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COMPLEX FACTORIZATION BY CHEBYSEV POLYNOMIALS

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Let $\{a_i\},\{b_i\}$ be real numbers for $0 \leqslant i \leqslant r-1$, and define a *r-periodic sequence* $\{v_n\}$ with initial conditions v_0, v_1 and recurrences $v_n = a_t v_{n-1} + b_t v_{n-2}$ where $n \equiv t \,(\bmod r)$ $(n \geqslant 2)$. In this paper, by aid of Chebyshev polynomials, we introduce a new method to obtain the complex factorization of the sequence $\{v_n\}$ so that we extend some recent results and solve some open problems. Also, we provide new results by obtaining the binomial sum for the sequence $\{v_n\}$ by using Chebyshev polynomials.

Let $\{a_i\}$ and $\{b_i\}$ be real numbers for $0 \leqslant i \leqslant r-1$, and define a sequence $\{v_n\}$ with initial conditions v_0, v_1, and for $n \geqslant 2$,

$$v_n = \begin{cases} a_0 v_{n-1} + b_0 v_{n-2}, & \text{if } n \equiv 0 \,(\bmod r)\,, \\ a_1 v_{n-1} + b_1 v_{n-2}, & \text{if } n \equiv 1 \,(\bmod r)\,, \\ \vdots & \vdots \\ a_{r-1} v_{n-1} + b_{r-1} v_{n-2}, & \text{if } n \equiv r-1 \,(\bmod r)\,. \end{cases} \tag{1}$$

We call $\{v_n\}$ as a *r-periodic sequences*. It is studied in [7] by Panario et. al. and they find the generating function and Binet's like formula for the sequence $\{v_n\}$ via generalized continuant. Petronilho obtain the same Binet's like formula by using tools from ortogonal polynomials in [8].

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For $r = 2$ and initial values $v_0 = 1$, $v_1 = a_1$, Cooper and Parry [3] called the sequence $\{v_n\}$ as *the period two second order linear recurrence system*, and gave the complex factorization of odd terms of this sequence by determining the eigenvalues and eigenvectors of certain tridiagonal matrices. The problems remained unsolved in [3] are determining the complex factorization of even terms of the period two second order linear recurrence system and determining the complex factorization of the sequence $\{v_n\}$ for given general r. Also, for $r = 2$ and initial values $v_0 = 0$, $v_1 = 1$, Jun [5] give a connection between the sequence $\{v_n\}$ and Chebyshev polynomials of the second kind $\{U_n(x)\}$. By using the factorization of $\{U_n(x)\}$, Jun derive the complex factorization of the sequence $\{v_n\}$ with initial values $v_0 = 0$ and $v_1 = 1$ for $r = 2$.

In Section 3, we solve the open problems in [3] for the sequence $\{v_n\}$ with initial values $v_0 = 0$ and $v_1 = 1$ by using Chebyshev polynomials. Also, since we will get the complex factorization for any r, our results are a generalization of [5].

In Section 4, we provide new results by obtaining the binomial sum for the sequence $\{v_n\}$ by using Chebyshev polynomials of the second kind $\{U_n(x)\}$.

1. Chebyshev polynomials $\{T_n(x)\}$ and $\{U_n(x)\}$

Chebyshev polynomials of the first and second kinds are the polynomials $T_n(x)$ and $U_n(x)$, respectively, such that

$$T_n(x) = cos\left(n\cos^{-1}x\right)$$

and

$$U_n(x) = \frac{\sin((n+1)\cos^{-1}x)}{\sin(\cos^{-1}x)}.$$

Note that both formulas hold for all x where they make sense and they are defined by continuity for other values of x (since both formulas define polynomials in the variable x at least on the interval $-1 < x < 1$). Also, $T_n(x)$ and $U_n(x)$ both satisfy the following second order recurrence

$$y_{n+1}(x) = 2xy_n(x) - y_{n-1}(x), \quad n \geq 0,$$

with initial conditionas $T_{-1}(x) = x$, $T_0(x) = 1$, $T_1(x) = x$ and $U_{-1}(x) = 0$, $U_0(x) = 1$, $U_1(x) = 2x$. The complex factorization of Chebyshev polynomials is given as follows:

$$T_n(x) = 2^{n-1}\prod_{k=1}^{n}\left(x - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right), \quad n \geq 1 \tag{2}$$

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{n+1} \right) \right) \tag{3}$$

and it is well known that the binomial sums for Chebyshev polynomials are

$$T_n(x) = \frac{n}{2} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j}{n-j} \binom{n-j}{j} (2x)^{n-2j}. \tag{4}$$

and

$$U_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} x^{n-2j}. \tag{5}$$

Also, a well known relation between Chebsyshev polynomials is

$$T_n(x) = U_n(x) - xU_{n-1}(x). \tag{6}$$

(See [1], [2], [6], [9], [10] and [11] for details).

2. The Connection between $\{v_n\}$ and $\{U_n(x)\}$

In this section, we give a connection between the sequence $\{v_n\}$ and Chebyshev polynomials of second kind $\{U_n(x)\}$. We need to remind that some definitions from [8] to obtain our results.

Consider the determinant of a tridiagonal matrix

$$\Delta_{\mu,\xi} := \begin{vmatrix} a_\mu & 1 & & & \\ -b_{\mu+1} & a_{\mu+1} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{\xi-1} & a_{\xi-1} & 1 \\ & & & -b_\xi & a_\xi \end{vmatrix} \quad \text{if} \quad 0 \le \mu < \xi \le r$$

and if $\mu \ge \xi$,

$$\Delta_{\mu,\xi} := \begin{cases} 0, & \text{if} \quad \mu > \xi + 1 \\ 1, & \text{if} \quad \mu = \xi + 1 \\ a_\mu, & \text{if} \quad \mu = \xi. \end{cases}$$

Then, we consider

$$\Delta_r := \begin{vmatrix} a_2 & 1 & & & & 1 \\ -b_3 & a_3 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -b_{r-1} & a_{r-1} & 1 & \\ & & & -b_0 & a_0 & 1 \\ -b_2 & & & & -b_1 & a_1 \end{vmatrix}.$$

Also, recall the following definitions from [8]:

$$b := (-1)^r \prod_{i=0}^{r-1} b_i,$$

$$c := (-1)^r (b_2 + b/b_2)$$

and

$$\widetilde{U}_n(x) := d^n U_n\left(\frac{x-c}{2d}\right), \quad n \geq 0,$$

where d is one of the square roots of b. (k in [8] corresponds to our r).

Now, we can establish the connection between the sequence $\{v_n\}$ and Chebyshev polynomials of the second kind $\{U_n\}$.

Lemma 2.1. *For $r \geq 3$, the terms of the sequence $\{v_n\}$ are given by in terms of Chebyshev polynomials of the second kind $\{U_n\}$ as follow:*

$$v_{nr} = \Delta_{2,r}\widetilde{U}_{n-1}(\Delta_r),$$

and for $1 \leq t \leq r-1$,

$$v_{nr+t} = \Delta_{2,t}\widetilde{U}_n(\Delta_r) + (-1)^t \left(\prod_{i=2}^{t+1} b_i\right) \Delta_{t+2,r}\widetilde{U}_{n-1}(\Delta_r).$$

Proof. We will use the results from [8] in the proof. Let $\{R_{n+1}(x)\}$ be the sequence of polynomials defined by the recurrence relation

$$R_{n+1}(x) = (x-\beta_n)R_n(x) - \gamma_n R_{n-1}(x), \quad n \geq 0,$$

with initial conditions $R_{-1}(x) = 0$ and $R_0(x) = 1$ where

$$\beta_{nr+j} := -a_{j+2}, \gamma_{nr+j} := -b_{j+2}, \quad 0 \leq j \leq r-1, \quad n \geq 0.$$

Then clearly,

$$v_n = R_{n-1}(0), \quad n \geq 0. \tag{7}$$

Let $\Delta_{\mu,\xi}(x)$ be a polynomial of degree $\xi - \mu + 1$ obtained by replacing a_i by $x + a_i$ in the definition of $\Delta_{\mu,\xi}$. Similarly, let $\varphi_r(x)$ be a polynomial of degree r obtained by replacing a_i by $x + a_i$ in the definition of Δ_r. In this case, $\Delta_{\mu,\xi} = \Delta_{\mu,\xi}(0)$ and $\Delta_r = \varphi_r(0)$.

We can obtain

$$R_{nr+j}(x) = \Delta_{2,j+1}(x)\widetilde{U}_n(\varphi_r(x)) + (-1)^{j+1}\left(\prod_{i=2}^{j+2} b_i\right)\Delta_{j+3,r}(x)\widetilde{U}_{n-1}(\varphi_r(x)), \tag{8}$$

where $0 \leq j \leq r-1, n \geq 0$, by using Theorem 5.1 in [4].

If we use (7) and (8) then we get

$$\begin{aligned} v_{nr} &= R_{nr-1}(0) \\ &= R_{(n-1)r+(r-1)}(0) \quad (\text{Take } j = r-1 \text{ and } n = n-1 \text{ in } (8)) \\ &= \Delta_{2,r}\widetilde{U}_{n-1}(\varphi_r(0)) + (-1)^r \left(\prod_{i=2}^{r+1} b_i\right) \Delta_{r+2,r}(0)\widetilde{U}_{n-2}(\varphi_r(0)) \\ &= \Delta_{2,r}\widetilde{U}_{n-1}(\Delta_r) + (-1)^r \left(\prod_{i=2}^{r+1} b_i\right) \Delta_{r+2,r}\widetilde{U}_{n-2}(\Delta_r). \end{aligned}$$

Then, since $\Delta_{r+2,r} = 0$, we get the first equality in the hypothesis of theorem as follow:

$$v_{nr} = \Delta_{2,r}\widetilde{U}_{n-1}(\Delta_r).$$

Now, again if we use (7) and (8) for $1 \leq t \leq r-1$, we get the desired result

$$\begin{aligned} v_{nr+t} &= R_{nr+t-1}(0) \quad (\text{Take } j = t-1 \text{ in } (8)) \\ &= \Delta_{2,t}\widetilde{U}_n(\varphi_r(0)) + (-1)^t \left(\prod_{i=2}^{t+1} b_i\right) \Delta_{t+2,r}(0)\widetilde{U}_{n-1}(\varphi_r(0)) \\ &= \Delta_{2,t}\widetilde{U}_n(\Delta_r) + (-1)^t \left(\prod_{i=2}^{t+1} b_i\right) \Delta_{t+2,r}\widetilde{U}_{n-1}(\Delta_r). \end{aligned}$$

$\square$

Example 2.2. We combine Fibonacci, Jacobsthal and a second order recurrence equations to get the following sequence $\{v_n\}$:

$$v_n = \begin{cases} v_{n-1} + v_{n-2}, & \text{if } n \equiv 0 \,(\text{mod } 3), \\ v_{n-1} + 2v_{n-2}, & \text{if } n \equiv 1 \,(\text{mod } 3), \\ 3v_{n-1} - 2v_{n-2}, & \text{if } n \equiv 2 \,(\text{mod } 3). \end{cases}$$

A few terms of the sequence $\{v_n\}$ are $\{0,1,3,4,10,22,32,76,164,240,568, 1224,\ldots\}$ and we have $r = 3, a_0 = a_1 = b_0 = 1$, $b_1 = 2$, $a_2 = 3$ and $b_2 = -2$. We need to compute Δ_3, $\Delta_{2,1}$, $\Delta_{2,2}$, $\Delta_{2,3}$, b, c and d to establish the connection. By using the definitions from Section 2, we get

$$\Delta_3 = \begin{vmatrix} a_2 & 1 & 1 \\ -b_0 & a_0 & 1 \\ -b_2 & -b_1 & a_1 \end{vmatrix} = 12,\ \Delta_{2,3} = \begin{vmatrix} a_2 & 1 \\ -b_0 & a_0 \end{vmatrix} = 4,\ \Delta_{2,1} = 1,\ \Delta_{2,2} = a_2 = 3,$$

Table 1:

n	v_{3n}	$2^{n+1}U_{n-1}(x)$
1	4	4
2	32	$16x$
3	240	$64x^2-16$
4	1792	$256x^3-128x$
5	13376	$1024x^4-768x^2+64$
6	99840	$4096x^5-4096x^3+768x$

$$b=(-1)^3 b_0 b_1 b_2 = 4, \quad d=\sqrt{b}=2, \quad c=(-1)^3\left(b_2+\frac{b}{b_2}\right)=4.$$

Now, substituting these values in Lemma 2.1, we obtain

$$\begin{aligned} v_{3n} &= \Delta_{2,3}\widetilde{U}_{n-1}(\Delta_3) \\ &= \Delta_{2,3} d^{n-1} U_{n-1}\left(\frac{\Delta_3 - c}{2d}\right) \\ &= 2^2 2^{n-1} U_{n-1}\left(\frac{12-4}{4}\right) \\ &= 2^{n+1} U_{n-1}(2). \end{aligned}$$

We show this connection in Table 1 by calculating a few terms of the sequence of $\{v_n\}$ and Chebsyev polynomials of second kind $\{U_n(x)\}$. We use a symbolic programming language to calculate the terms in the table.

Similarly, we can obtain the connections for $\{v_{3n+1}\}$ and $\{v_{3n+2}\}$ by using Lemma 2.1.

3. The Complex Factorization of the sequence $\{v_n\}$

Theorem 3.1. *For $r \geq 3$,*

$$v_{nr} = \Delta_{2,r}(2d)^{n-1}\prod_{k=1}^{n-1}\left(\frac{\Delta_r - c}{2d} - \cos\left(\frac{k\pi}{n}\right)\right).$$

Proof. If we use Lemma 2.1 and $\widetilde{U}_n(x) := d^n U_n\left(\frac{x-c}{2d}\right)$, we obtain

$$\begin{aligned} v_{nr} &= \Delta_{2,r}\widetilde{U}_{n-1}(\Delta_r) \\ &= \Delta_{2,r} d^{n-1} U_{n-1}\left(\frac{\Delta_r - c}{2d}\right). \end{aligned}$$

Now, if we use (3), that is the complex factorization of Chebyshev polynomials of the second kind $\{U_n\}$, we get the desired result

$$v_{nr} = \Delta_{2,r}(2d)^{n-1}\prod_{k=1}^{n-1}\left(\frac{\Delta_r - c}{2d} - \cos\left(\frac{k\pi}{n}\right)\right).$$

□

Example 2.2 (continued). We get the connection between $\{v_{3n}\}$ and $\{U_n\}$. So, using Theorem 3.1, we obtain the complex factorization of $\{v_{3n}\}$ as follow:

$$\begin{aligned} v_{3n} &= \Delta_{2,3}(2d)^{n-1}\prod_{k=1}^{n-1}\left(\frac{\Delta_3 - c}{2d} - \cos\left(\frac{k\pi}{n}\right)\right) \\ &= 4^{n-1}\prod_{k=1}^{n-1}\left(\frac{12-4}{4} - \cos\left(\frac{k\pi}{n}\right)\right) \\ &= 2^{2n-2}\prod_{k=1}^{n-1}\left(2 - \cos\left(\frac{k\pi}{n}\right)\right). \end{aligned}$$

■

Theorem 3.2. *For $r \geq 3$, if the equality*

$$\Delta_{2,t}\left(c - \Delta_r\right) = 2\left(-1\right)^t \prod_{i=2}^{t+1} b_i \Delta_{t+2,r}, \quad 1 \leq t \leq r-1$$

holds for some t then

$$i. v_{nr+t} = \Delta_{2,t} d^n T_n\left(\frac{\Delta_r - c}{2d}\right)$$

$$ii.. v_{nr+t} = 2^{n-1} d^n \Delta_{2,t}\prod_{k=1}^{n}\left(\frac{\Delta_r - c}{2d} - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right).$$

Proof. We can obtain the following connection by using Lemma 2.1:

$$\begin{aligned} v_{nr+t} &= \Delta_{2,t} d^n U_n\left(\frac{\Delta_r - c}{2d}\right) + (-1)^t\prod_{i=2}^{t+1} b_i \Delta_{t+2,r} d^{n-1} U_{n-1}\left(\frac{\Delta_r - c}{2d}\right) \\ &= d^n \Delta_{2,t}\left(U_n\left(\frac{\Delta_r - c}{2d}\right) + \frac{(-1)^t \prod_{i=2}^{t+1} b_i \Delta_{t+2,r}}{d\Delta_{2,t}} U_{n-1}\left(\frac{\Delta_r - c}{2d}\right)\right). \end{aligned}$$

Also, if we substitute the equality

$$\Delta_{2,t}\left(c-\Delta_r\right)=2\left(-1\right)^t\prod_{i=2}^{t+1}b_i\Delta_{t+2,r}\quad 1\le t\le r-1,$$

on the statement of theorem in the above equation, we obtain

$$v_{nr+t}=d^n\Delta_{2,t}\left(U_n\left(\frac{\Delta_r-c}{2d}\right)-\frac{\Delta_r-c}{2d}U_{n-1}\left(\frac{\Delta_r-c}{2d}\right)\right).$$

Now, if we use the well known identity $T_n(x)=U_n(x)-xU_{n-1}(x)$ in the last equation, we get the part (i) of the theorem:

$$v_{nr+t}=d^n\Delta_{2,t}T_n\left(\frac{\Delta_r-c}{2d}\right).$$

Now, by using Equation (2), that is the complex factorization of Chebyshev polynomials of first kind we get the part (ii) of the theorem as follow:

$$\begin{aligned} v_{nr+t} &= d^n\Delta_{2,t}T_n\left(\frac{\Delta_r-c}{2d}\right)\\ &= 2^{n-1}d^n\Delta_{2,t}\prod_{k=1}^{n}\left(\frac{\Delta_r-c}{2d}-\cos\left(\frac{(2k-1)\pi}{2n}\right)\right). \end{aligned}$$

□

Example 3.3. Let us consider the following 3-periodic sequence $\{v_n\}$

$$v_n=\begin{cases} -7v_{n-1}+6v_{n-2}, & \text{if } n\equiv 0\,(\text{mod } 3)\,,\\ 3v_{n-1}-2v_{n-2}, & \text{if } n\equiv 1\,(\text{mod } 3)\,,\\ v_{n-1}+v_{n-2}, & \text{if } n\equiv 2\,(\text{mod } 3)\,. \end{cases}$$

A few terms of the sequence $\{v_n\}$ are $\{0,1,1,-1,-5,-6,12,48,60,-132,-516,$ $-648,1440\ldots\}$ and we have $r=3, a_0=-7, a_1=3, a_2=1=b_2, b_0=6$ and $b_1=-2$. We want to get the complex factorization of $\{v_{3n+2}\}$ by using Theorem 3.2. By using the definitions from Section 2, we get

$$\Delta_3=\begin{vmatrix} a_2 & 1 & 1\\ -b_0 & a_0 & 1\\ -b_2 & -b_1 & a_1 \end{vmatrix}=-25,\quad \Delta_{2,2}=\Delta_{4,3}=1,$$

$$b=(-1)^3b_0b_1b_2=12,\quad d=\sqrt{b}=2\sqrt{3},\quad c=(-1)^3\left(b_2+\frac{b}{b_2}\right)=-13.$$

For $t=2$, since

$$\begin{aligned}\Delta_{2,t}(c-\Delta_r)-2(-1)^t\prod_{i=2}^{t+1}b_i\Delta_{t+2,r}, &= \Delta_{2,2}(c-\Delta_3)-2(-1)^2\prod_{i=2}^{3}b_i\Delta_{4,3}\\ &= 1.(-13+25)-2.1.b_2b_3.1)\\ &= 12-2.b_2b_0\\ &= 0,\end{aligned}$$

the condition of Theorem 3.2 is satisfied. So, if we use Theorem 3.2, we obtain

$$\begin{aligned}v_{3n+2} &= \Delta_{2,t}d^nT_n\left(\frac{\Delta_r-c}{2d}\right)\\ &= \Delta_{2,2}(2\sqrt{3})^nT_n\left(\frac{\Delta_3-(-13)}{4\sqrt{3}}\right)\\ &= (2\sqrt{3})^nT_n(-\sqrt{3}).\end{aligned}$$

and we get the complex factorization

$$\begin{aligned}v_{3n+2} &= 2^{n-1}d^n\Delta_{2,t}\prod_{k=1}^{n}\left(\frac{\Delta_r-c}{2d}-\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)\\ &= \Delta_{2,2}2^{n-1}d^n\prod_{k=1}^{n}\left(\frac{\Delta_3-c}{2d}-\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)\\ &= \Delta_{2,2}2^{n-1}(2\sqrt{2})^n\prod_{k=1}^{n}\left(\frac{-25+13}{4\sqrt{2}}-\cos\left(\frac{(2k-1)\pi}{2n}\right)\right)\\ &= 2^{(5n-2)/2}\prod_{k=1}^{n}\left(\frac{-3}{\sqrt{2}}-\cos\left(\frac{(2k-1)\pi}{2n}\right)\right).\end{aligned}$$

4. The Binomial Sum for the sequence $\{v_n\}$

Theorem 4.1. *For $r>3$, $\{v_{nr}\}$ can be defined in terms of sums*

$$v_{nr} = \Delta_{2,r}d^{n-1}\sum_{j=0}^{[(n-1)/2]}(-1)^j\binom{n-1-j}{j}\left(\frac{\Delta_r-c}{2d}\right)^{n-1-2j}.$$

Proof. We have the connection

$$v_{nr}=\Delta_{2,r}d^{n-1}U_{n-1}\left(\frac{\Delta_r-c}{2d}\right)$$

by Lemma 2.1. If we make a substitution using (5) in this connection, we obtain the desired result

$$v_{nr} = \Delta_{2,r} d^{n-1} \sum_{j=0}^{[(n-1)/2]} (-1)^j \binom{n-1-j}{j} \left(\frac{\Delta_r - c}{2d}\right)^{n-1-2j}.$$

□

We can get the binomial sum for the sequence $\{v_{3n}\}$ in the Example 2.2 as an example:

Example 2.2 (continued). Bu using Theorem 4.1, we can write the sequence $\{v_{3n}\}$ in terms of sums as follow:

$$\begin{aligned} v_{nr} &= \Delta_{2,r} d^{n-1} \sum_{j=0}^{[(n-1)/2]} (-1)^j \binom{n-1-j}{j} \left(\frac{\Delta_r - c}{2d}\right)^{n-1-2j} \\ &= \Delta_{2,3} 2^{n-1} \sum_{j=0}^{[(n-1)/2]} (-1)^j \binom{n-1-j}{j} \left(\frac{\Delta_3 - 4}{4}\right)^{n-1-2j} \\ &= 2^{n+1} \sum_{j=0}^{[(n-1)/2]} (-1)^j \binom{n-1-j}{j} 2^{n-1-2j}. \end{aligned}$$

■

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