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INT-SOFT STRUCTURES APPLIED TO ORDERED SEMIHYPERGROUPS

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Molodtsov introduced the theory of soft sets, which can be seen as a new mathematical approach to vagueness. The main goal of this paper is to introduce and study some classes of ordered semihypergroups and to investigate some interesting characterizations theorems of these classes in terms of int-soft hyperideals. In this respect, we characterize weakly regular ordered semihypergroups for example (see Theorems 3.4, 3.5 and 3.7) intra-regular and left weakly-regular ordered semihypergroups (see Theorems 4.2 and 4.4) and semisimple ordered semihypergroups (see Theorems 5.5 and 5.7) in terms of int-soft hyperideals. In this regard, we study semisimple ordered semihypergroups and characterize it in terms of int-soft hyperideals. We also characterize intra-regular and weakly-regular ordered semihypergroups in terms of int-soft hyperideals.

1. Introduction

The theory of hyperstructures was introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [8]. Marty introduced hypergroups as a generalization of groups. He published some papers on hypergroups,

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using them in different contexts as algebraic functions, rational fractions, non commutative groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In [15] Corsini and Leoreanu-Fotea collected numerous applications of algebraic hyperstructures such as: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many scholars have studied different aspects of semihypergroups. The concept of ordering hypergroups investigated by Chvalina [22], as a special class of hypergroups and studied by him and many others. Heidari and Davvaz [16], applied the theory of hyperstructures to ordered semigroups and introduced the concepts of ordered semihypergroups, which is a generalization of ordered semigroups. Heidari and Davvaz, also studied hyperideals of ordered semihypergroups in [16]. Changphas and Davvaz introduced the concepts of bi-hyperideals and quasi-hyperideals in ordered semihypergroups [20]. Pibaljommee et al. [17] introduced the notions of fuzzy hyperideals, fuzzy bi-hyperideals and fuzzy quasi-hyperideals of ordered semihypergroups. Tipachot and Pibaljommee in [18], and Tang et al. in [19], studied the notion of fuzzy interior hyperideals in ordered semihypergroups. Tang et al. [21], introduced the notions of hyperfilters and fuzzy hyperfilters of ordered semihypergroups.

Problems in many fields involve data that contain uncertainties. Uncertainties may be dealt with using a wide range of existing theories such as theory of probability, fuzzy set theory [25], intuitionistic fuzzy set [23], vague set [24], theory of interval mathematics [26], rough set theory [27], etc. All of these theories have their own difficulties which are pointed out in [6]. To overcome these difficulties, Molodtsov [6], introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties. In [6, 28], Molodtsov pointed out several directions for the applications of soft sets, such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability, theory of measurement and so on. Maji et al. [29], described the application of soft set theory to a decision making problem. Maji et al. [30], also studied several operations on the theory of soft sets. Cagman and Enginoglu [31], introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Feng [32], considered the application of soft rough approximations in multicriteria group decision making problems. In fact, in the aspect of algebraic structures, the soft set theory has been applied to groups [33], semirings [2], ordered semigroups [35] and hemirings [5, 7] and so on. Feng et al. [3], discussed soft relations in semigroups. In [12], Naz and Shabir applied soft set theory to semihypergroups. Farooq et al. [14], applied soft set theory to ordered semihypergroups.

The purpose of this paper is to characterize weakly-regular, intra-regular and semisimple ordered semihypergroups by the properties of their int-soft hyperideals. We have shown that an ordered semihypergroup is left weakly-regular if and only if every int-soft left hyperideal of S is idempotent and S is semisimple if and only if every int-soft two-sided hyperideal of S is idempotent.

2. Preliminaries

By an ordered semihypergroup we mean a structure $(S,\circ,\leq)$ in which the following conditions are satisfied:

(OS1) $(S,\circ)$ is a semihypergroup.

(OS2) $(S,\leq)$ is a poset.

(OS3) $(\forall a,b,x\in S)$ $a\leq b$ implies $x\circ a\leq x\circ b$ and $a\circ x\leq b\circ x$.

For $A\subseteq S$,we denote $(A]:=\{t\in S: t\leq h$ for some $h\in A\}$.

For $A,B\subseteq S$, we have $A\circ B:=\bigcup\{a\circ b: a\in A,\, b\in B\}$.

A nonempty subset A of an ordered semihypergroup S is called a subsemihypergroup of S if $A^2\subseteq A$.

A nonempty subset A of S is called a left (resp. right) hyperideal of S if it satisfies the following conditions:

(i) $S\circ A\subseteq A$ (resp. $A\circ S\subseteq A$).

(ii) If $a\in A, b\in S$ and $b\leq a$, then $b\in A$.

By a two sided hyperideal or simply a hyperideal of S we mean a nonempty subset of S which is both a left hyperideal and a right hyperideal of S.

A subsemihypergroup A of S is called an interior hyperideal of S if it satisfies the following conditions:

(i) $S\circ A\circ S\subseteq A$.

(ii) If $a\in A, b\in S$ and $b\leq a$, then $b\in A$.

A nonempty subset B of an ordered semihypergroup S is called a bi-hyperideal of S if it satisfies the following conditions:

(i) $B\circ S\circ B\subseteq B$.

(ii) If $a\in B, b\in S$ and $b\leq a$, then $b\in B$.

A nonempty subset Q of an ordered semihypergroup S is called a quasi-hyperideal of S if it satisfies the following conditions:

(i) $(Q\circ S]\cap(S\circ Q]\subseteq Q$.

(ii) If $a\in Q, b\in S$ and $b\leq a$, then $b\in Q$.

We denote by $R(a)$ (resp. $L(a)$, $I(a)$, $\mathcal{I}(a)$, $B(a)$ and $Q(a)$) the right (resp. left, two-sided, interior, bi- and quasi-) hyperideal of S generated by a $(a \in S)$. We obtain

$$\begin{aligned}
R(a) &= (a \cup a \circ S], \\
L(a) &= (a \cup S \circ a], \\
I(a) &= (a \cup a \circ S \cup S \circ a \cup S \circ a \circ S], \\
\mathcal{I}(a) &= \left(a \cup a^2 \cup S \circ a \circ S\right], \\
B(a) &= \left(a \cup a^2 \cup a \circ S \circ a\right], \\
Q(a) &= (a \cup (a \circ S \cap S \circ a)].
\end{aligned}$$

An ordered semihypergroup $(S, \circ, \leq)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq a \circ x \circ a$. Equivalent Definitions: (1) $A \subseteq (A \circ S \circ A]$ $\forall A \subseteq S$. (2) $a \in (a \circ S \circ a]$ $\forall a \in S$.

An ordered semihypergroup S is called intra-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq x \circ a^2 \circ y$. Equivalent Definitions: (1) $A \subseteq (S \circ A^2 \circ S]$ $\forall A \subseteq S$. (2) $a \in (S \circ a^2 \circ S]$ $\forall a \in S$.

An ordered semihypergroup S is called left (resp. right) weakly-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq x \circ a \circ y \circ a$ (resp. $a \leq a \circ x \circ a \circ y$). Equivalent Definitions: $a \in \left((S \circ a)^2\right]$ $\left(\text{resp. } a \in \left((a \circ S)^2\right]\right)$ $\forall a \in S$. (2) $A \subseteq \left((S \circ A)^2\right]$ (resp. $A \subseteq \left((A \circ S)^2\right]$ $\forall A \subseteq S$.

If S is left weakly-reguar and right-weakly regular then it is called weakly-regular. Thus, if S is commutative and weakly-regular, then S is regular.

An ordered semigroup S is called semisimple if for every $a \in S$, there exist $x, y, z \in S$ such that $a \leq x \circ a \circ y \circ a \circ z$. Equivalent Definitions: (1) $a \in (S \circ a \circ S \circ a \circ S]$ $\forall a \in S$. (2) $A \subseteq (S \circ A \circ S \circ A \circ S]$ $\forall A \subseteq S$.

For subsets A and B of an ordered semihypergroup S we obtain

$A \subseteq (A]$ and if $A \subseteq B$, then $(A] \subseteq (B]$, $(A] \circ (B] \subseteq (A \circ B]$, $((A]] = (A]$.

For the sake of simplicity throughout this paper, we denote $a^n = \underbrace{a \circ \ldots \circ a}_{n\text{-copies}}$.

2.1. Basic concepts of soft sets

In [34], Sezgin and Atagun introduced some new operations on soft set theory. They defined soft sets in the following manner.

In what follows, we take $E = S$ as the set of parameters, which is an ordered semihypergroup, unless otherwise specified.

From now on, U is an initial universe set, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C ... \subseteq E$.

Definition 2.1. (see [34]). A *soft set* f_A over U is defined as $f_A : E \longrightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Hence, f_A is also called an *approximation function.*

A soft set f_A over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}.$$

It is clear that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2. (see [34]). Let $f_A, f_B \in S(U)$. Then, f_A is called a *soft subset* of f_B, denoted by $f_A \widetilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$. Two soft sets f_A and f_B are said to be equal soft sets if $f_A \widetilde{\subseteq} f_B$ and $f_B \widetilde{\subseteq} f_A$ and is denoted by $f_A \widetilde{=} f_B$.

Definition 2.3. (see [34]). Let $f_A, f_B \in S(U)$. Then, the *soft union* of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B = f_{A\cup B}$, is defined by $\left(f_A \widetilde{\cup} f_B\right)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.

Definition 2.4. (see [34]). Let $f_A, f_B \in S(U)$. Then, the *soft intersection* of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B = f_{A\cap B}$, is defined by $\left(f_A \widetilde{\cap} f_B\right)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

For $x \in S$, we define

$$A_x = \{(y,z) \in S \times S \mid x \leq y \circ z\}.$$

Definition 2.5. (see [9]). Let f_A and g_B be two soft sets of an ordered semihypergroup S over U. Then, the int-soft product, denoted by $f_A \widetilde{*} g_B$, is defined by

$$f_A \widetilde{*} g_B : S \longrightarrow P(U), x \longmapsto (f_A \widetilde{*} g_B)(x) = \begin{cases} \bigcup\limits_{(y,z)\in A_x} \{f_A(y) \cap g_B(z)\}, & \text{if } A_x \neq \emptyset, \\ \emptyset, & \text{if } A_x = \emptyset, \end{cases}$$

for all $x \in S$.

For a nonempty subset A of S the characteristic soft set is defined to be the soft set $\mathcal{S}_{\mathcal{A}}$ of A over U in which $\mathcal{S}_{\mathcal{A}}$ is given by

$$\mathcal{S}_{\mathcal{A}} : S \longmapsto P(U). \quad x \longmapsto \begin{cases} U, & \text{if} \quad x \in A \\ \emptyset, & \text{otherwise} \end{cases}$$

For an ordered semihypergroup S, the soft set "$\mathcal{S}_{\mathcal{S}}$" of S over U is defined as follows:

$$\mathcal{S}_{\mathcal{S}} : S \longrightarrow P(U), x \longmapsto \mathcal{S}_{\mathcal{S}}(x) = U \text{ for all } x \in S.$$

The soft set "$\mathcal{S}_{\mathcal{S}}$" of an ordered semihypergroup S over U is called the whole soft set of S over U.

Definition 2.6. (see [9]). A soft set f_A of an ordered semihypergroup S over U is called *an int-soft subsemihypergroup of S* over U if:

$$(\forall x, y \in S) \bigcap_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(x) \cap f_A(y).$$

Definition 2.7. (see [9]). Let f_A be a soft set of S over U. Then, f_A is called an int-*soft left (*resp. right) *hyperideal* of S over U if it satisfies the following conditions:

(1) $(\forall x, y \in S) \bigcap_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(y)$ (resp. $\bigcap_{\alpha \in x \circ y} f_A(\alpha) \supseteq f_A(x)$).
(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f_A(x) \supseteq f_A(y)$.

A soft set f_A of an ordered semihypergroup S over U is called an int-soft hyperideal (or int-soft two-sided hyperideal) if it is both an int-soft left hyperideal and an int-soft right hyperideal of S over U.

Definition 2.8. (see [10]). An int-soft subsemihypergroup f_A of an ordered semihypergroup S over U is called an int-soft interior hyperideal of S over U if it satisfies the following conditions:

(1) $(\forall x, y, a \in S) \bigcap_{\alpha \in x \circ a \circ y} f_A(\alpha) \supseteq f_A(a)$.
(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f_A(x) \supseteq f_A(y)$.

Definition 2.9. (see [11]). An int-soft subsemihypergroup f_A of an ordered semihypergroup S over U is called an int-soft bi-*hyperideal* of S over U if it satisfies the following conditions:

(1) $(\forall x, y, z \in S) \bigcap_{\alpha \in x \circ y \circ z} f_A(\alpha) \supseteq f_A(x) \cap f_A(z)$.
(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f_A(x) \supseteq f_A(y)$.

Definition 2.10. (see [13]). A soft set f_A of an ordered semihypergroup S over U is called an int-soft quasi-*hyperideal* of S over U if it satisfies the following conditions:

(1) $(f_A \widetilde{*} \mathcal{S}_S) \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A) \widetilde{\subseteq} f_A$.
(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f_A(x) \supseteq f_A(y)$.

Definition 2.11. (see [11]). A soft set f_A of an ordered semihypergroup S over U is called idempotent if

$$f_A \widetilde{*} f_A = f_A.$$

Proposition 2.12. *(see [9]). Let S be an ordered semihypergroup. Let $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ be soft sets of S over U where A and B are nonempty subsets of S. Then,*

$$\mathcal{S}_{\mathcal{A}} \widetilde{*} \mathcal{S}_{\mathcal{B}} = \mathcal{S}_{(A \circ B]}.$$

Lemma 2.13. *(see [10, 14]). Let $(S,\circ,\leq)$ be an ordered semihypergroup. A nonempty subset A of S is a left (resp. right, interior) hyperideal of S if and only if the characteristic function $\mathcal{S}_{\mathcal{A}}$ of A is an int-soft left (resp. right, interior) hyperideal of S over U.*

Lemma 2.14. *(see [13, 14]). Let $(S,\circ,\leq)$ be an ordered semihypergroup. A nonempty subset A of S is a quasi-(resp. bi-) hyperideal of S if and only if the characteristic function $\mathcal{S}_{\mathcal{A}}$ of A is an int-soft quasi- (resp. bi-) hyperideal of S over U.*

Proposition 2.15. *(see [10]). Let $(S,\circ,\leq)$ be an ordered semihypergroup and f_A be an int-soft hyperideal of S over U. Then, f_A is an int-soft interior hyperideal of S over U.*

Proposition 2.16. *(see [9]). Let $(S,\circ,\leq)$ be an ordered semihypergroup, f_A an int-soft right hyperideal and g_B an int-soft left hyperideal of S over U. Then,*

$$f_A\widetilde{*}g_B\widetilde{\subseteq}f_A\widetilde{\cap}g_B.$$

Proposition 2.17. *(see [9]). Let $(S,\circ,\leq)$ be an ordered semihypergroup, f_A an int-soft left (resp. right) hyperideal of S over U. Then, $\mathcal{S}_{\mathcal{S}}\widetilde{*}f_A\widetilde{\subseteq}f_A$ (resp. $f_A\widetilde{*}\mathcal{S}_{\mathcal{S}}\widetilde{\subseteq}f_A$).*

Proposition 2.18. *(see [9]). Let $(S,\circ,\leq)$ be an ordered semihypergroup, f_A an int-soft left (resp. right) hyperideal of S over U. Then,*

$$f_A\widetilde{*}f_A\widetilde{\subseteq}f_A.$$

Corollary 2.19. *Let $(S,\circ,\leq)$ be an ordered semihypergroup, f_A an int-soft hyperideal of S over U. Then,*

$$f_A\widetilde{*}f_A\widetilde{\subseteq}f_A.$$

Proposition 2.20. *(see [13]). Let $(S,\circ,\leq)$ be an ordered semihypergroup. Then, every one sided hyperideal is a quasi-hyperideal.*

Proposition 2.21. *(see [13]).Let $(S,\circ,\leq)$ be an ordered semihypergroup. Then, every one sided int-soft hyperideal is an int-soft quasi-hyperideal S over U.*

3. Characterizations of weakly-regular ordered semihypergroups in terms of int-soft hyperideals

Lemma 3.1. *Let $(S,\circ,\leq)$ be an ordered semihypergroup. Then, the following are equivalent:*

(1) S is left (resp. right) weakly-regular.

(2) $\left(L^2\right] = L$ (resp. $\left(R^2\right] = R$) for every left hyperideal L, (resp. right hyperideal R) of S.

(3) $\left(L(a)^2\right] = L(a)$ $\left(\text{resp. } \left(R(a)^2\right] = R(a)\right)$ for every $a \in S$.

Proof. $(1) \Longrightarrow (2)$. Let L be a left hyperideal of a weakly-regular ordered semihypergroup S. Then, $\left(L^2\right] \subseteq (S \circ L] \subseteq (L] = L$. For the reverse inclusion let $a \in L$. Since S is left weakly-regular, it follows that there exist $x, y \in S$ such that $a \leq x \circ a \circ y \circ a \subseteq (S \circ L) \circ (S \circ L) \subseteq L \circ L \subseteq \left(L^2\right]$. Thus, $\left(L^2\right] = L$.

$(2) \Longrightarrow (3)$. Obvious.

$(3) \Longrightarrow (1)$. Suppose that $a \in S$. Then,

$$\begin{aligned} a &\in L(a) = \left(L(a)^2\right] \\ &= ((a \cup S \circ a] \circ (a \cup S \circ a]] \\ &\subseteq (((a \cup S \circ a) \circ (a \cup S \circ a)]] \\ &= ((a \cup S \circ a) \circ (a \cup S \circ a)] \\ &= \left(a^2 \cup a \circ S \circ a \cup S \circ a^2 \cup S \circ a \circ S \circ a\right]. \end{aligned}$$

Then, $a \leq a^2$ or $a \leq a \circ x \circ a$ or $a \leq x \circ a^2$ or $a \leq x \circ a \circ y \circ a$ for some $x, y \in S$. If $a \leq a^2$ then $a \leq a^2 = a \circ a \leq a^2 \circ a^2 = a \circ a \circ a \circ a$. If $a \leq x \circ a^2$ then $a \leq x \circ a^2 = x \circ a \circ a \leq x \circ a \circ x \circ a^2 = x \circ a \circ (x \circ a) \circ a = x \circ a \circ y \circ a$ where $y \in x \circ a$. If $a \leq a \circ x \circ a$ then $a \leq a \circ x \circ a \leq (a \circ x) \circ a \circ x \circ a = y \circ a \circ x \circ a$ where $y \in a \circ x$. Thus, S is left weakly-regular. □

Proposition 3.2. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Let f_A be a soft set of S over U. Then, $\mathcal{S}_S \widetilde{*} f_A$ (resp. $f_A \widetilde{*} \mathcal{S}_S$) is an int-soft left (resp. right) hyperideal of S over U.*

Proof. Straightforward. □

Corollary 3.3. *Let $(S, \circ, \leq)$ be an ordered semihypergroup with identity element 1. Let f_A be a soft set of S over U. Then, $\mathcal{S}_S \widetilde{*} f_A$ (resp. $f_A \widetilde{*} \mathcal{S}_S$) is the smallest int-soft left (resp. right) hyperideal of S over U containing f_A.*

Proof. By Proposition 3.2, $\mathcal{S}_S \widetilde{*} f_A$ is an int-soft left hyperideal of S over U. If $x \in S$, then $(1, x) \in A_x$.

$$\begin{aligned} \left(\mathcal{S}_S \widetilde{*} f_A\right)(x) &= \bigcup_{(a,b) \in A_x} \{\mathcal{S}_S(a) \cap f_A(b)\} \\ &\supseteq \{\mathcal{S}_S(1) \cap f_A(x)\} \\ &= U \cap f_A(x) \\ &= f_A(x). \end{aligned}$$

Hence, $f_A \widetilde{\subseteq} \mathcal{S}_S \widetilde{*} f_A$. Let g_B be an int-soft left hyperideal of S over U such that $f_A \widetilde{\subseteq} g_B$. Then, $\mathcal{S}_S \widetilde{*} f_A \widetilde{\subseteq} \mathcal{S}_S \widetilde{*} g_B = g_B$ by Proposition 3.2. Hence, $\mathcal{S}_S \widetilde{*} f_A$ is the smallest int-soft left hyperideal of S over U containing f_A. □

Theorem 3.4. *An ordered semihypergroup S is left weakly-regular if and only if for every int-soft left hyperideal f_A of S over U, we have*

$$f_A \widetilde{*} f_A = f_A.$$

Proof. Let S be a left weakly-regular ordered semihypergroup, f_A be an int-soft left hyperideal of S over U and $a \in S$. Then,

$$(f_A \widetilde{*} f_A)(a) = f_A(a).$$

Since S is left weakly-regular, it follows that there exist $x, y \in S$ such that $a \leq (x \circ a) \circ (y \circ a)$. So there exist $u \in x \circ a$ and $v \in y \circ a$ such that $a \leq u \circ v$. Then, $(u, v) \in A_a$. Since $A_a \neq \emptyset$, it follows that

$$\begin{aligned} (f_A \widetilde{*} f_A)(a) &= \bigcup_{(p,q) \in A_a} \{f_A(p) \cap f_A(q)\} \\ &\supseteq \{f_A(u) \cap f_A(v)\}. \end{aligned}$$

Since f_A is an int-soft left hyperideal of S over U, it follows that

$$\bigcap_{u \in x \circ a} f_A(u) \supseteq f_A(a) \text{ and } \bigcap_{v \in y \circ a} f_A(v) \supseteq f_A(a).$$

Hence, $f_A(u) \supseteq f_A(a)$ and $f_A(v) \supseteq f_A(a)$. Thus,

$$\begin{aligned} (f_A \widetilde{*} f_A)(a) &\supseteq \{f_A(u) \cap f_A(v)\} \\ &\supseteq \{f_A(a) \cap f_A(a)\} \\ &= f_A(a). \end{aligned}$$

Thus, $f_A \widetilde{\subseteq} f_A \widetilde{*} f_A$. For the reverse inclusion, since f_A is an int-soft left hyperideal of S over U, so by Proposition 2.18, it follows that $f_A \widetilde{*} f_A \widetilde{\subseteq} f_A$. Thus, $f_A \widetilde{*} f_A = f_A$.

Conversely, assume that $f_A \widetilde{*} f_A = f_A$ for every int-soft left hyperideal f_A of S over U. Then, S is left weakly-regular. In fact, it is enough to prove that

$$L(a) = \left(L(a)^2\right] \text{ for all } a \in S.$$

Let $a \in S$ and $b \in L(a)$. Then, $b \in \left(L(a)^2\right]$. Indeed, $L(a)$ is left hyperideal of S generated by a. Then, $\mathcal{S}_{L(a)}$ is an int-soft left hyperideal of S over U. Then, by hypothesis

$$\left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{L(a)}\right)(b) = \mathcal{S}_{L(a)}(b).$$

Since $b \in L(a)$, it follows that $\mathcal{S}_{L(a)}(b) = U$. This implies that

$$\left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{L(a)}\right)(b) = U.$$

But by Proposition 2.12, we obtain $\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{L(a)} = \mathcal{S}_{\left(L(a)^2\right]}$. Thus,

$$\mathcal{S}_{\left(L(a)^2\right]}(b) = U \Longrightarrow b \in \left(L(a)^2\right].$$

Therefore, $L(a) \subseteq \left(L(a)^2\right]$. On the other hand, $\left(L(a)^2\right] \subseteq L(a)$ always true. Thus, $L(a) = \left(L(a)^2\right]$. □

Similarly, we can prove the following theorem.

Theorem 3.5. *An ordered semihypergroup S is right weakly-regular if and only if for every int-soft right hyperideal f_A of S over U, we obtain*

$$f_A \widetilde{*} f_A = f_A.$$

Lemma 3.6. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, the following are equivalent:*

(1) S is weakly-regular.
(2) $Q = (Q \circ S]^2 \cap (S \circ Q]^2$ for every quasi-hyperideal Q of S.

Proof. $(1) \Longrightarrow (2)$. Let S be a weakly-regular ordered semihypergroup and Q a quasi-hyperideal of S. Then, the left hyperideal $(S \circ Q]$ and right hyperideal $(Q \circ S]$ are idempotents, by Lemma 3.1. Thus, we obtain

$$(Q \circ S]^2 \cap (S \circ Q]^2 = (Q \circ S] \cap (S \circ Q] \subseteq Q.$$

For the reverse inclusion, let $a \in Q$. Since S is left weakly-regular, it follows that there exist $x, y \in S$ such that $a \leq x \circ a \circ y \circ a \subseteq (S \circ Q) \circ (S \circ Q) \subseteq (S \circ Q]^2$. Similarly, we can prove that $a \in (Q \circ S]^2$. Thus, $a \in (Q \circ S]^2 \cap (S \circ Q]^2$. Therefore, $Q \subseteq (Q \circ S]^2 \cap (S \circ Q]^2$. Hence, $Q = (Q \circ S]^2 \cap (S \circ Q]^2$.

$(2) \Longrightarrow (1)$. Let R be any right hyperideal of S. Then, R is a quasi-hyperideal of S by Proposition 2.20. By (2) we obtain ,

$$R = (R \circ S]^2 \cap (S \circ R]^2 \subseteq (R \circ S]^2 \subseteq (R]^2 \subseteq \left(R^2\right] \subseteq (R] = R.$$

Thus, $\left(R^2\right] = R$, and so S is right weakly-regular ordered semihypergroup. On the same way we can prove that S is left weakly-regular. □

Theorem 3.7. *An ordered semihypergroup S is weakly-regular if and only if for every int-soft quasi-hyperideal f_A of S over U, we obtain*

$$f_A = (f_A \widetilde{*} \mathcal{S}_S)^2 \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A)^2 .$$

Proof. Let S be a weakly-regular ordered semihypergroup, f_A an int-soft quasi-hyperideal of S over U. Since f_A is an int-soft quasi-hyperideal of S over U, so by Proposition 3.2, $f_A \widetilde{*} \mathcal{S}_S$ is an int-soft right hyperideal and $\mathcal{S}_S \widetilde{*} f_A$ is an int-soft left hyperideal of S over U. Since S is weakly-regular, by Theorems 3.4 and 3.5, we have $\mathcal{S}_S \widetilde{*} f_A$ and $f_A \widetilde{*} \mathcal{S}_S$ are idempotents. Hence,

$$(f_A \widetilde{*} \mathcal{S}_S)^2 \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A)^2 = (f_A \widetilde{*} \mathcal{S}_S) \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A) \widetilde{\subseteq} f_A$$

(since f_A is an int-soft quasi-hyperideal).

In order to prove the reverse inclusion, let $a \in S$. Since S is right weakly-regular, it follows that there exist $x, y \in S$ such that $a \leq (a \circ x) \circ (a \circ y)$. Then, there exist $\alpha \in a \circ x$ and $\beta \in a \circ y$ such that $a \leq \alpha \circ \beta$. Hence, $(\alpha, \beta) \in A_a$. Since $A_a \neq \emptyset$, it follows that

$$\begin{aligned}
(f_A \widetilde{*} \mathcal{S}_S)^2 (a) &= \bigcup_{(p,q) \in A_a} \{(f_A \widetilde{*} \mathcal{S}_S)(p) \cap (f_A \widetilde{*} \mathcal{S}_S)(q)\} \\
&\supseteq \{(f_A \widetilde{*} \mathcal{S}_S)(\alpha) \cap (f_A \widetilde{*} \mathcal{S}_S)(\beta)\} \\
&= \left[\bigcup_{(u,v) \in A_\alpha} \{f_A(u) \cap \mathcal{S}_S(v)\} \right] \cap \left[\bigcup_{(u,v) \in A_\beta} \{f_A(u) \cap \mathcal{S}_S(v)\} \right] \\
&\supseteq \{f_A(a) \cap \mathcal{S}_S(x)\} \cap \{f_A(a) \cap \mathcal{S}_S(y)\} \\
&= \{f_A(a) \cap U\} \cap \{f_A(a) \cap U\} \\
&= f_A(a) \cap f_A(a) = f_A(a).
\end{aligned}$$

Thus, we obtain $f_A \widetilde{\subseteq} (f_A \widetilde{*} \mathcal{S}_S)^2$. Similarly, we can show that $f_A \widetilde{\subseteq} (\mathcal{S}_S \widetilde{*} f_A)^2$. Thus, $f_A \widetilde{\subseteq} (f_A \widetilde{*} \mathcal{S}_S)^2 \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A)^2$. Hence,

$$f_A = (f_A \widetilde{*} \mathcal{S}_S)^2 \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A)^2 .$$

Conversely, assume that, f_A is an int-soft right hyperideal of S over U. By Proposition 2.21, f_A is an int-soft quasi-hyperideal of S over U. By assumption and Proposition 2.17, we obtain

$$f_A = (f_A \widetilde{*} \mathcal{S}_S)^2 \widetilde{\cap} (\mathcal{S}_S \widetilde{*} f_A)^2 \widetilde{\subseteq} (f_A \widetilde{*} \mathcal{S}_S)^2 \widetilde{\subseteq} f_A \widetilde{*} f_A \widetilde{\subseteq} f_A.$$

Hence, $f_A \widetilde{*} f_A = f_A$. Thus by Theorem 3.5, S is right weakly-regular. By the same way we can prove that S is left weakly-regular. $\square$

4. Characterizations of intra-regular and left weakly-regular ordered semihypergroups in terms of int-soft hyperideals

In this paragraph we characterize intra-regular and left weakly-regular ordered semihypergroups in terms of their int-soft left (resp. right, quasi- and bi-) hyperideals

Lemma 4.1. *Let* $(S,\circ,\leq)$ *be an ordered semihypergroup with identity element* 1. *Then, the following are equivalent:*

(1) S is both intra-regular and left weakly-regular.

(2) $L\cap R\cap Q\subseteq(L\circ R\circ Q]$ for every quasi-hyperideal Q, every left hyperideal L and every right hyperideal R of S.

(3) $L(a)\cap R(a)\cap Q(a)\subseteq(L(a)\circ R(a)\circ Q(a)]$ for every $a\in S$.

Proof. $(1)\Longrightarrow(2)$. Let S be both an intra-regular and left weakly-regular ordered semihypergroup. Then, for every left hyperideal L, right hyperideal R and quasi-hyperideal Q of S, we have

$$L\cap R\cap Q\subseteq(L\circ R\circ Q].$$

In fact, if $a\in L\cap R\cap Q$, then $a\in L$, $a\in R$ and $a\in Q$. Since S is intra-regular, it follows that there exist $x,y\in S$ such that $a\leq x\circ a^2\circ y$ and since S is left weakly-regular, it follows that there exist $u,v\in S$ such that $a\leq u\circ a\circ v\circ a$. Hence,

$$\begin{aligned}
a &\leq u\circ a\circ v\circ a\leq u\circ(x\circ a\circ a\circ y)\circ v\circ a\\
&= ((u\circ x)\circ a)\circ((a\circ y\circ v)\circ a)\\
&\subseteq (S\circ L)\circ(R\circ S)\circ Q\\
&\subseteq (L\circ R\circ Q)\\
&\subseteq (L\circ R\circ Q].
\end{aligned}$$

$(2)\Longrightarrow(3)$. If $a\in S$, then $L(a)$ the left hyperideal, $R(a)$ right hyperideal and $Q(a)$ quasi-hyperideal of S generated by a respectively. By (2) we have

$$L(a)\cap R(a)\cap Q(a)\subseteq(L(a)\circ R(a)\circ Q(a)].$$

$(3) \Longrightarrow (1)$. Suppose that $a \in S$. Then,

$$\begin{array}{rcl} a & \in & L(a) \cap R(a) \cap Q(a) \\ & \subseteq & (L(a) \circ R(a) \circ Q(a)] \\ & \subseteq & (L(a) \circ R(a) \circ S] \\ & \subseteq & (L(a) \circ R(a)] \\ & = & ((S \circ a] \circ (a \circ S]] \\ & = & (((S \circ a) \circ (a \circ S)]] \\ & = & ((S \circ a) \circ (a \circ S)] \\ & = & \left(S \circ a^2 \circ S\right]. \end{array}$$

Thus, S is intra-regular ordered semihypergroup. Again, we have

$$\begin{array}{rcl} a & \in & L(a) \cap R(a) \cap Q(a) \\ & \subseteq & (L(a) \circ R(a) \circ Q(a)] \\ & = & ((S \circ a] \circ (a \circ S] \circ (S \circ a \cap a \circ S]] \\ & \subseteq & ((S \circ a) \circ (a \circ S) \circ (S \circ a \cap a \circ S]] \\ & = & ((S \circ a) \circ (a \circ S) \circ (S \circ a \cap a \circ S)] \\ & = & \left(\left(S \circ a^2 \circ S\right) \circ (S \circ a \cap a \circ S)\right] \\ & = & \left(S \circ a^2 \circ S^2 \circ a \cap S \circ a^2 \circ S \circ a \circ S\right] \\ & \subseteq & (S \circ a \circ S \circ a \cap S \circ a \circ S \circ a \circ S] \\ & \subseteq & (S \circ a \circ S \circ a]. \end{array}$$

Thus, S is left-weakly regular. □

Theorem 4.2. *An ordered semihypergroup S with identity element 1, is both intra-regular and left weakly-regular if and only if for every int-soft left hyperideal f_A, every int-soft right hyperideal g_B and every int-soft quasi-hyperideal h_C of S over U, we have*

$$f_A \widetilde{\cap} g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A \widetilde{*} g_B \widetilde{*} h_C.$$

Proof. Let S be both intra-regular and left weakly-regular ordered semihypergroup. Let f_A be an int-soft left hyperideal, g_B an int-soft right hyperideal and h_C an int-soft quasi-hyperideal of S over U. Then, for each $a \in S$, we obtain

$$\left(f_A \widetilde{\cap} g_B \widetilde{\cap} h_C\right)(a) \widetilde{\subseteq} \left(f_A \widetilde{*} g_B \widetilde{*} h_C\right)(a).$$

Since S is intra-regular, it follows that there exist $x, y \in S$ such that $a \leq x \circ a^2 \circ y$. Since S is left weakly-regular, it follows that there exist $u, v \in S$, such that

$a \leq u \circ a \circ v \circ a$. Then, $a \leq u \circ a \circ v \circ a \leq u \circ \left(x \circ a^2 \circ y\right) \circ v \circ a = ((u \circ x) \circ a) \circ (a \circ (y \circ v) \circ a)$. So there exist $\alpha \in (u \circ x) \circ a$ and $\beta \in a \circ (y \circ v) \circ a$ such that $a \leq \alpha \circ \beta$. Then, $(\alpha, \beta) \in A_a$. Since $A_a \neq \emptyset$, it follows that

$$\begin{aligned}
\left(f_A \widetilde{*} g_B \widetilde{*} h_C\right)(a) &= \bigcup_{(p,q) \in A_a} \left\{f_A(p) \cap \left(g_B \widetilde{*} h_C\right)(q)\right\} \\
&\supseteq \left\{f_A(\alpha) \cap \left(g_B \widetilde{*} h_C\right)(\beta)\right\} \\
&= \left\{ f_A(\alpha) \cap \bigcup_{(p_1,q_1) \in A_\beta} \left(g_B(p_1) \cap h_C(q_1)\right) \right\} \\
&\supseteq f_A(\alpha) \cap g_B(\gamma) \cap h_C(a).
\end{aligned}$$

Since $\beta \in a \circ (y \circ v) \circ a = (a \circ y \circ v) \circ a$, it follows that there exists $\gamma \in (a \circ y \circ v)$ such that $\beta \leq \gamma \circ a$. Since f_A is an int-soft left hyperideal and g_B is an int-soft right hyperideal of S over U, then $\bigcap_{\alpha \in (u \circ x) \circ a} f_A(\alpha) \supseteq f_A(a)$ and $\bigcap_{\gamma \in a \circ (y \circ v)} g_B(\gamma) \supseteq g_B(a)$. Hence, $f_A(\alpha) \supseteq f_A(a)$ and $g_B(\gamma) \supseteq g_B(a)$. Thus, we obtain ,

$$\begin{aligned}
\left(f_A \widetilde{*} g_B \widetilde{*} h_C\right)(a) &\supseteq f_A(\alpha) \cap g_B(\gamma) \cap h_C(a) \\
&\supseteq f_A(a) \cap g_B(a) \cap h_C(a) \\
&= \left(f_A \widetilde{\cap} g_B \widetilde{\cap} h_C\right)(a).
\end{aligned}$$

Conversely, assume that $f_A \widetilde{\cap} g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A \widetilde{*} g_B \widetilde{*} h_C$ for every int-soft left hyperideal f_A, every int-soft right hyperideal g_B and every int-soft quasi-hyperideal h_C of S over U. Then, S is both intra-regular and left weakly-regular. In fact, by Lemma 4.1, it is enough to prove that

$$L(a) \cap R(a) \cap Q(a) \subseteq (L(a) \circ R(a) \circ Q(a)] \text{ for all } a \in S.$$

Let $a \in S$, $b \in L(a) \cap R(a) \cap Q(a)$. Then, $b \in (L(a) \circ R(a) \circ Q(a)]$. Indeed, $L(a)$ is a left hyperideal, $R(a)$ a right hyperideal and $Q(a)$ a quasi-hyperideal of S generated by a respectively. Then by Lemma 2.13, $\mathcal{S}_{L(a)}$ is an int-soft left hyperideal, $\mathcal{S}_{R(a)}$ an int-soft right hyperideal and by Lemma 2.14, $\mathcal{S}_{Q(a)}$ an int-soft quasi-hyperideal of S over U. Then by hypothesis,

$$\left(\mathcal{S}_{L(a)} \widetilde{\cap} \mathcal{S}_{R(a)} \widetilde{\cap} \mathcal{S}_{Q(a)}\right)(b) \widetilde{\subseteq} \left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{Q(a)}\right)(b).$$

Since $\left(\mathcal{S}_{L(a)} \widetilde{\cap} \mathcal{S}_{R(a)} \widetilde{\cap} \mathcal{S}_{Q(a)}\right)(b) = \left\{\mathcal{S}_{L(a)}(b) \cap \mathcal{S}_{R(a)}(b) \cap \mathcal{S}_{Q(a)}(b)\right\}$, it follows that

$$\left\{\mathcal{S}_{L(a)}(b) \cap \mathcal{S}_{R(a)}(b) \cap \mathcal{S}_{Q(a)}(b)\right\} \subseteq \left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{Q(a)}\right)(b).$$

Since $b \in L(a)$, $b \in R(a)$ and $b \in Q(a)$, it follows that $\mathcal{S}_{L(a)}(b) = U$, $\mathcal{S}_{R(a)}(b) = U$ and $\mathcal{S}_{Q(a)}(b) = U$. Thus, we obtain $\left\{\mathcal{S}_{L(a)}(b) \cap \mathcal{S}_{R(a)}(b) \cap \mathcal{S}_{Q(a)}(b)\right\} = U$ and so

$$\left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{Q(a)}\right)(b) = U.$$

But from Proposition 2.12, it follows that

$$\mathcal{S}_{L(a)}\widetilde{*}\mathcal{S}_{R(a)}\widetilde{*}\mathcal{S}_{Q(a)}=\mathcal{S}_{(L(a)\circ R(a)\circ Q(a)]}.$$

Thus, $\mathcal{S}_{(L(a)\circ R(a)\circ Q(a)]}(b)=U$ implies that $b\in(L(a)\circ R(a)\circ Q(a)]$. Therefore by Lemma 4.1, it follows that S is both intra-regular and left weakly-regular. □

Lemma 4.3. *Let $(S,\circ,\leq)$ be an ordered semihypergroup with identity element 1. Then, the following are equivalent:*

(1) S is both intra-regular and left weakly-regular.

(2) $L\cap R\cap B\subseteq(L\circ R\circ B]$ for every left hyperideal L, every right hyperideal R and every bi-hyperideal B of S.

(3) $L(a)\cap R(a)\cap B(a)\subseteq(L(a)\circ R(a)\circ B(a)]$ for every $a\in S$.

Proof. $(1)\Longrightarrow(2)$. Let S be both intra-regular and left weakly-regular ordered semihypergroup. Then, $L\cap R\cap B\subseteq(L\circ R\circ B]$ for every left hyperideal L, right hyperideal R and bi-hyperideal B of S. In fact, if $a\in L\cap R\cap B$, then $a\in L$, $a\in R$ and $a\in B$. Since S is intra-regular, it follows that there exist $x,y\in S$ such that $a\leq x\circ a^2\circ y$ and since S is left weakly-regular, there exist $u,v\in S$ such that $a\leq u\circ a\circ v\circ a$. Hence,

$$\begin{aligned}
a &\leq u\circ a\circ v\circ a\leq u\circ(x\circ a\circ a\circ y)\circ v\circ a\\
&= ((u\circ x)\circ a)\circ(a\circ(y\circ v)\circ a)\subseteq(S\circ L)\circ(R\circ S)\circ B\\
&\subseteq (L\circ R\circ B)\subseteq(L\circ R\circ B].
\end{aligned}$$

$(2)\Longrightarrow(3)$. Obvious.

$(3)\Longrightarrow(1)$. Suppose that $a\in S$. Then,

$$\begin{aligned}
a &\in L(a)\cap R(a)\cap B(a)\\
&\subseteq (L(a)\circ R(a)\circ B(a)]\\
&\subseteq (L(a)\circ R(a)\circ S]\\
&\subseteq (L(a)\circ R(a)]\\
&= ((S\circ a]\circ(a\circ S]]\\
&= ((S\circ a)\circ(a\circ S)]\\
&= \left(S\circ a^2\circ S\right].
\end{aligned}$$

Then, S is intra-regular ordered semihypergroup. Also, we have

$$\begin{aligned} a &\in L(a)\cap R(a)\cap B(a) \\ &\subseteq (L(a)\circ R(a)\circ B(a)] \\ &= ((S\circ a]\circ(a\circ S]\circ(a\circ S\circ a]] \\ &= (((S\circ a)\circ(a\circ S)\circ(a\circ S\circ a)]] \\ &= ((S\circ a)\circ(a\circ S)\circ(a\circ S\circ a)] \\ &= \left(\left(S\circ a^2\circ S\right)\circ(a\circ S\circ a)\right] \\ &\subseteq \left(S\circ a^2\circ S\circ a\circ S\circ a\right] \\ &\subseteq (S\circ a\circ S\circ a]. \end{aligned}$$

Hence, S is left weakly-regular. □

Theorem 4.4. *An ordered semihypergroup S is both intra-regular and left weakly-regular if and only if for every int-soft left hyperideal f_A, every int-soft right hyperideal g_B and every int-soft bi-hyperideal h_C of S over U, we have*

$$f_A\widetilde{\cap}g_B\widetilde{\cap}h_C\widetilde{\subseteq}f_A\widetilde{*}g_B\widetilde{*}h_C.$$

Proof. Let S be both intra-regular and left weakly-regular ordered semihypergroup. Let f_A be an int-soft left hyperideal, g_B an int-soft right hyperideal and h_C an int-soft bi-hyperideal of S over U. Then, for each $a\in S$, we obtain

$$\left(f_A\widetilde{\cap}g_B\widetilde{\cap}h_C\right)(a)\widetilde{\subseteq}\left(f_A\widetilde{*}g_B\widetilde{*}h_C\right)(a).$$

Since S is intra-regular, it follows that there exist $x,y\in S$ such that $a\leq x\circ a^2\circ y$. Since S is left weakly-regular, it follows that there exist $u,v\in S$, such that $a\leq u\circ a\circ v\circ a$. Then,

$$a\leq u\circ a\circ v\circ a\leq u\circ\left(x\circ a^2\circ y\right)\circ v\circ a=((u\circ x)\circ a)\circ(a\circ(y\circ v)\circ a).$$

So there exist $\alpha\in(u\circ x)\circ a$ and $\beta\in a\circ(y\circ v)\circ a$ such that $a\leq\alpha\circ\beta$. Then, $(\alpha,\beta)\in A_a$. Since $A_a\neq\emptyset$, it follows that

$$\begin{aligned} \left(f_A\widetilde{*}g_B\widetilde{*}h_C\right)(a) &= \bigcup_{(p,q)\in A_a}\left\{f_A(p)\cap\left(g_B\widetilde{*}h_C\right)(q)\right\} \\ &\supseteq \left\{f_A(\alpha)\cap\left(g_B\widetilde{*}h_C\right)(\beta)\right\} \\ &= \left\{f_A(\alpha)\cap\bigcup_{(p_1,q_1)\in A_\beta}\left(g_B(p_1)\cap h_C(q_1)\right)\right\} \\ &\supseteq f_A(\alpha)\cap g_B(\gamma)\cap h_C(a). \end{aligned}$$

Since $\beta \in a \circ (y \circ v) \circ a = (a \circ y \circ v) \circ a$, it follows that there exists $\gamma \in a \circ y \circ v$ such that $\beta \leq \gamma \circ a$. Since f_A is an int-soft left hyperideal and g_B is an int-soft right hyperideal of S over U, we have $\bigcap\limits_{\alpha \in (u \circ x) \circ a} f_A(\alpha) \supseteq f_A(a)$ and $\bigcap\limits_{\gamma \in a \circ (y \circ v)} g_B(\gamma) \supseteq g_B(a)$. Hence, $f_A(\alpha) \supseteq f_A(a)$ and $g_B(\gamma) \supseteq g_B(a)$. Thus,

$$\begin{aligned}
(f_A \widetilde{*} g_B \widetilde{*} h_C)(a) &\supseteq f_A(\alpha) \cap g_B(\gamma) \cap h_C(a) \\
&\supseteq f_A(a) \cap g_B(a) \cap h_C(a) \\
&= \left(f_A \widetilde{\cap} g_B \widetilde{\cap} h_C\right)(a).
\end{aligned}$$

Conversely, assume that $f_A \widetilde{\cap} g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A \widetilde{*} g_B \widetilde{*} h_C$ for every int-soft left hyperideal f_A, every int-soft right hyperideal g_B and every int-soft bi-hyperideal h_C of S over U. Then S is both intra-regular and left weakly-regular. In fact, by Lemma 4.3, it is enough to prove that

$$L(a) \cap R(a) \cap B(a) \subseteq (L(a) \circ R(a) \circ B(a)] \text{ for all } a \in S.$$

Let $a \in S$, $b \in L(a) \cap R(a) \cap B(a)$. Clearly, $b \in (L(a) \circ R(a) \circ B(a)]$. Since $L(a)$ is a left hyperideal, $R(a)$ a right hyperideal and $B(a)$ a bi-hyperideal of S generated by a respectively, by Lemma 2.13, we have that $\mathcal{S}_{L(a)}$ is an int-soft left hyperideal, $\mathcal{S}_{R(a)}$ an int-soft right hyperideal and by Lemma 2.14, $\mathcal{S}_{B(a)}$ an int-soft bi-hyperideal of S over U. Hence by hypothesis,

$$\left(\mathcal{S}_{L(a)} \widetilde{\cap} \mathcal{S}_{R(a)} \widetilde{\cap} \mathcal{S}_{B(a)}\right)(b) \widetilde{\subseteq} \left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{B(a)}\right)(b).$$

Since

$$\left(\mathcal{S}_{L(a)} \widetilde{\cap} \mathcal{S}_{R(a)} \widetilde{\cap} \mathcal{S}_{B(a)}\right)(b) = \left\{\mathcal{S}_{L(a)}(b) \cap \mathcal{S}_{R(a)}(b) \cap \mathcal{S}_{B(a)}(b)\right\},$$

we obtain

$$\left\{\mathcal{S}_{L(a)}(b) \cap \mathcal{S}_{R(a)}(b) \cap \mathcal{S}_{B(a)}(b)\right\} \subseteq \left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{B(a)}\right)(b).$$

Since $b \in L(a)$, $b \in R(a)$ and $b \in B(a)$, hence $\mathcal{S}_{L(a)}(b) = U$, $\mathcal{S}_{R(a)}(b) = U$ and $\mathcal{S}_{B(a)}(b) = U$, thus we obtain $\left\{\mathcal{S}_{L(a)}(b) \cap \mathcal{S}_{R(a)}(b) \cap \mathcal{S}_{B(a)}(b)\right\} = U$ and so

$$\left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{B(a)}\right)(b) = U.$$

From Proposition 2.12, it follows that

$$\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{R(a)} \widetilde{*} \mathcal{S}_{B(a)} = \mathcal{S}_{(L(a) \circ R(a) \circ B(a)]}.$$

Thus, $\mathcal{S}_{(L(a) \circ R(a) \circ B(a)]}(b) = U$ implies that $b \in (L(a) \circ R(a) \circ B(a)]$. Therefore by Lemma 4.3, it follows that S is both intra-regular and left weakly-regular. $\square$

5. Characterizations of semisimple ordered semihypergroups in terms of int-soft hyperideals

In this paragraph, we prove that an ordered semihypergroup S is semisimple if and only if for every int-soft two-sided hyperideal f_A of S over U, we have, $f_A \widetilde{*} f_A = f_A$. We prove that in semisimple ordered semihypergroups the concepts of int-soft hyperideals and int-soft interior hyperideals coincide.

Proposition 5.1. *Let $(S,\circ,\leq)$ be a semisimple ordered semihypergroup, f_A be an int-soft interior hyperideal of S over U. Then, f_A an int-soft two-sided hyperideal of S over U.*

Proof. Let f_A be an int-soft interior hyperideal of S over U. Let $a,b \in S$. Since S is semisimple, it follows that there exist $x,y,z \in S$ such that $a \leq x \circ a \circ y \circ a \circ z$. Thus, $a \circ b \leq x \circ a \circ y \circ a \circ z \circ b = (x \circ a \circ y) \circ a \circ (z \circ b)$. Then, there exist $\alpha \in a \circ b$, $\beta \in x \circ a \circ y$, $\gamma \in z \circ b$ and $\delta \in \beta \circ a \circ \gamma$ such that $\alpha \leq \delta$. Since f_A is an int-soft interior hyperideal of S over U, it follows that $f_A(\alpha) \supseteq f_A(\delta) \supseteq \bigcap\limits_{\delta \in \beta \circ a \circ \gamma} f_A(\delta) \supseteq f_A(a)$. Thus, $\bigcap\limits_{\alpha \in a \circ b} f_A(\alpha) \supseteq f_A(a)$. Therefore, f_A is an int-soft right hyperideal of S over U. Similarly we can prove that f_A is an int-soft left hyperideal of S over U. Thus, f_A is an int-soft hyperideal of S over U. $\square$

The following proposition is a special case of Proposition 5.1.

Proposition 5.2. *Let $(S,\circ,\leq)$ be a semisimple ordered semihypergroup, I an interior hyperideal of S. Then, I is a two-sided hyperideal of S.*

Combining Propositions 2.15 and 5.2, we have the following:

Theorem 5.3. *In semisimple ordered semihypergroups the concepts of int-soft hyperideals and int-soft interior hyperideals coincide.*

Lemma 5.4. *Let $(S,\circ,\leq)$ be an ordered semihypergroup with identity element* 1. *Then, the following are equivalent:*

(1) S is semisimple.
(2) $I_1 \cap I_2 = (I_1 \circ I_2]$ for all hyperideals I_1, I_2 of S.
(3) $I = \left(I^2\right]$ for every hyperideal I of S.
(4) $I(a) = \left(I(a)^2\right]$ for every $a \in S$.

Proof. $(1) \Longrightarrow (2)$. Let I_1 and I_2 be the hyperideals of S, and $a \in I_1 \cap I_2$. Clearly, $a \in I_1$ and $a \in I_2$. Since S is semisimple, it follows that there exist $x,y,z \in S$ such that $a \leq x \circ a \circ y \circ a \circ z$. Thus,

$$a \in (S \circ a \circ S \circ a \circ S] \subseteq ((S \circ I_1) \circ (S \circ I_2 \circ S)] \subseteq (I_1 \circ I_2].$$

On the other hand, $(I_1 \circ I_2] \subseteq I_1 \cap I_2$ always true. Thus,

$$I_1 \cap I_2 = (I_1 \circ I_2].$$

$(2) \Longrightarrow (3)$. Take $I_1 = I_2 = I$ then $I = I_1 \cap I_2 = (I_1 \circ I_2] = \left(I^2\right]$.

$(3) \Longrightarrow (4)$. Let $a \in S$. Then, $I(a)$ be a two sided hyperideal of S generated by a. By (2), we obtain ,

$$\left(I(a)^2\right] = I(a).$$

$(4) \Longrightarrow (1)$. Suppose that $a \in S$. Then,

$$\begin{aligned} a &\in I(a) = \left(I(a)^2\right] \\ &= ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]] \\ &= (((S \circ a \circ S) \circ (S \circ a \circ S)]] \\ &= ((S \circ a \circ S) \circ (S \circ a \circ S)] \\ &\subseteq (S \circ a \circ S \circ a \circ S]. \end{aligned}$$

Thus, S is a semisimple ordered semihypergroup. □

Theorem 5.5. *Let $(S, \circ, \leq)$ be an ordered semihypergroup with identity element 1. Then, S is semisimple if and only if for every int-soft two-sided hyperideal f_A of S over U, we obtain ,*

$$f_A \widetilde{*} f_A = f_A.$$

Proof. Let S be a semisimple ordered semihypergroup, and $a \in S$. Then,

$$(f_A \widetilde{*} f_A)(a) = f_A(a).$$

In fact, since S is semisimple, there exist $x, y, z \in S$ such that $a \leq (x \circ a \circ y) \circ (a \circ z)$. Then, for some $\alpha \in x \circ a \circ y$ and $\beta \in a \circ z$, we have $a \leq \alpha \circ \beta$. Then, $(\alpha, \beta) \in A_a$. Since $A_a \neq \emptyset$, it follows that

$$\begin{aligned} (f_A \widetilde{*} f_A)(a) &= \bigcup_{(p,q) \in A_a} \{f_A(p) \cap f_A(q)\} \\ &\supseteq \{f_A(\alpha) \cap f_A(\beta)\}. \end{aligned}$$

Since f_A is an int-soft two-sided hyperideal of S over U, it follows that

$$\begin{aligned} \bigcap_{\alpha \in x \circ a \circ y} f_A(\alpha) &= \bigcap_{\substack{\alpha \in x \circ u \\ u \in a \circ y}} f_A(\alpha) \\ &\supseteq f_A(u) \\ &\supseteq \bigcap_{u \in a \circ y} f_A(u) \\ &\supseteq f_A(a) \end{aligned}$$

and

$$\bigcap_{\beta \in a \circ z} f_A(\beta) \supseteq f_A(a).$$

Hence, $f_A(\alpha) \supseteq f_A(a)$ and $f_A(\beta) \supseteq f_A(a)$. Thus, we obtain ,

$$\begin{aligned} (f_A \widetilde{*} f_A)(a) &\supseteq \{f_A(\alpha) \cap f_A(\beta)\} \\ &\supseteq f_A(a) \cap f_A(a) \\ &= f_A(a). \end{aligned}$$

For the reverse inclusion, since f_A is an int-soft hyperideal of S over U, by Corollary 2.19, it follows that $f_A \widetilde{*} f_A \widetilde{\subseteq} f_A$. Thus, $f_A \widetilde{*} f_A = f_A$.

Conversely, assume that $f_A \widetilde{*} f_A = f_A$ for every int-soft two-sided hyperideal f_A of S over U. Then, S is semisimple. In fact: By Lemma 5.4, it is enough to prove that

$$I(a) = \left(I(a)^2\right] \ \forall a \in S.$$

Let $a \in S$, $b \in I(a)$. By Lemma 2.13, since $I(a)$ is a hyperideal of S generated by a, then $\mathcal{S}_{I(a)}$ is an int-soft hyperideal of S over U. By hypothesis

$$\left(\mathcal{S}_{I(a)} \widetilde{*} \mathcal{S}_{I(a)}\right)(b) = \mathcal{S}_{I(a)}(b).$$

Since $b \in I(a)$, it follows that $\mathcal{S}_{I(a)}(b) = U$. Hence, we have

$$\left(\mathcal{S}_{I(a)} \widetilde{*} \mathcal{S}_{I(a)}\right)(b) = U.$$

By Proposition 2.12, we have,

$$\mathcal{S}_{I(a)} \widetilde{*} \mathcal{S}_{I(a)} = \mathcal{S}_{\left(I(a)^2\right]}.$$

Thus, $\mathcal{S}_{\left(I(a)^2\right]}(b) = U \Longrightarrow b \in \left(I(a)^2\right]$. Consequently, $I(a) \subseteq \left(I(a)^2\right]$. On the other hand, $\left(I(a)^2\right] \subseteq I(a)$ always true. Hence, $I(a) = \left(I(a)^2\right]$. Therefore, S is semisimple. □

Lemma 5.6. *Let* $(S, \circ, \leq)$ *be an ordered semihypergroup with identity element* 1. *Then, the following are equivalent:*

(1) S is semisimple.
(2) $R \cap I \subseteq (I \circ R]$ (resp. $L \cap I \subseteq (L \circ I]$)
for each right hyperideal R (resp. each left hyperideal L) and two-sided hyperideal I of S.
(3) $R(a) \cap I(a) \subseteq (I(a) \circ R(a)]$ (resp. $L(a) \cap I(a) \subseteq (L(a) \circ I(a)]$) for every $a \in S$.

Proof. $(1) \Longrightarrow (2)$. Let S be a semisimple ordered semihypergroup. Let $a \in R \cap I$. Clearly, $a \in R$ and $a \in I$. Since $a \in S$ and S is semisimple, it follows that there exist $x, y, z \in S$ such that $a \leq (x \circ a \circ y) \circ (a \circ z) \subseteq ((S \circ I \circ S) \circ (R \circ S)] \subseteq (I \circ R]$.

$(2) \Longrightarrow (3)$. Suppose that $a \in S$. Then, $R(a)$ a right hyperideal and $I(a)$ a two-sided hyperideal of S generated by a, respectively. By (2) we obtain ,

$$R(a) \cap I(a) \subseteq (I(a) \circ R(a)].$$

$(3) \Longrightarrow (1)$. Suppose that $a \in S$. Then,

$$\begin{aligned} a &\in R(a) \cap I(a) \subseteq (I(a) \circ R(a)] \\ &= ((S \circ a \circ S] \circ (a \circ S]] \\ &\subseteq (((S \circ a \circ S) \circ (a \circ S)]] \\ &= (S \circ a \circ S \circ a \circ S]. \end{aligned}$$

Thus, S is semisimple. □

Theorem 5.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup with identity element 1. Then, S is semisimple if and only if for every int-soft left hyperideal f_A and every int-soft two-sided hyperideal g_B of S over U, we obtain ,*

$$f_A \widetilde{\cap} g_B \widetilde{\subseteq} f_A \widetilde{*} g_B.$$

Proof. Let S be a semisimple ordered semihypergroup, and $a \in S$. Since S is semisimple, it follows that there exist $x, y, z \in S$ such that $a \leq (x \circ a) \circ (y \circ a \circ z)$. Then, for some $\alpha \in x \circ a$ and $\beta \in y \circ a \circ z$, we have $a \leq \alpha \circ \beta$ and $(\alpha, \beta) \in A_a$. Since $A_a \neq \emptyset$, it follows that

$$\begin{aligned} (f_A \widetilde{*} g_B)(a) &= \bigcup_{(p,q) \in A_a} \{f_A(p) \cap g_B(q)\} \\ &\supseteq \{f_A(\alpha) \cap f_A(\beta)\}. \end{aligned}$$

Since f_A is an int-soft left hyperideal and g_B an int-soft two-sided hyperideal of S over U, we obtain $\bigcap_{\alpha \in x \circ a} f_A(\alpha) \supseteq f_A(a)$ and $\bigcap_{\beta \in y \circ a \circ z} g_B(\beta) = \bigcap_{\substack{\beta \in y \circ u \\ u \in a \circ z}} g_B(\beta) \supseteq g_B(u) \supseteq \bigcap_{u \in a \circ z} g_B(u) \supseteq g_B(a)$. Hence, $f_A(\alpha) \supseteq f_A(a)$ and $g_B(\beta) \supseteq g_B(a)$. Thus,

$$\begin{aligned} (f_A \widetilde{*} g_B)(a) &\supseteq \{f_A(\alpha) \cap f_A(\beta)\} \\ &\supseteq f_A(a) \cap g_B(a) \\ &= (f_A \widetilde{\cap} g_B)(a). \end{aligned}$$

Hence, $f_A \widetilde{\cap} g_B \widetilde{\subseteq} f_A \widetilde{*} g_B$.

Conversely, assume that $f_A \widetilde{\cap} g_B \widetilde{\subseteq} f_A \widetilde{*} g_B$ for every int-soft left hyperideal f_A and int-soft two-sided hyperideal g_B of S over U. Suppose that $a \in S$. Then, S is semisimple. Indeed, by Lemma 5.6, we show that

$$R(a) \cap I(a) \subseteq (I(a) \circ R(a)] \ \forall a \in S.$$

Let $a \in S$ and $b \in R(a) \cap I(a)$. Then, $b \in (I(a) \circ R(a)]$. Indeed, $L(a)$ is a left hyperideal and $R(a)$ a right hyperideal of S generated by a respectively. By Lemma 2.13, $\mathcal{S}_{L(a)}$ is an int-soft left hyperideal and $\mathcal{S}_{I(a)}$ is an int-soft two-sided hyperideal of S over U, and by hypothesis

$$\left(\mathcal{S}_{L(a)} \widetilde{\cap} \mathcal{S}_{I(a)}\right)(b) \widetilde{\subseteq} \left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{I(a)}\right)(b).$$

Since $\left(\mathcal{S}_{L(a)} \widetilde{\cap} \mathcal{S}_{I(a)}\right)(b) = \left(\mathcal{S}_{L(a)}(b) \widetilde{\cap} \mathcal{S}_{I(a)}(b)\right)$, we obtain

$$\left(\mathcal{S}_{L(a)}(b) \widetilde{\cap} \mathcal{S}_{I(a)}(b)\right) \widetilde{\subseteq} \left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{I(a)}\right)(b).$$

Since $b \in L(a)$ and $b \in I(a)$, hence $\mathcal{S}_{L(a)}(b) = U$ and $\mathcal{S}_{I(a)}(b) = U$, then we have, $\left(\mathcal{S}_{L(a)}(b) \widetilde{\cap} \mathcal{S}_{I(a)}(b)\right) = U$, and hence $\left(\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{I(a)}\right)(b) = U$. But from Proposition 2.12, it follows that

$$\mathcal{S}_{L(a)} \widetilde{*} \mathcal{S}_{I(a)} = \mathcal{S}_{(L(a) \circ I(a)]}.$$

Thus, $\mathcal{S}_{(L(a) \circ I(a)]}(b) = U \Longrightarrow b \in (L(a) \circ I(a)]$. Thus by Lemma 5.6, it follows that S is semisimple. $\square$

6. Conclusion

We have considered the following items.

1. To characterize weakly regular ordered semihypergroups by means of int-soft left (right) hyperideals and int-soft quasi-hyperideals.

2. To characterize intra-regular and left weakly-regular ordered semihypergroups by means of int-soft left (right) hyperideals, int-soft bi-hyperideals and quasi-hyperideals.

3. To characterize semisimple ordered semihypergroups by means of int-soft two-sided hyperideal and int-soft left hyperideals

Work is on going. Some important issues for future work are

1. To develop strategies for obtaining more valuable results.

2. To apply these notions and results for studying related notions in other soft algebraic structures.

REFERENCES

[1] H. Aktas - N. Cağman, *Soft sets and soft groups*, Information Sciences 177 (2007), 2726–2735.

[2] F. Feng - Y.B. Jun - X. Zhao, *Soft semirings*, Computers & Mathematics with Applications 56 (2008) 2621–2628.

[3] F. Feng - M.I. Ali, and M. Shabir, *Soft relations applied to semigroups*, Filomat 27 (2013), 1183–1196.

[4] F. Feng - Y.M. Li, *Soft subsets and soft product operations*, Information Sciences 232 (2013), 44–57.

[5] X. Ma - J. Zhan, *Characterizations of three kinds of hemirings by fuzzy soft h-ideals*, J. Intel. Fuzzy Syst. 24 (2013), 535-548.

[6] D. Molodtsov, *Soft set theory–first results*, Computers & Mathematics with Applications, 37 (1999), 19–31.

[7] J. Zhan - N. Čağman - A.S. Sezer, *Applications of soft union sets to hemirings via SU-h-ideals*, Journal of Intelligent & Fuzzy Systems 26 (2014), 1363-1370.

[8] F. Marty, *Sur Une generalization de la notion de group*, 8^{iem}congress, Math Scandinaves Stockholm (1934), 45-49.

[9] A. Khan - M. Farooq - B. Davvaz, *Int-soft left (right) hyperideals of ordered semihypergroups*, Submitted.

[10] A. Khan - M. Farooq - B. Davvaz, *Int-soft interior hyperideals of ordered semihypergroups*, International Journal of Analysis and Applications 14 (2017), 193–202.

[11] A. Khan - M. Farooq - B. Davvaz, *Prime Int-soft bi-hyperideals of ordered semihypergroups*, Submitted.

[12] S. Naz - M. Shabir, *On soft semihypergroups*, Journal of Intelligent & Fuzzy Systems 26 (2014), 2203-2213.

[13] A. Khan - M. Farooq - B. Davvaz, *Int-soft quasi-hyperideals of ordered semihypergroups*, Submitted.

[14] M. Farooq - A. Khan - B. Davvaz, *Characterizations of ordered semihypergroups by the properties of their intersectional-soft generalized bi-hyperideals*, Soft Comput., DOI 10.1007/s00500-017-2550-6.

[15] P. Corsini - V. Leoreanu-Fotea, *Applications of hyperstructure theory*, Advanced in Mathematics, Kluwer Academic Publisher, (2003).

[16] D. Heidari - B. Davvaz, *On Ordered Hyperstructures*, U.P.B. Sci. Bull. Series A 73 (2011), 85–96.

[17] B. Pibaljommee - K. Wannatong - B. Davvaz, *An investigation on fuzzy hyperideals of ordered semihypergroups*, Quasigroups and Related Systems 23 (2015), 297–308.

[18] N. Tipachot - B. Pibaljommee, *Fuzzy interior hyperideals in ordered semihypergroups*, Italian Journal of Pure and Applied Mathematics 36 (2016), 859–870.

[19] J. Tang - B. Davvaz - Y.F. Luo *A study on fuzzy interior hyperideals in ordered*

semihypergroups, Italian Journal of Pure and Applied Mathematics 36 (2016), 125–146.

[20] T. Changphas - B. Davvaz, *Bi-hyperideals and quasi-hyperideals in ordered semihypergroups*, Italian Journal of Pure and Applied Mathematics 35 (2015), 493–508.

[21] J. Tang - B. Davvaz - Y. Lau, *Hyperfilters and fuzzy hyperfilters of ordered semihypergroups*, Journal of Intelligent & Fuzzy Systems 29 (2015), 75–84.

[22] J. Chvalina, *Commutative hypergroups in the sense of Marty and ordered sets*, General algebra and ordered sets (Horn$\widetilde{a}$Lipova, 1994), 19–30.

[23] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), 87–96.

[24] W.L. Gau - D.J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man and Cybernetics 23 (1993), 610–614.

[25] L.A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965), 338–353.

[26] M.B. Gorzalzany, *A method of inference in approximate reasoning based on interval-valued fuzzy sets*, Fuzzy Sets and Systems 21 (1987), 1–17.

[27] Z. Pawlak, *Rough sets*, International Journal of Information and Computer Sciences 11 (1982), 341–356.

[28] D. Molodtsov, *The Theory of Soft Sets*, URSS Publishers, Moscow, 2004 (in Russian).

[29] P.K. Maji - A. R. Roy - R. Biswas, *An application of soft sets in a decision making problem*, Computers & Mathematics with Applications 44 (2002), 1077–1083.

[30] P.K. Maji, R. Biswas, and A.R.Roy, *Soft set theory*, Computers & Mathematics with Applications 45 (2003), 555–562.

[31] N. Cağman - S. Enginoglu, *FP-soft set theory and its applications*, Annals of Fuzzy Mathematics and Informatics 2 (2011), 219–226.

[32] F. Feng, *Soft rough sets applied to multicriteria group decision making*, Annals of Fuzzy Mathematics and Informatics 2 (2011), 69–80.

[33] H. Aktas - N. Cağman, *Soft sets and soft groups*, Information Sciences 177 (2007), 2726–2735.

[34] A. Sezgin - A.O. Atagun, *On operations of soft sets*, Comput Math. Appl. 61 (2011), 1457–1467.

[35] Y. B. Jun - S.Z. Song - G. Muhiuddin, *Concave soft sets, critical soft points, and union-soft ideals of ordered semigroups*, The Scientific World Journal, 2014, Article ID 467968, 11 pages.

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