\mathcal{J}$ is unmixed[1], see [33, Prop. 4.3.1, page 109]. It is well-known that if $f_1, \ldots, f_k \in \mathcal{P}$ is a regular sequence then $\langle f_1, \ldots, f_k \rangle$ is unmixed, see e.g. [7, Thm. 2.1.6].

Let us recall the definitions of satiety and Castelnuovo-Mumford regularity of a homogeneous ideal. Let $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$ be the unique maximal homogeneous ideal of $\mathcal{P}$. The *saturation* of $\mathcal{J}$ is defined as $\mathcal{J} : \mathfrak{m}^\infty = \bigcup_{i=1}^{\infty} \mathcal{J} : \mathfrak{m}^i$ and is denoted by $\mathcal{J}^{\mathrm{sat}}$. The ideal $\mathcal{J}$ is called saturated if $\mathcal{J} = \mathcal{J}^{\mathrm{sat}}$. Equivalently, $\mathcal{J}$

[1] An ideal is unmixed if all associated primes of the ideal share the same dimension.

is saturated iff the maximal homogeneous ideal $\langle x_1,\dots,x_n\rangle$ is not an associated prime of $\mathcal{J}$.

**Definition 3.4.** The *satiety* of $\mathcal{J}$, denoted by $\mathrm{sat}(\mathcal{J})$, is the smallest positive integer $m$ such that $\mathcal{J}_s=\mathcal{J}_s^{\mathrm{sat}}$ for all $s\geq m$.

From Noetherianity of $\mathcal{P}$, it follows that the satiety of a homogeneous ideal is always finite.

**Definition 3.5.** A homogeneous ideal $\mathcal{J}\subset\mathcal{P}$ is *m-regular*, if there exists a minimal graded free resolution:

$$0\longrightarrow\bigoplus_j\mathcal{P}(e_{rj})\longrightarrow\cdots\longrightarrow\bigoplus_j\mathcal{P}(e_{1j})\longrightarrow\bigoplus_j\mathcal{P}(e_{0j})\longrightarrow\mathcal{J}\longrightarrow 0$$

of $\mathcal{J}$ such that $e_{ij}-i\leq m$ for each $i,j$. The *Castelnuovo-Mumford regularity* of $\mathcal{J}$ is the smallest $m$ such that $\mathcal{J}$ is $m$-regular; we note it by $\mathrm{reg}(\mathcal{J})$.

For more details on the regularity, we return to [2, 4, 17, 44]. By a well-known result, $\mathrm{reg}(\mathcal{I})$ is an upper bound for the degrees of the Gröbner basis elements, in generic coordinates and for the degree reverse lexicographic ordering. This upper bound is reached if the characteristic of $\mathbb{K}$ is zero, see [2]. From [24], it is helpful to highlight an important relation between Hilbert regularity, satiety and Castelnuovo-Mumford regularity by the equality

$$\begin{aligned}\mathrm{reg}(\mathcal{J})&=\max\{\mathrm{sat}(\mathcal{J}),\mathrm{sat}(\mathcal{J}+\langle L_1\rangle),\dots,\mathrm{sat}(\mathcal{J}+\langle L_1,\dots,L_D\rangle)\}\\&=\max\{\mathrm{hilb}(\mathcal{J}),\mathrm{hilb}(\mathcal{J}+\langle L_1\rangle),\dots,\mathrm{hilb}(\mathcal{J}+\langle L_1,\dots,L_D\rangle)\}\end{aligned}$$

where $L_1,\dots,L_D$ is an almost regular sequence of linear polynomials for $\mathcal{P}/\mathcal{J}$.

We conclude this section by listing some helpful properties of the ideals in quasi stable position from [25, Thm. 16]. Assume that $\mathcal{J}$ is a homogeneous ideal (in quasi stable position) having a finite Pommaret basis $H$. Then, we have $\mathrm{reg}(\mathcal{J})$ equals the maximal degree of the elements of $H$. In addition, $\mathrm{depth}(\mathcal{J})=\mathrm{depth}(\mathrm{LT}(\mathcal{J}))$ is given by $n$ minus the maximal class of an element of $H$. Also, $\mathcal{J}^{\mathrm{sat}}=\mathcal{J}:x_n^{\infty}$ and $\mathrm{sat}(\mathcal{J})=\mathrm{sat}(\mathrm{LT}(\mathcal{J}))$ which equals the maximal degree of an element of class $n$ in $H$. Finally, $\mathcal{P}/\mathcal{J}$ is Cohen-Macaulay, if and only if $\mathcal{P}/\mathrm{LT}(\mathcal{J})$ is Cohen-Macaulay. For more details on the theory of Pommaret bases and its applications, we refer the reader to [47–49].

## 4. H-bases

In this section, we recall briefly the definition of an H-basis and the main properties, that we will need. H-bases can also be defined similar to Gröbner bases, if one replaces the concept of *leading term* by the *leading form*. Given a polynomial $f \in \mathcal{P}$, the leading form of $f$, denoted by $\mathrm{LF}(f)$, is the highest degree homogeneous part of $f$. Indeed, $\mathrm{LF}(f)$ is the sum of all monomials appearing in $f$ with the highest degree. In addition, for a given finite set $F = \{f_1,\dots,f_k\}$ the leading form of $F$ is denoted by $\mathrm{LF}(F) = \{\mathrm{LF}(f_1),\dots,\mathrm{LF}(f_k)\}$. The *leading form ideal* of $\mathcal{I}$ is defined as $\mathrm{LF}(\mathcal{I}) = \langle \mathrm{LF}(f) \mid 0 \neq f \in \mathcal{I}\rangle$.

**Definition 4.1.** A finite set $H = \{h_1,\dots,h_t\} \subset \mathcal{I}$ is called an *H-basis* for $\mathcal{I}$ if

$$\mathrm{LF}(\mathcal{I}) = \langle \mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)\rangle. \tag{1}$$

In the literature these bases are known also under the name *Macaulay* bases, see e.g. [31, page 43]. For more details on the theory of H-bases, we refer to [31, 40, 41, 46]. It can be seen that there is a very close relationship between H-bases and Gröbner bases. It was remarked in [40] that any Gröbner basis w.r.t. any degree compatible term ordering (i.e., for two polynomials $p,q$, if $\deg(p) < \deg(q)$ then $p \prec q$) is also an H-basis. This approach may lead to a naive degree upper bound for H-bases that we discuss in the following. Keeping the notations of Sec. 2, we let $\mathcal{I} = \langle f_1,\dots,f_k\rangle$. Following [18, Def. 31, page 112] we consider a *good extension* $\prec_h$ of $\prec$ defined by the degree reverse lexicographic ordering with $x_{n+1} \prec x_n \prec \cdots \prec x_1$. Suppose that $\{G_1,\dots,G_m\}$ is a Gröbner basis of $\tilde{\mathcal{I}}$ w.r.t. $\prec_h$. By [18, Prop. 34, page 113], we know that $G = \{G_1|_{x_{n+1}=1},\dots,G_m|_{x_{n+1}=1}\}$ forms a Gröbner basis for $\mathcal{I}$ w.r.t. $\prec$. Moreover, from [31, Thm. 4.3.19], it follows that $G$ remains an H-basis for $\mathcal{I}$. Assume that $U(n,d)$ is a function presenting a degree upper bound for the Gröbner basis of a homogeneous ideal $\mathcal{J}$. Then, these arguments entail the naive upper bound $U(n+1,d)$ for the degrees of the polynomials in an H-basis of $\mathcal{I}$. Remark that it is not true that every H-basis is obtained from a Gröbner basis as it was shown in [41], see also Exam. 6.6. This is the crucial property allowing us to present sharper degree upper bounds for H-bases in the next sections.

Below, we give a brief review of the construction of H-bases. This algorithm is based on a *reduction procedure* which is a fairly straightforward extension of the well-known division algorithm. Assume that $F = \{f_1,\dots,f_k\} \subset \mathcal{P}$ and $f \in \mathcal{P}$. Then, we say that $f$ is reducible modulo $F$ and write $f \longrightarrow_F r$ if $r = f - \sum_{i=1}^{k} g_i f_i, \deg(r) < \deg(f)$ and $\deg(g_i f_i) \leq \deg(f)$. In addition, we say $r$ is a remainder of the division of $f$ by $F$ and write $f \xrightarrow{*}_F r$ if there exists a sequence of polynomials $r_1,\dots,r_m \in \mathcal{P}$ with $f \longrightarrow_F r_1 \longrightarrow_F \cdots \longrightarrow_F r_m = r$ and $r$ is no longer reducible modulo $F$. If a polynomial $f$ is not reducible by $F$, we say that it is *reduced* w.r.t. $F$. We refer the reader to [45] for an efficient reduction process.

**Definition 4.2.** An H-basis $H = \{h_1,\dots,h_t\} \subset \mathcal{P}$ is *reduced* if for each $i$, $h_i$ is reduced w.r.t. $H \setminus \{h_i\}$.

It is worth noting that there may not exist a unique reduced H-basis for a given ideal. As a toy example, let $\mathcal{I} = \langle x^2 - y^2, x^2 + y^2 \rangle$. Then, the sets $\{x^2 - y^2, x^2 + y^2\}$ and $\{x^2, y^2\}$ are two reduced H-bases for $\mathcal{I}$. This generalized reduction provides a characterization of H-bases by means of the *module of syzygies* (referred to also as syzygy module). Let us briefly recall this notion.

**Definition 4.3.** For a given finite sequence $(F_1, \dots, F_k) \in \mathcal{P}^k$ of homogeneous polynomials, the *(first) module of syzygies* is defined as

$$\mathrm{Syz}(F_1, \dots, F_k) = \{(G_1, \dots, G_k) \in \mathcal{P}^k \mid \sum_{i=1}^{k} G_i F_i = 0\}.$$

We shall note that the study of syzygies may be traced back to [27]. It is well-known that $\mathrm{Syz}(F_1, \dots, F_k)$ as an $\mathcal{P}$-module is finitely generated, see [50]. The classical method to construct a basis for such a module by using a Gröbner basis of $\langle F \rangle$, was described in [8]. The following result [46] is useful to characterize H-bases.

**Theorem 4.4.** *Let $H = \{h_1, \dots, h_t\} \subset \mathcal{P}$ and $G$ a basis for* $\mathrm{Syz}(\mathrm{LF}(H))$. *Then $H$ is an H-basis for $\langle H \rangle$ if and only if for each $(G_1, \dots, G_t) \in G$ we have $\sum_{i=1}^{t} G_i h_i \xrightarrow{*}_H 0$.*

According to this theorem, we are able to describe a straightforward extension of Buchberger's algorithm, referred to as the H-Basis algorithm, to compute H-bases, see [46].

**Algorithm 1** H-Basis

**Require:** $H \subset \mathcal{P}$; a finite set
**Ensure:** An H-basis for $\langle H \rangle$
    $G$:=A basis for $\mathrm{Syz}(\mathrm{LP}(H))$
    **while** $G \neq \{\}$ **do**
        select and remove $(G_p \mid p \in H)$ from $G$
        let $\sum_{p \in H} G_p p \xrightarrow{*}_H h$
        **if** $h \neq 0$ **then**
            $H := H \cup \{h\}$
            $G$:=A basis for $\mathrm{Syz}(\mathrm{LF}(H))$
        **end if**
    **end while**
    **Return** $(H)$

## 5. Degree bounds for H-bases

In this section, we prove upper bounds for the degrees of elements in an H-basis of a given ideal. We maintain the notations of Sec. 2. Let us fix some further notations. Throughout this section, for polynomials $h_1,\dots,h_t \in \mathcal{P}$, we denote by $\mathcal{J}$ the homogeneous ideal generated by $\mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)$. For an arbitrary $s$ let

$$\mathcal{J}_s = \{\textstyle\sum_{i=1}^t G_i \mathrm{LF}(h_i) \mid G_i \in \mathcal{P}_{s-\deg(h_i)}\} ,$$

$$\mathrm{Syz}_s(\mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)) = \{(G_1,\dots,G_t) \mid \textstyle\sum_{i=1}^t G_i \mathrm{LF}(h_i) = 0, G_i \in \mathcal{P}_{s-\deg(h_i)}\} .$$

Indeed, the ideal generated by $h_1,\dots,h_t$ can eventually contain polynomials $f$ such that $\mathrm{LF}(f)$ is not included in $\mathcal{J}$. This leads to the definition of H-bases. Following [32], we define below the notion of *truncated H-bases*.

**Definition 5.1.** The set $\{h_1,\dots,h_t\} \subset \mathcal{I}$ is called an *H-basis* for $\mathcal{I}$ up to (degree) $K$, if one of the following equivalent conditions holds for all non-negative integers $s \le K$.

(1) If $f \in \mathcal{I}_{\le s}$ then $f = \sum_{i=1}^t p_i h_i$ with $p_i \in \mathcal{P}_{\le s-\deg(h_i)}$ for $i = 1,\dots,t$.

(2) If $p \in \mathcal{I}$ and $\deg(p) = s$, then $\mathrm{LF}(p) \in \mathcal{J}_s$.

(3) For each $(G_1,\dots,G_t) \in \mathrm{Syz}_s(\mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t))$, there exists $p_1,\dots,p_t$ in $\mathcal{P}$ such that $\sum_{i=1}^t G_i h_i = \sum_{i=1}^t p_i h_i$ with $p_i h_i \in \mathcal{P}_{\le s-1}$.

In particular, the set $\{h_1,\dots,h_t\}$ is an H-basis for $\mathcal{I}$, if it is an H-basis up to $K$ for all $K$.

The main obstacle in constructing H-bases is to find conditions for an H-basis up to a fixed $K$, which allow to conclude that it is already an H-basis.

In the sequel we shall need the following lemmatas in which $L$ (resp. $\ell$) denotes always a (resp. not necessarily) homogeneous linear polynomial in $\mathcal{P}$.

**Lemma 5.2.** *Let $H = \{h_1,\dots,h_t\} \subset \mathcal{I}$, $\mathcal{J} = \langle \mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)\rangle \subset \mathcal{P}$ and $K$ an arbitrary positive number. Let $L \in \mathcal{P}_1$ be a homogeneous linear non-zero divisor on $\mathcal{P}/\mathcal{J}$. If $H$ is an H-basis for $\mathcal{I}$ up to $K$ then $\{L,h_1,\dots,h_t\}$ is also an H-basis for the ideal it generates up to $K$ .*

*Proof.* We prove the assertion by showing that (3) of Def. 5.1 holds, i.e., for every

$$(G_0,G_1,\dots,G_t) \in \mathrm{Syz}_s(L,\mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t))$$

and $s \le K$ we have to show that there exist $p_0,p_1,\dots,p_t$ such that we have $G_0 L + \sum_{i=1}^t G_i h_i = p_0 L + \sum_{i=1}^t p_i h_i$ with $p_0 \in \mathcal{P}_{\le s-2},\ p_i h_i \in \mathcal{P}_{\le s-1}$. Now, two cases may occur: Case 1 Let $G_0 = 0$, then $(G_1,\dots,G_t) \in \mathrm{Syz}_s(\mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t))$. Since $H$ is an H-basis up to $K$, from (3) of Def. 5.1 it follows that there are $g_1,\dots,g_t$ such that $G_0 L + \sum_{i=1}^t G_i h_i = 0\cdot L + \sum_{i=1}^t g_i h_i$ with $g_i h_i \in \mathcal{P}_{\le s-1}$. Thus in this case the assertion holds. Case 2 If $G_0 \ne 0$ then $G_0 L = -\sum_{i=1}^t G_i \mathrm{LF}(h_i) \in \mathcal{J}$.

Therefore from assumption we have $G_0 \in \mathcal{J}$. Since $G_0 \in \mathcal{P}_{s-1}$, it has a representation of the form $G_0 = \sum_{i=1}^{t} U_i \mathrm{LF}(h_i)$ where $U_i \in \mathcal{P}_{s-1-\deg(h_i)}$ is homogeneous. Substituting this $G_0$ in the representation for $G_0 L$ gives

$$(U_1 L + G_1, \ldots, U_t L + G_t) \in \mathrm{Syz}_s(\mathrm{LF}(h_1), \ldots, \mathrm{LF}(h_t)) .$$

Using the fact that that $H$ is an H-basis up to $K$, we find $g_i \in \mathcal{P}_{\leq s-1-\deg(h_i)}$, such that $\sum_{i=1}^{t}(U_i L + G_i) h_i = \sum_{i=1}^{t} g_i h_i$. It is clear that $G_0 = \mathrm{LF}(\sum_{i=1}^{t} U_i h_i)$, and in turn we can write

$$G_0 L + \sum_{i=1}^{t} G_i h_i = (G_0 - \sum_{i=1}^{t} U_i h_i) L + \sum_{i=1}^{t} g_i h_i .$$

This entails that in both cases that condition $(3)$ of Def. 5.1 holds true. Therefore $H \cup \{L\}$ is an H-basis for the ideal it generates up to $K$. □

**Lemma 5.3.** *Suppose that $H = \{h_1, \ldots, h_t\} \subset \mathcal{I}$ and $\ell \in \mathcal{P}$ is a linear non-zero divisor on $\mathcal{P}/\langle h_1, \ldots, h_t \rangle$. If $H \cup \{\ell\}$ is an H-basis for the ideal it generates then $H$ is an H-basis for $\mathcal{I}$.*

*Proof.* In contrary, assume that $H$ is not an H-basis for $\mathcal{I}$. By $(1)$ of Def. 5.1, there exists $f \in \mathcal{I}$ with $\deg(f) = s$ such that it cannot be written as a combination of degree at most $s$ of the $h_i$'s. We may assume that $f$ has minimal degree with this property. Since $\{\ell, h_1, \ldots, h_t\}$ is an H-basis, polynomials $p_0 \in \mathcal{P}_{\leq s-1}$ and $p_i \in \mathcal{P}_{\leq s-\deg(p_i)}$ for $i = 1, \ldots, t$ exist satisfying $f = p_0 \ell + \sum_{i=1}^{t} p_i h_i$. It follows from assumption of our lemma that $p_0 \in \langle h_1, \ldots, h_t \rangle$. Since $p_0 \in \mathcal{P}_{\leq s-1}$, by the choice of $f$, one obtains polynomials $u_1, \ldots, u_t$ so that $p_0 = \sum_{i=1}^{t} u_i h_i$, with $u_i h_i \in \mathcal{P}_{\leq s-1}$, $i = 1, \ldots, s$. Substituting this $p_0$ in the representation for $f$, one obtains $f = \sum_{i=1}^{t}(p_i + u_i \ell) h_i$ with $(p_i + u_i \ell) p_i \in \mathcal{P}_{\leq s}$ for $i = 1, \ldots, s$. Hence we will arrive at a contradiction. □

The following lemma may be well-known, however, since we could not find the exact statement that we need in the literature, we give a proof for the sake of completeness.

**Lemma 5.4.** *With the above notations, assume that $\mathcal{J}$ is saturated . Then, there exists a linear homogeneous polynomial which is a non-zero divisor on $\mathcal{P}/\mathcal{J}$.*

*Proof.* Let $\{P_1, \ldots, P_m\}$ be the set of all associated primes of $\mathcal{J}$. From assumption, we may infer that the maximal homogeneous ideal of $\mathcal{P}$ does not belong to this set. Consider the $\mathbb{K}$-linear space $S = \mathbb{K}^n$ and for each integer $i = 1, \ldots, m$ we define the subspace

$$S_i = \{(a_1, \ldots, a_n) \in S \mid a_1 x_1 + \cdots + a_n x_n \in P_i\}$$

It is easy to see that $S_i$ is a subspace of $S$. We claim that $S_i \neq S$ for each $i$. Otherwise, $x_1,\dots,x_n \in S_i$ and it follows that $P_i = \langle x_1,\dots,x_n\rangle$ which leads to a contradiction. Since $\mathbb{K}$ is infinite, $S \neq S_1 \cup \cdots \cup S_m$ by an elementary result from linear algebra. This implies that there exists a linear homogeneous polynomial which is an non-zero divisor on $\mathcal{P}/\mathcal{J}$. □

Below, we show that a non-zero divisor for the leading form ideal of a given ideal remains a non-zero divisor for the ideal, cf. [31, Prop. 5.6.34].

**Lemma 5.5.** *Let $H = \{h_1,\dots,h_t\} \subset \mathcal{I}$ be an H-basis for $\mathcal{I}$ up to $K$ where $K$ is an arbitrary positive number and $\mathcal{J} = \langle \mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)\rangle$. If a linear homogeneous polynomial $L \in \mathcal{P}$ is a non-zero divisor on $\mathcal{P}_{<K}/\mathcal{J}_{<K}$ then it is a non-zero divisor on $\mathcal{P}_{<K}/\mathcal{I}_{<K}$, too.*

*Proof.* We proceed by reductio ad absurdum. Assume that there exists $f \notin \mathcal{I}$ of degree $< K$ so that $L.f \in \mathcal{I}$. Without loss of generality, we may assume that $f$ is reduced w.r.t. $H$. From the membership $\mathrm{LF}(L.f) = L.\mathrm{LF}(f) \in \mathcal{J}$ and the assumptions, we conclude that $\mathrm{LF}(f) \in \mathcal{J}$. It follows that $f$ is reducible by $H$ which leads to a contradiction and this proves the assertion. □

Let us review a few concepts related to Hilbert function before proving the main results of this section. If $H = \{h_1,\dots,h_t\}$ is an H-basis for $\mathcal{I}$ up to $K$ and generates the ideal $\mathcal{I}$, then the Hilbert functions of the ideals $\mathcal{I}$ and $\mathcal{J} = \langle \mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)\rangle$ are correlated by ${}^a\mathrm{HF}_{\mathcal{J}}(s) = {}^a\mathrm{HF}_{\mathcal{I}}(s)$ for all $s \leq K$, see [31, Prop. 5.6.3]. Thus, we can write

$$\mathrm{HF}_{\mathcal{J}}(s) = {}^a\mathrm{HF}_{\mathcal{I}}(s) - {}^a\mathrm{HF}_{\mathcal{I}}(s-1)$$

for all $s \leq K$. Based on this property, and following [7, page 148], we define *backward differences* of the affine Hilbert function of $\mathcal{I}$, inductively, by

$$\begin{aligned} \nabla_0{}^a\mathrm{HF}_{\mathcal{I}}(s) &= {}^a\mathrm{HF}_{\mathcal{I}}(s), \\ \nabla_{i+1}{}^a\mathrm{HF}_{\mathcal{I}}(s) &= \nabla_i{}^a\mathrm{HF}_{\mathcal{I}}(s) - \nabla_i{}^a\mathrm{HF}_{\mathcal{I}}(s-1) \end{aligned}$$

for each $i = 0,1,2,\dots$.

**Proposition 5.6.** *Suppose that $H = \{h_1,\dots,h_t\}$ generates $\mathcal{I}$ and is an H-basis up to $K-1$ for some $K$. Furthermore, suppose that $\mathcal{J} = \langle \mathrm{LF}(h_1),\dots,\mathrm{LF}(h_t)\rangle$ is saturated and $\nabla_{m+1}{}^a\mathrm{HF}_{\mathcal{I}}(K) = 0$ for some $m \geq 0$. Then $\mathcal{I}$ has an H-basis containing only polynomials of degree $\leq K$ and $\nabla_{m+1}{}^a\mathrm{HF}_{\mathcal{I}}(s) = 0$ for all $s > K$.*

*Proof.* We proceed by induction on $m$. Let $m = 0$, i.e.; $\nabla{}^a\mathrm{HF}_{\mathcal{I}}(K) = 0$. Then ${}^a\mathrm{HF}_{\mathcal{I}}(K) = {}^a\mathrm{HF}_{\mathcal{I}}(K-1)$ and hence

$$\dim_{\mathbb{K}}(\mathcal{I}_{\leq K}) - \dim_{\mathbb{K}}(\mathcal{I}_{\leq K-1}) = \binom{K+n-1}{n-1} = \dim_{\mathbb{K}}(\mathcal{J}_K).$$

Therefore, for every monomial $u = x_1^{i_1} \cdots x_n^{i_n}$ of degree $K$, there is a polynomial $p_u \in \mathcal{I}$ with $\mathrm{LF}(p_u) = u$, such that $\{\mathrm{LF}(p_u) \mid u \in \mathcal{P}_K\}$ is a basis of $\mathcal{P}_K$ and of $\mathcal{J}_K$. Hence we have $\mathcal{J}_K = \mathcal{P}_K$. Multiplication of such $p_u$ by arbitrary monomials of degree $s-K$ gives by an analogous argument that $\mathcal{J}_s = \mathcal{P}_s$ for all $s \geq K$ and so and $\nabla^a \mathrm{HF}_{\mathcal{I}}(s) = 0$ for all $s > K$. Therefore every $f \in \mathcal{I}$ can be reduced to a polynomial $p \in \mathcal{I}$ of degree $< K$, $f = \sum_u q_u p_u + p$ with $\deg(p_u) \leq \deg(f) - K$. On the other hand, $p$ has a representation $p = \sum_{i=1}^t p_i' h_i$ with $\deg(p_i' h_i) < K$ by condition (1) of Def. 5.1. Inserting it into $f = \sum_u q_u p_u + p$ gives a representation (1) of Def. 5.1 which confirms that $\{h_1, \ldots, h_t\} \cup \{p_u \mid u = x_1^{i_1} \cdots x_n^{i_n},\ \deg(u) = K\}$ is an H-basis for $\mathcal{I}$ and the degrees of the elements of this basis is at most $K$.

Assume the induction hypothesis holds for $m-1$. From assumptions and by Lemmatas 5.4 and 5.5, there exists a linear homogeneous polynomial $L \in \mathcal{P}$ which is simultaneously a non-zero divisor on $\mathcal{P}/\mathcal{I}_{<K}$ and $\mathcal{P}/\mathcal{J}$. So, for such an $L$, a classical property of the Hilbert function gives rise to ${}^a\mathrm{HF}_{\mathcal{I}+\langle L\rangle}(K) = {}^a\mathrm{HF}_{\mathcal{I}}(K) - {}^a\mathrm{HF}_{\mathcal{I}}(K-1) = \nabla_1 {}^a\mathrm{HF}_{\mathcal{I}}(K)$, and therefore we can write

$$\nabla_m {}^a\mathrm{HF}_{\mathcal{I}+\langle L\rangle}(K) = \nabla_m \nabla_1 {}^a\mathrm{HF}_{\mathcal{I}}(K) = \nabla_{m+1} {}^a\mathrm{HF}_{\mathcal{I}}(K) = 0\ .$$

Since $L$ is a non-zero divisor on $\mathcal{P}/\mathcal{J}$ then by Lem. 5.2, we have $\{L, h_1, \ldots, h_t\}$ is an H-basis up to $K-1$. Apply then the induction hypothesis to $\mathcal{I} + \langle L\rangle = \langle h_1, \ldots, h_t, L\rangle$ (we note that since $L$ is a non-zero divisor on $\mathcal{P}/\mathcal{J}$ then the ideal $\mathcal{I} + \langle L\rangle$ remains saturated). This gives that there are polynomials $p_1, \ldots, p_r$ of degree at most $K$ so that $\{L, h_1, \ldots, h_t\} \cup \{p_1, \ldots, p_r\}$ is an H-basis and $\nabla_m {}^a\mathrm{HF}_{\mathcal{I}+\langle L\rangle}(s) = 0$ for each $s > K$. Without loss of generality, we may assume that $p_1, \ldots, p_r$ are reduced w.r.t. $L$. It follows that $p_1, \ldots, p_r \in \mathcal{I}$. Since $L$ is a non-zero divisor on the ring $\mathcal{P}/\mathcal{I}$ then, it is a non-zero divisor on the ring $\mathcal{P}/\langle h_1, \ldots, h_t, p_1, \ldots, p_r\rangle$ as well. Lem. 5.3 shows that $\{h_1, \ldots, h_t, p_1, \ldots, p_r\}$ is an H-basis for the ideal it generates, i.e.; $\mathcal{I}$. Finally, for all $s > K$ we have $\nabla_{m+1} {}^a\mathrm{HF}_{\mathcal{I}}(s) = \nabla_m \nabla_1 {}^a\mathrm{HF}_{\mathcal{I}}(s) = \nabla_m {}^a\mathrm{HF}_{\mathcal{I}+\langle L\rangle}(s) = 0$ and this ends the proof. □

**Theorem 5.7.** *The ideal $\mathcal{I}$ possesses an H-basis $H = \{h_1, \ldots, h_t\}$ so that for each i, we have* $\deg(h_i) \leq \max\{d, \max\{\mathrm{hilb}(\mathcal{I}), \mathrm{sat}(\mathrm{LF}(\mathcal{I}))\} + D + 1\}$.

*Proof.* Suppose that $\{G_1, \ldots, G_m\}$ is a Gröbner basis for $\tilde{\mathcal{I}}$ w.r.t. $x_{n+1} \prec \cdots \prec x_1$. As already discussed, $\{G_1|_{x_{n+1}=1}, \ldots, G_m|_{x_{n+1}=1}\} =: \{g_1, \ldots, g_m\}$ forms an H-basis of $\mathcal{I}$ and the set $\{{}^h(G_1|_{x_{n+1}=1})|_{x_{n+1}=0}, \ldots, {}^h(G_m|_{x_{n+1}=1})|_{x_{n+1}=0}\}$ generates $\mathcal{J} = \mathrm{LF}(\mathcal{I})$. For $T = \mathrm{sat}(\mathcal{J})$, consider the ideals $\mathcal{I}_1 = \langle {}^h g_1, \ldots, {}^h g_m\rangle_{\geq T}|_{x_{n+1}=1}$ and $\mathcal{J}_1 = \mathcal{J}_{\geq T}$. One sees readily that the set $\mathcal{I} \setminus \mathcal{I}_1$ has an H-basis $H_1$ of the elements of degree $< T$. Thus, it is enough to show that $\mathcal{I}_1$ has an H-basis of degree $\leq \max\{\mathrm{hilb}(\mathcal{I}), \mathrm{sat}(\mathrm{LF}(\mathcal{I}))\} + D + 1$. Note that $\mathcal{I}$ and $\mathcal{I}_1$ share the same dimension and $\mathcal{J}_1$ is saturated. We first claim that $\mathrm{hilb}(\mathcal{I}_1) \leq \max\{\mathrm{hilb}(\mathcal{I}), T\}$.

Since $\mathcal{I}$ and $\mathcal{I}_1$ differ only in degree $< T$ then, we can say that ${}^a\mathrm{HF}_{\mathcal{I}_1}(s) = {}^a\mathrm{HF}_{\mathcal{I}}(s) + a$ for an integer $a$ and for each $s \geq T$ which proves the claim.

On the other hand, we know that $\nabla_{D+1}{}^a\mathrm{HF}_{\mathcal{I}_1}(s) = 0$ for sufficiently large $s$, because the degree of the affine Hilbert polynomial of $\mathcal{I}_1$ is $D = \dim(\mathcal{I}_1)$ and every application of $\nabla$ to a polynomial reduces its degree by 1. To give a lower bound for $s$ from which this equality holds, we note that for the computation of $\nabla_i{}^a\mathrm{HF}_{\mathcal{I}_1}(s)$ one needs the values of ${}^a\mathrm{HF}_{\mathcal{I}_1}(s), {}^a\mathrm{HF}_{\mathcal{I}_1}(s-1), \ldots, {}^a\mathrm{HF}_{\mathcal{I}_1}(s-i)$. In addition, for $s \geq i + \mathrm{hilb}(\mathcal{I}_1)$, we have $\nabla_i{}^a\mathrm{HF}_{\mathcal{I}_1}(s)$ uses only values of the Hilbert polynomial. These arguments together show that $\nabla_{D+1}{}^a\mathrm{HF}_{\mathcal{I}_1}(s) = 0$ for $s \geq \max\{\mathrm{hilb}(\mathcal{I}), \mathrm{sat}(\mathrm{LF}(\mathcal{I}))\} + D + 1$. Let

$$K = \max\{d, \max\{\mathrm{hilb}(\mathcal{I}), \mathrm{sat}(\mathrm{LF}(\mathcal{I}))\} + D + 1\}$$

and $H_2$ an H-basis for $\mathcal{I}_1$ up to $K-1$. Since $\mathcal{J}_1$ is saturated then from Prop. 5.6, we know that $\mathcal{I}_1$ has an H-basis $H_2'$ of degree $\leq K$. This shows that $H_1 \cup H_2'$ is an H-basis (and a generating set) for $\mathcal{I}$ and this implies the desired conclusion. □

## 6. Analysis of the upper bound

Up to this point we have proven an upper bound for the degrees of the elements of an H-basis. In this section, we will discuss this bound for some special cases. In addition, we show that the maximal degree of the elements of an H-basis does not change if we transform the ideal it generates into generic position. In particular, using these properties we show that in general an H-basis may be reached earlier than a Gröbner basis. Let us first give some useful properties of H-bases.

**Lemma 6.1.** *Suppose that $H = \{h_1, \ldots, h_t\}$ and $G = \{g_1, \ldots, g_m\}$ are two reduced H-bases for $\mathcal{I}$. Then,* $\max\{\deg(h_i) \mid i = 1, \ldots, t\} = \max\{\deg(g_i) \mid i = 1, \ldots, m\}$

*Proof.* Without loss of generality, assume that $h_t$ (resp. $g_m$) has the maximal degree between the elements in $H$ (resp. $G$). Let us, in contrary, assume that $\deg(h_t) < \deg(g_m)$. Then, using the fact that $G$ is an H-basis for $\mathcal{I}$ we conclude that for each $i$ there exist $p_{ij} \in \mathcal{P}$ so that $h_i = \sum_{j=1}^{m-1} p_{ij}g_j$ and $\deg(p_{ij}g_j) \leq \deg(h_i)$. It follows that $g_m$ has a representation in terms of $g_1, \ldots, g_{m-1}$, contradicting the assumption that $G$ is reduced. □

Based on this lemma, we can give the next definition.

**Definition 6.2.** Given an ideal $\mathcal{I}$, $\mathrm{Hdeg}(\mathcal{I})$ denotes the maximal degree of the elements of a reduced H-basis of $\mathcal{I}$.

Now, let us study some properties of H-bases after performing a linear change of variables. For any linear change $\phi$, we define $\phi(\mathcal{I}) = \langle \phi(f) \mid f \in \mathcal{I} \rangle$. In addition, for a finite set $H \subset \mathcal{P}$, $\phi(H)$ stands for $\{\phi(h) \mid h \in H\}$.

**Lemma 6.3.** *Suppose that $H = \{h_1, \ldots, h_t\}$ is an H-basis for $\mathcal{I}$. Then, for any linear change of variables $\phi$, the set $\phi(H)$ remains an H-basis for the ideal $\phi(\mathcal{I})$. Moreover, if $H$ is reduced then $\phi(H)$ is reduced, too.*

*Proof.* Since any linear change of variables is a $\mathbb{K}$-linear automorphism of $\mathcal{P}$ preserving the degree and $H$ is an H-basis, then trivially for any $f \in \phi(\mathcal{I})$ of degree $s$, $\phi^{-1}(f) \in \mathcal{I}$ has a representation of the form $\phi^{-1}(f) = \sum_{i=1}^{t} p_i h_i$ with $p_i \in \mathcal{P}_{\leq s-\deg(h_i)}$ for $i = 1, \ldots, t$. Then applying $\phi$ on both sides of this equality yields a representation for $f$ and this shows the first assertion. To prove the second conclusion, arguing by reductio ad absurdum, suppose that $\phi(h_1)$ is reducible by $\{\phi(h_2), \ldots, \phi(h_t)\}$. Then, we can write $\phi(h_1) = \sum_{i=2}^{t} p_i \phi(h_i) + r$ for some $p_i, r \in \mathcal{P}$ with $\deg(r) < \deg(h_1)$. Since $\phi$ is an automorphism of $\mathcal{P}$, then, we conclude that $h_1$ is reducible by $\{h_2, \ldots, h_t\}$, which leads to a contradiction. □

This lemma allows us to assume, without loss of generality, that the ideal $\mathcal{I}$ (an in consequence $\mathrm{LF}(\mathcal{I})$) is in generic position and this assumption does not change the maximal degree of the elements of a reduced H-basis of $\mathcal{I}$. In this paper, as the notion of genericity, we are interested in ideals in quasi stable position introduced in Sec. 3. Remark that by [26], there exists an invertible linear change $\phi$ of the variables $x_1, \ldots, x_n$ such that $\mathcal{I}$ is in quasi stable position. In addition, the Hilbert function and therefore also the Hilbert series, the Hilbert polynomial, the Hilbert regularity and the dimension of an ideal $\mathcal{I}$ do not change after performing any linear change of variables. If $\mathcal{I}$ is a not necessarily homogeneous ideal then from $\mathrm{sat}(\mathcal{I})$ and $\mathrm{reg}(\mathcal{I})$ we mean $\mathrm{sat}(\mathrm{LF}(\mathcal{I}))$ and $\mathrm{reg}(\mathrm{LF}(\mathcal{I}))$, respectively. Thus, if $\mathcal{I}$ is in quasi stable position then from [25, Thm. 16] we conclude that $\mathrm{sat}(\mathcal{I}) = \mathrm{sat}(\mathrm{LT}(\mathcal{I}))$ and $\mathrm{reg}(\mathcal{I}) = \mathrm{reg}(\mathrm{LT}(\mathcal{I}))$.

**Theorem 6.4.** *Assume that $\phi$ is an invertible linear change $\phi$ of the variables $x_1, \ldots, x_n$ such that $\phi(\mathcal{I})$ is in quasi stable position. Then, we have* $\mathrm{Hdeg}(\mathcal{I}) \leq \max\{d, \max\{\mathrm{hilb}(\phi(\mathcal{I})), \mathrm{sat}(\phi(\mathcal{I}))\} + D + 1\}$. *In consequence it follows that* $\mathrm{Hdeg}(\mathcal{I}) \leq \max\{d, \mathrm{reg}(\mathcal{I}) + D + 1\}$.

*Proof.* From Thm. 5.7, we know that $\phi(\mathcal{I})$ has an H-basis $H_1$ with

$$\deg(H_1) \leq \max\{d, \max\{\mathrm{hilb}(\phi(\mathcal{I})), \mathrm{sat}(\mathrm{LF}(\phi(\mathcal{I})))\} + D + 1\}.$$

We may assume that $H_1$ is reduced. From assumption, we conclude that $\mathrm{LF}(\mathcal{I})$ is in quasi stable position and therefore $\mathrm{sat}(\mathrm{LF}(\phi(\mathcal{I}))) = \mathrm{sat}(\mathrm{LT}(\mathrm{LF}(\phi(\mathcal{I}))))$.

From definition of H-bases, we know that $\mathrm{LT}(\mathrm{LF}(\phi(\mathcal{I}))) = \mathrm{LT}(\phi(\mathcal{I}))$ which shows that $\deg(H_1) \leq \max\{d, \max\{\mathrm{hilb}(\phi(\mathcal{I})), \mathrm{sat}(\phi(\mathcal{I}))\} + D + 1\}$. Let $H$ be a reduced H-basis for $\mathcal{I}$. From Lem. 6.3, $\phi(H)$ is a reduced H-basis for $\phi(\mathcal{I})$ and by Lem. 6.1 we have $\deg(\phi(H)) = \deg(\phi(H_1))$. Then, the first assertion will follow from the fact that $\deg(\phi(H)) = \deg(H)$. The second assertion is a consequence of the fact that $\max\{\mathrm{hilb}(\phi(\mathcal{I})), \mathrm{sat}(\phi(\mathcal{I}))\} \leq \mathrm{reg}(\mathcal{I})$. □

**Proposition 6.5.** *Suppose that the ideal $\mathcal{I}$ generated by $f_1, \ldots, f_k$ is unmixed and $D > 0$. Then,* $\mathrm{Hdeg}(\mathcal{I}) \leq \max\{d, \mathrm{reg}(\mathcal{I})\}$.

*Proof.* Since any linear change of variables $\phi$ is a $\mathbb{K}$-linear automorphism of $\mathcal{P}$ then trivially the ideal generated by $\phi(f_1), \ldots, \phi(f_k)$ remains unmixed. Thus, using Lem. 6.3, and without loss of generality we may assume that $\mathrm{LT}(\mathcal{I})$ is in quasi stable. If $\mathcal{I} = Q_1 \cap \cdots \cap Q_t$ is a primary decomposition of $\mathcal{I}$ and $Q_i$ for each $i$ is $P_i$-primary then [18, Prop. 27] yields that ${}^h\mathcal{I} = {}^hQ_1 \cap \cdots \cap {}^hQ_t$ is a primary decomposition of ${}^h\mathcal{I}$ and ${}^hQ_i$ for each $i$ is ${}^hP_i$-primary. Since $\mathcal{I}$ is unmixed then ${}^h\mathcal{I}$ is unmixed and in consequence $\mathcal{P}/{}^h\mathcal{I}$ is Cohen-Macaulay. In addition, ${}^h\mathcal{I}$ is in quasi stable position with $\mathrm{depth}({}^h\mathcal{I}) = D + 1$ and $x_{n-D+1}, \ldots, x_{n+1}$ is a regular sequence on $\mathcal{P}/{}^h\mathcal{I}$. Since $\mathrm{LT}(\mathcal{I})$ is in quasi stable then $\mathrm{LF}(\mathcal{I})$ is in this position too and its depth is $D + 1$, see Sec. 3 (note $\mathcal{I}$ and $\mathrm{LF}(\mathcal{I})$ share the same leading term ideal). The assumption $D > 0$ implies that $\mathrm{depth}(\mathrm{LF}(\mathcal{I}))$ is positive and therefore $\mathrm{LF}(\mathcal{I})$ is saturated and its satiety is zero. Hence, from Thm. 5.7, we have $\mathrm{Hdeg}(\mathcal{I}) \leq \max\{d, \mathrm{hilb}(\mathcal{I}) + D + 1\}$. Applying the fact that $\mathcal{P}/\mathrm{LF}(\mathcal{I})$ is Cohen-Macaulay and using [3, Rem. 2.3], we deduce that $\mathrm{reg}(\mathrm{LF}(\mathcal{I})) = \mathrm{hilb}(\mathrm{LF}(\mathcal{I})) + D + 1$ and this completes the proof. □

As already mentioned, a classical way to compute an H-basis for $\mathcal{I}$ is to compute a Gröbner basis for ${}^h\mathcal{I}$ (see the above notations). By [2, Prop. 2.9] $\mathrm{reg}({}^h\mathcal{I})$ is the maximal degree of the elements of the reduced Gröbner basis of ${}^h\mathcal{I}$ provided that ${}^h\mathcal{I}$ is in generic position. Therefore, by this theorem we expect that in general H-bases may be reached earlier than Gröbner bases, as is illustrated in the next example.

**Example 6.6.** Consider the ideal $\mathcal{I} = \langle x_1^{n+1} - x_2x_3^{n-1}, x_1x_2^{n-1} - x_3^n, x_1^nx_3 - x_2^n \rangle \subset \mathbb{K}[x_1, x_2, x_3]$ known as the Lazard-Mora example, cf. [32]. Then, we can see $\dim(\mathcal{I}) = 1$, $\mathcal{I}$ is a prime ideal (which is unmixed) and $\mathrm{hilb}(\mathcal{I}) = 2n - 2$. Therefore, by Prop. 6.5, $\mathcal{I}$ has an H-basis of degree $\leq 2n - 2 + 1 + 1 = 2n$. Computing an H-basis for $\mathcal{I}$, we observe that the generating set of $\mathcal{I}$ forms an H-basis for $\mathcal{I}$ and therefore $\mathrm{Hdeg}(\mathcal{I}) = n + 1$ while the maximal degree of the elements of the reduced Gröbner basis of $\tilde{\mathcal{I}}$ is $n^2 + 1$. Note that all these general results have been obtained by carrying out the computations over several choices of $n$ using the software MAPLE.

Below, we will give an upper bound for $\text{Hdeg}(\mathcal{I})$ when $\mathcal{I}$ is either an ideal generated by a regular sequence or a zero-dimensional ideal. For this purpose, we state the next lemma due to Sombra [51, Lem. 3.15] which allows us to extract a generating set for an ideal generated by a regular sequence which remains regular after homogenization. Let us denote by $\mathcal{I}_i$, for $i = 1,\dots,k$, the ideal generated by $f_1,\dots,f_i$ with the convention $\mathcal{I}_0 = \langle 0\rangle$.

**Lemma 6.7.** *Let $f_1,\dots,f_k$ be a regular sequence. Then there exist polynomials $p_1,\dots,p_k$ and $q_1,\dots,q_k$ in $\mathcal{P}$ such that for each i the following conditions hold:*

- ${}^hp_i = x_{n+1}^{c_i}\,{}^hf_i + q_i$ *with* $q_i \in {}^h\mathcal{I}_{i-1}$ *and* $c_i \le \max\{\deg({}^h\mathcal{I}_{i-1}), \deg(f_i)\}$
- $\deg(p_i) \le \max\{\deg({}^h\mathcal{I}_{i-1}), \deg(f_i)\}$
- ${}^hp_1,\dots,{}^hp_k$ *forms a regular sequence in* ${}^h\mathcal{P}$.

One observes readily that $p_1,\dots,p_k$ generate the ideal generated by the $f_i$'s. Lazard in his paper [32] investigated upper bounds for the maximal degree of the Gröbner basis of a (affine) zero-dimensional ideal using the technique of homogenization. The main obstacle in this direction is that for a given ideal $\mathcal{I}$, $\tilde{\mathcal{I}}$ may contain *alien* components including $x_{n+1}$ with higher dimensions and this may prevent us from applying the results for homogeneous ideals of dimension one. In the following theorem, see [32, Thm. 2], a simple upper bound is given for the degree of the Gröbner basis of $\mathcal{I}$ provided that $\mathcal{I}$ has no such an alien component.

**Theorem 6.8.** *Assume that $\mathcal{I}$ is a zero-dimensional ideal such that* $\dim(\tilde{\mathcal{I}}) = 1$. *Then the maximal degree of the elements of the Gröbner basis of $\mathcal{I}$ is $\le d_1 + \cdots + d_{n+1} - n$ with $d_{n+1} = 1$ if $k = n$.*

**Theorem 6.9.** *Suppose that $f_1,\dots,f_k$ is a regular sequence. Then,* $\text{Hdeg}(\mathcal{I}) \le d^k$. *Moreover, if $H$ is a reduced H-basis of $\mathcal{I}$ then for each $h \in H$ there exist $q_1,\dots,q_k \in \mathcal{P}$ such that $h = q_1f_1 + \cdots + q_kf_k$ with* $\deg(q_if_i) \le 2d^k$.

*Proof.* Since after performing a linear change of variables a regular sequence remains a regular sequence then by Lem. 6.3 we may assume that $\mathcal{I}$ is in generic position. Lem. 6.7 implies the existence of polynomials $p_1,\dots,p_k$ such that their homogenizations is a regular sequence in ${}^h\mathcal{P}$ and $\deg({}^hp_i) \le \max\{\deg({}^h\mathcal{I}_{i-1}), \deg(f_i)\}$. On the other hand, we know that ${}^h\mathcal{I}_{i-1} = \langle {}^hf_1,\dots,{}^hf_{i-1}\rangle : x_{n+1}^\infty$. Furthermore, the degree of a homogeneous ideal is equal to the sum of the degrees of its primary components of the dimension of the ideal, cf. [6, page 282]. These arguments show that $\deg({}^h\mathcal{I}_{i-1}) \le \deg(\langle {}^hf_1,\dots,{}^hf_{i-1}\rangle)$ which is less than or equal to $d_1\cdots d_{i-1}$ by [5, Thm. 4.5]. It follows that $\deg({}^hp_1) \le d$ and $\deg({}^hp_i) \le d^{i-1}$ for each $i > 1$. Let $\mathcal{K} = \langle {}^hp_1,\dots,{}^hp_k\rangle$. Since $\mathcal{K} \subset {}^h\mathcal{P}$

is unmixed then, ${}^h\mathcal{I}$ is unmixed too and ${}^h\mathcal{P}/{}^h\mathcal{I}$ is Cohen-Macaulay. Hence $\dim({}^h\mathcal{I}) = \text{depth}({}^h\mathcal{I}) = D+1$. On the other hand, using the assumptions, we can conclude that ${}^h\mathcal{I}$ (and in consequence $\text{LT}({}^h\mathcal{I})$) is in quasi stable position and this shows that the sequence $x_{n-D+1},\dots,x_n,x_{n+1}$ is regular on ${}^h\mathcal{P}/\text{LT}({}^h\mathcal{I})$, see e.g. [49, Prop. 5.2.7]. Obviously, $x_{n-D+1},\dots,x_{n+1}$ (which forms a regular sequence on $\mathcal{P}/\mathcal{K}$) do not appear in the reduced Göbner basis of ${}^h\mathcal{I}$ and in turn $x_{n-D+1},\dots,x_n$ do not appear in the leading terms of the the elements of the Gröbner basis of $\mathcal{K}$. Therefore, the ideal $\mathcal{K}+\langle x_{n-D+1},\dots,x_n\rangle \subset {}^h\mathcal{P}$ is one-dimensional and by Thm. 6.8 has a Gröbner basis w.r.t. $\prec_h$ of degree at most $\sum_{i=1}^k \deg({}^hp_i)-k+1 \le d+d+d^2+\cdots+d^{k-1}-k+1$. Using a simple induction and the fact that $2\le d$, we obtain $d+d+d^2+\cdots+d^{k-1}\le d^k$. Since $x_{n-D+1},\dots,x_n$ do not appear in the leading terms of the elements of this basis then $\text{Hdeg}(\mathcal{I}) \le d^k-k+1\le d^k$. The second assertion follows immediately from the fact that (under the assumptions of the theorem) for each $f\in\mathcal{I}$ there exist $q_1,\dots,q_k\in\mathcal{P}$ such that $f=q_1f_1+\cdots+q_kf_k$ with $\deg(q_if_i)\le d^k+\deg(f)$, cf. [15, Thm. 5.1]. □

**Remark 6.10.** As a direct consequence of this theorem, we infer that if $f_1,\dots,f_k$ is a regular sequence and $\text{LT}(\mathcal{I})$ is quasi stable then the maximal degree of the elements of the Gröbner basis of $\mathcal{I}$ w.r.t. $\prec$ is bounded above by $d^k$ and each polynomial in this basis has a representation in terms of the $f_i$'s of degree at most $2d^k$.

To study the degree upper bound for the H-bases of zero-dimensional ideals, we shall need also the following known lemma, see [29]

**Lemma 6.11.** *Suppose $\mathcal{I}$ is zero-dimensional. For $i=2,\dots,n$ let $g_i=f_i+a_{i,i+1}f_{i+1}+\cdots+a_{i,k}f_k$ where $a_{i,j}\in\mathbb{K}$. For almost all choices of the $a_{i,j}$ the sequence $f_1,g_2,\dots,g_n$ is regular and* $\deg(g_i)=\deg(f_i)$.

**Theorem 6.12.** *Suppose that the ideal $\mathcal{I}=\langle f_1,\dots,f_k\rangle$ is zero-dimensional. Then,* $\text{Hdeg}(\mathcal{I})\le nd^n-n$. *Furthermore, if $H$ is a reduced H-basis of $\mathcal{I}$ then for each $h\in H$ there exist $q_1,\dots,q_k\in\mathcal{P}$ such that $h=q_1f_1+\cdots+q_kf_k$ with* $\deg(q_if_i)\le(n+2)d^n-n+d$.

*Proof.* Following Lem. 6.11, without loss of generality, we may assume that $f_1,\dots,f_n$ is a regular sequence. Let $\mathcal{I}_1=\langle f_1,\dots,f_n\rangle$. From Thm. 6.9, $\mathcal{I}_1$ has a reduced H-basis $H_1=\{h_1,\dots,h_t\}$ and a Gröbner basis $G_1$ of degree $\le d^n$. Assume that $H_2$ is a reduced H-basis for $\mathcal{I}$. Reduce the elements of $H_2$ by $H_1$ and call the new set $H_2'$. It is easy to check that $H=H_1\cup H_2'$ is an H-basis for $\mathcal{I}$. Assume in contrary that $H$ contains a polynomial $h$ with $\deg(h)\ge nd^n-n+1$. It is well-known that $\dim_{\mathbb{K}}(\mathcal{P}/\mathcal{I}_1)\le d^n$ which implies that $\mathcal{I}_1$ contains univariate polynomial $g_i\in\mathbb{K}[x_i]$ of degree $\le d^n$ for each $i$. Thus using the zero-dimensionality of $G_1$ we have $\mathrm{LF}(h) \in \langle \mathrm{LT}(G_1) \rangle$. The set $H_1$ is an H-basis for $\mathcal{I}_1$ which yields that $h$ is reducible by $H_1$, leading to a contradiction. The proof of the the second assertion was inspired by the proof of [15, Thm. 3.3]. Let us consider $h \in H$. Then, we can write $h = \sum_{i=1}^{k} p_i f_i$ with $p_i \in \mathcal{P}$. Dividing $p_1, \ldots, p_k$ by $H$ we get $p_i = \sum_{j=1}^{t} q_{ij} h_j + \tilde{p}_i$ with $\deg(\tilde{p}_i) \le nd^n - n$. Therefore,

$$h = \sum_{i=1}^{k} (\sum_{j=1}^{t} q_{ij} h_j + \tilde{p}_i) f_i.$$

Let us consider $g = \sum_{i=1}^{k} \sum_{j=1}^{t} q_{ij} h_j f_i$. It is clear that $g \in \mathcal{I}_1$ and $\deg(g) \le nd^n - n + d$. Thus, we can write $g = \sum_{i=1}^{t} r_i h_i$ with $\deg(r_i h_i) \le nd^n - n + d$. On the other hand, from Thm. 6.9 we know that $h_i = \sum_{i=1}^{k} s_{ij} f_i$ with $\deg(s_{ij} f_i) \le 2d^n$. Therefore, $h$ can be written of the form $h = \sum_{i=1}^{k} (\sum_{j=1}^{t} s_{ij} r_j + \tilde{p}_i) f_i$ with $\deg((\sum_{j=1}^{t} s_{ij} r_j + \tilde{p}_i) f_i) \le (n+2)d^n - n + d$. □

**Remark 6.13.** Similar results to Thm. 6.12 hold for Gröbner bases by repeating the same argument. Assume that the zero-dimensional ideal $\mathcal{I}$ is generated by the polynomials $f_1, \ldots, f_k$. Then the maximal degree of the elements of the Gröbner basis of $\mathcal{I}$ is bounded above by $nd^n - n$ and each polynomial in this basis has a representation in terms of the $f_i$'s of degree at most $(n+2)d^n - n + d$. It is worth noting that the existing degree upper bounds for these purposes are $nd^n$ and $nd^{2n} + d^n + d$, respectively, cf. [15, Thm. 3.3].

We finish the paper by commenting that Mayr and Ritscher [38] gave the upper bound $2(1/2d^{n-D} + d)^{2^{D-1}}$ for the maximal degree of the elements of the reduced Gröbner basis of a homogeneous ideal generated by polynomials of degree $\le d$. Using this result, we can give an upper bound for $\mathrm{Hdeg}(\mathcal{I})$. By the above notations, $\mathrm{Hdeg}(\mathcal{I})$ is less than or equal to the maximal degree of the elements of the reduced Gröbner basis of $\tilde{\mathcal{I}}$. This shows that $\mathrm{Hdeg}(\mathcal{I}) \le 2(1/2d^{n+1-D'} + d)^{2^{D'-1}}$ where $D' = \dim(\tilde{\mathcal{I}})$.

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