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CALDERÓN'S REPRODUCING FORMULAS FOR THE SPHERICAL MEAN L^2-MULTIPLIER OPERATORS

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First we study the spherical mean L^2-multiplier operators on $[0,+\infty[\times\mathbb{R}^n$. Next, we give for these operators Calderón's reproducing formulas and best approximation formulas.

1. Introduction

In the Euclidean case the multiplier operator T_m associated with a bounded function m on $\mathbb{R}^n$ is defined by $\widehat{T_m f} = m\widehat{f}$, where $\widehat{f}$ denotes the classical Fourier transform. Many authors [5, 9, 24] have been interested to extend the L^p Fourier-multipliers on several hypergroups and to show similarly its L^p-boundedness. Recently, these operators are studied in [25] where the author established some applications (Calderón's reproducing formulas, best approximation formulas and extremal functions...) .

The spherical mean operator $\mathscr{R}$ is defined, for a function f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable [15], by

$$\mathscr{R}(f)(r,x) = \int_{S^n} f(r\eta, x+r\xi)\, d\sigma_n(\eta,\xi), \quad (r,x)\in\mathbb{R}\times\mathbb{R}^n,$$

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where S^n is the unit sphere of $\mathbb{R}\times\mathbb{R}^n$ and $d\sigma_n$ is the surface measure on S^n normalized to have total measure one.
The dual of the spherical mean operator ${}^t\mathscr{R}$ is defined by

$$ {}^t\mathscr{R}(g)(r,x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} g(\sqrt{r^2+|x-y|^2},y)dy. $$

The spherical mean operator $\mathscr{R}$ and its dual have many important physical applications, namely in image processing of so-called synthetic aperture radar (SAR) data [6, 7, 23, 28], or in the linearized inverse scattering problem in acoustics [4].

The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined for every integrable function f on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_{n+1}$, by

$$ \forall(s,y)\in\Upsilon,\ \mathscr{F}(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\mathscr{R}\big(\cos(s.)e^{-i\langle y|.\rangle}\big)(r,x)d\nu_{n+1}(r,x), $$

where $d\nu_{n+1}$ is the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$ d\nu_{n+1}(r,x) = \frac{r^n}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}dr\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}}, $$

$\|.\|_{p,\nu_{n+1}}$ its norm, and Υ is the set given by

$$ \Upsilon = \mathbb{R}\times\mathbb{R}^n \cup \big\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ |r|\leqslant|x|\big\}. \tag{1.1} $$

Many harmonic analysis results related to the Fourier transform $\mathscr{F}$ have already been proved by Dziri, Jlassi, Nessibi, Rachdi and Trimche [3, 8, 15, 18] or also by Peng and Zhao [17, 30]. Recently, Baccar, Omri and Rachdi [2] studied the generalized Fock spaces associated with the spherical mean operator $\mathscr{R}$, and Msehli, Rachdi and Omri [13, 14, 16] established several uncertainty principles for the Fourier transform $\mathscr{F}$.

Let m be a function in the Lebesgue space $L^2(d\nu_{n+1})$. We define the spherical mean L^2-multiplier operators on $[0,+\infty[\times\mathbb{R}^n$, for regular functions

$$ T_{m,\varepsilon}f = \mathscr{F}^{-1}((m_\varepsilon\circ\theta)\mathscr{F}(f)),\quad \varepsilon>0, $$

where m_ε is the function given by

$$ m_\varepsilon(r,x) = m(\varepsilon r,\varepsilon x), \tag{1.2} $$

and θ is the bijective function, defined on the set

$$\Upsilon_+ = [0,+\infty[\times\mathbb{R}^n \cup \big\{(is,y)\ ;\ (s,y)\in[0,+\infty[\times\mathbb{R}^n;\ s\leqslant|y|\big\}$$

by,

$$\theta(s,y) = (\sqrt{s^2+|y|^2},y). \tag{1.3}$$

Our purpose in this work is to study the multiplier $T_{m,\varepsilon}$, for which we shall prove an analogue of the Calderón's reproducing formulas by using the theory of the Fourier transform $\mathscr{F}$ and the convolution product $*$.

Next, we use the theory of reproducing kernels to give best approximation of these operators and a Calderón's reproducing formula of the associated extremal function. This paper is organized as follows, in the second section we recall some harmonic analysis results related to the spherical mean operator $\mathscr{R}$ and its associated Fourier transform $\mathscr{F}$.

In the third section we study the spherical mean L^2-multiplier operators $T_{m,\varepsilon}$, and for these operators we establish Calderón's reproducing formulas.

The last section of this paper is devoted to giving best approximation for every function $m\in L^\infty(d\nu_{n+1})$ of the operators $T_{m,\varepsilon}$.

2. The spherical mean operator

In [15], Nessibi, Rachdi and Trimèche showed that for every $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, the function $\varphi_{(\mu,\lambda)}$ defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$\varphi_{(\mu,\lambda)}(r,x) = \mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\right)(r,x), \tag{2.1}$$

is the unique infinitely differentiable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, satisfying the following system

$$\begin{cases} \dfrac{\partial u}{\partial x_j}(r,x_1,...,x_n) = -i\lambda_j u(r,x_1,...,x_n), & 1\leqslant j\leqslant n,\\ \ell_{\frac{n-1}{2}}u(r,x_1,...,x_n) - \Delta u(r,x_1,...,x_n) = -\mu^2 u(r,x_1,...,x_n), & \\ u(0,...,0)=1, & \\ \dfrac{\partial u}{\partial r}(0,x_1,...,x_n) = 0, & (x_1,\dots,x_n)\in\mathbb{R}^n, \end{cases}$$

where $\ell_{\frac{n-1}{2}}$ is the Bessel operator, defined by $\ell_{\frac{n-1}{2}} = \dfrac{\partial^2}{\partial r^2}+\dfrac{n}{r}\dfrac{\partial}{\partial r}$, and Δ denotes the usual Laplacian operator defined by $\Delta=\displaystyle\sum_{j=1}^{n}\dfrac{\partial^2}{\partial x_j^2}$. The authors proved also that the eigenfunction $\varphi_{(\mu,\lambda)}$ defined by relation (2.1), is explicitly given by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad \varphi_{(\mu,\lambda)}(r,x)=j_{\frac{n-1}{2}}(r\sqrt{\mu^2+|\lambda|^2})e^{-i\langle\lambda|x\rangle},\tag{2.2}$$

where $j_{\frac{n-1}{2}}$ is the modified Bessel function defined by

$$j_{\frac{n-1}{2}}(z)=2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})\frac{J_{\frac{n-1}{2}}(z)}{z^{\frac{n-1}{2}}}=\Gamma(\frac{n+1}{2})\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\frac{n+1}{2}+k)}(\frac{z}{2})^{2k},\ z\in\mathbb{C},$$

and $J_{\frac{n-1}{2}}$ is the Bessel function of the first kind and index $\dfrac{n-1}{2}$ (see [1, 11] and [29]).
The modified Bessel function $j_{\frac{n-1}{2}}$ has the following integral representation

$$\forall z\in\mathbb{C},\ j_{\frac{n-1}{2}}(z)=\frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}\int_0^1(1-t^2)^{\frac{n}{2}-1}\cos(zt)dt.\tag{2.3}$$

Relation (2.3) shows in particular that, for every $z\in\mathbb{C}$ and for every $k\in\mathbb{N}$, we have

$$\left|j^{(k)}_{\frac{n-1}{2}}(z)\right|\leqslant e^{|Im(z)|}.$$

From the properties of the modified Bessel function $j_{\frac{n-1}{2}}$, we deduce that the eigenfunction $\varphi_{(\mu,\lambda)}$ is bounded on $\mathbb{R}\times\mathbb{R}^n$ if, and only if, (μ,λ) belongs to the set Υ given by relation (1.1), and in this case

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}\left|\varphi_{(\mu,\lambda)}(r,x)\right|=1.\tag{2.4}$$

In the following we shall define the translation operators, the convolution product and the Fourier transform $\mathscr{F}$ associated with the operator $\mathscr{R}$. For this we denote by

• $\mathscr{B}_{\Upsilon_+}$ the σ-algebra defined on Υ_+ by,

$$\mathscr{B}_{\Upsilon_+}=\left\{\theta^{-1}(B)\ ,\ B\in\mathscr{B}_{Bor}([0,+\infty[\times\mathbb{R}^n)\right\},$$

where θ is the function, given by relation (1.3).

• γ_{n+1} the measure defined on $\mathscr{B}_{\Upsilon_+}$ by, $\gamma_{n+1}(B)=\nu_{n+1}(\theta(B))$.

• $L^p(d\gamma_{n+1})$, $p\in[1,+\infty]$ the Lebesgue space of measurable functions f on Υ_+, such that $\|f\|_{p,\gamma_{n+1}}<+\infty$.

We have the following properties (see [15] and [26])

i) For every nonnegative measurable function g on Υ_+, we have

$$\iint_{\Upsilon_+} g(\mu,\lambda)\,d\gamma_{n+1}(\mu,\lambda) = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} g(\mu,\lambda)(\mu^2+|\lambda|^2)^{\frac{n-1}{2}}\mu\,d\mu\,d\lambda + \int_{\mathbb{R}^n}\int_0^{|\lambda|} g(i\mu,\lambda)(|\lambda|^2-\mu^2)^{\frac{n-1}{2}}\mu\,d\mu\,d\lambda\Big).$$

ii) For every nonnegative measurable function f on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_{n+1}$), $fo\theta$ is a measurable nonnegative function on Υ_+, (respectively integrable on Υ_+ with respect to the measure $d\gamma_{n+1}$) and we have

$$\iint_{\Upsilon_+} (fo\theta)(\mu,\lambda)\,d\gamma_{n+1}(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\,d\nu_{n+1}(r,x).$$

Moreover, the function f belongs to $L^p(d\nu_{n+1})$, $p\in[1,+\infty]$ if and only if $fo\theta$ belongs to $L^p(d\gamma_{n+1})$ and we have

$$\|f\|_{p,\nu_{n+1}} = \|fo\theta\|_{p,\gamma_{n+1}}. \tag{2.5}$$

According to Rachdi, Nessibi and Trimèche (see [15, 26] and [27]), we have the following definition and properties for the translation operator associated with the spherical mean operator

Definition 2.1. *i*) For every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the spherical mean operator is defined on $L^p(d\nu_{n+1})$, $p\in[1,+\infty]$, by

$$\mathcal{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}\int_0^{\pi} f(\sqrt{r^2+s^2+2rs\cos\theta},x+y)\sin^{n-1}(\theta)d\theta.$$

ii) The convolution product of measurable functions f and g on $[0,+\infty[\times\mathbb{R}^n$, is defined by

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n;\ f*g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n}\mathcal{T}_{(r,x)}(f)(s,-y)g(s,y)d\nu_{n+1}(s,y),$$

whenever the integral of the right-hand side is defined.

For every $(r,x)\in]0,+\infty[\times\mathbb{R}^n$, and by a standard change of variables, we have

$$\forall (s,y)\in]0,+\infty[\times\mathbb{R}^n,\ \mathcal{T}_{(r,x)}(f)(s,y) = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\int_0^{+\infty} f(t,x+y)\mathscr{W}_n(r,s,t)t^n dt,$$

where the kernel $\mathscr{W}_n$, is given by

$$\mathscr{W}_n(r,s,t)=\frac{\Gamma(\frac{n+1}{2})^2}{2^{\frac{n-3}{2}}\Gamma(\frac{n}{2})\sqrt{\pi}}\frac{\left((r+s)^2-t^2\right)^{\frac{n}{2}-1}\left(t^2-(r-s)^2\right)^{\frac{n}{2}-1}}{(rst)^{n-1}}\mathbf{1}_{]|r-s|,r+s[}(t).$$

Also, the coming properties are satisfied

- For every $f\in L^p(d\nu_{n+1})$, $p\in[1,+\infty]$, and $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_{n+1})$ and we have

$$||\mathcal{T}_{(r,x)}(f)||_{p,\nu_{n+1}}\leqslant||f||_{p,\nu_{n+1}}. \tag{2.6}$$

- Let $p,q,r\in[1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. Then for every $f\in L^p(d\nu_{n+1})$ and $g\in L^q(d\nu_{n+1})$, the function $f*g$ belongs to the space $L^r(d\nu_{n+1})$, and we have the following Young's inequality

$$||f*g||_{r,\nu_{n+1}}\leqslant||f||_{p,\nu_{n+1}}||g||_{q,\nu_{n+1}}.$$

In the following, we shall define the Fourier transform $\mathscr{F}$ connected with the spherical mean operator, and we recall some of its properties that we need in the next sections.

Definition 2.2. The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined on $L^1(d\nu_{n+1})$ by [15]

$$\forall(\mu,\lambda)\in\Upsilon\;;\;\mathscr{F}(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\varphi_{(\mu,\lambda)}(r,x)\,d\nu_{n+1}(r,x),$$

where $\varphi_{(\mu,\lambda)}$ is the eigenfunction given by relation (2.2), and Υ is the set defined by relation (1.1).

Then, according to [15], we have
For every $f,g\in L^1(d\nu_{n+1})$,

$$\mathscr{F}(f*g)=\mathscr{F}(f)\mathscr{F}(g),$$

and $(\mu,\lambda)\in\Upsilon$

$$\mathscr{F}(\mathcal{T}_{(r,-x)}(f))(\mu,\lambda)=\varphi_{(\mu,\lambda)}(r,x)\mathscr{F}(f)(\mu,\lambda). \tag{2.7}$$

Moreover, relation (2.4) implies that the Fourier transform $\mathscr{F}$ is a bounded linear operator from $L^1(d\nu_{n+1})$ into $L^\infty(d\gamma_{n+1})$, and that for every $f\in L^1(d\nu_{n+1})$, we have

$$\|\mathscr{F}(f)\|_{\infty,\gamma_{n+1}}\leqslant\|f\|_{1,\nu_{n+1}}. \tag{2.8}$$

For every positive real number ε and for every $m \in L^p(d\nu_{n+1}), p \in [1,+\infty[$, the function m_ε defined by relation (1.2), belongs to $L^p(d\nu_{n+1})$ and we have

$$\|m_\varepsilon\|_{p,\nu_{n+1}} = \frac{1}{\varepsilon^{\frac{2n+1}{p}}}\|m\|_{p,\nu_{n+1}}. \tag{2.9}$$

In [15], Rachdi, Nessibi and Trimèche, established the following inversion formula and Plancherel theorem for the Fourier transform $\mathscr{F}$.

Theorem 2.3 (Inversion formula)**.** *Let $f \in L^1(d\nu_{n+1})$ such that $\mathscr{F}(f) \in L^1(d\gamma_{n+1})$, then for almost every $(r,x) \in \mathbb{R}\times\mathbb{R}^n$*

$$f(r,x) = \iint_{\Upsilon_+} \mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}\,d\gamma_{n+1}(\mu,\lambda).$$

Theorem 2.4 (Plancherel theorem)**.** *The Fourier transform $\mathscr{F}$ can be extended to an isometric isomorphism from $L^2(d\nu_{n+1})$ onto $L^2(d\gamma_{n+1})$. In particular, for every $f \in L^2(d\nu_{n+1})$*

$$\|\mathscr{F}(f)\|_{2,\gamma_{n+1}} = \|f\|_{2,\nu_{n+1}}.$$

Corollary 2.5. *For all functions f and g in $L^2(d\nu_{n+1})$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\overline{g(r,x)}d\nu_{n+1}(r,x) = \iint_{\Upsilon_+} \mathscr{F}(f)(\mu,\lambda)\overline{\mathscr{F}(g)(\mu,\lambda)}d\gamma_{n+1}(\mu,\lambda).$$

Remark 2.6. (i) For every $f,g \in L^2(d\nu_{n+1})$; the function $f*g$ belongs to the space $C_{e,0}(\mathbb{R}\times\mathbb{R}^n)$ consisting of continuous functions h on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable and such that $\lim\limits_{r^2+|x|^2\longrightarrow+\infty} h(r,x) = 0$.

Moreover,

$$f*g = \mathscr{F}^{-1}(\mathscr{F}(f)\mathscr{F}(g)), \tag{2.10}$$

where $\mathscr{F}^{-1}$ is the mapping defined on $L^1(d\gamma_{n+1})$ by

$$\mathscr{F}^{-1}(g)(r,x) = \iint_{\Upsilon_+} g(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}d\gamma_{n+1}(\mu,\lambda).$$

(ii) Let $f,g \in L^2(d\nu_{n+1})$, the function $f*g$ belongs to $L^2(d\nu_{n+1})$ if and only if $\mathscr{F}(f)\mathscr{F}(g)$ belongs to $L^2(d\gamma_{n+1})$, and we have

$$\mathscr{F}(f*g) = \mathscr{F}(f)\mathscr{F}(g).$$

(iii) Let $f,g \in L^2(d\nu_{n+1})$, then

$$\|\mathscr{F}(f)\mathscr{F}(g)\|_{2,\gamma_{n+1}} = \|f*g\|_{2,\nu_{n+1}}. \tag{2.11}$$

(iv) For every $g \in L^1(d\gamma_{n+1})$, $\mathscr{F}^{-1}(g)$ belongs to $L^\infty(d\nu_{n+1})$, and we have

$$\|\mathscr{F}^{-1}(g)\|_{\infty,\nu_{n+1}} \leqslant \|g\|_{1,\gamma_{n+1}}.$$

3. The Spherical mean L^2-multiplier operators

In this section we study the spherical mean L^2-multiplier operators on $[0,+\infty[\times\mathbb{R}^n$ and for these operators we establish Calderón's reproducing formulas.

Definition 3.1. Let m be a function in $L^2(d\nu_{n+1})$ and let ε be a positive real number. The spherical mean L^2-multiplier operators is defined for regular functions f on $[0,+\infty[\times\mathbb{R}^n$, by

$$\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\quad T_{m,\varepsilon}f(r,x)=\mathscr{F}^{-1}\big((m_\varepsilon\circ\theta)\mathscr{F}(f)\big)(r,x),\qquad (3.1)$$

where m_ε is the function given by relation (1.2) and θ is the function defined by (1.3).

Proposition 3.2. *(i) For every $m\in L^2(d\nu_{n+1})$, and $f\in L^1(d\nu_{n+1})$, the function $T_{m,\varepsilon}f$ belongs to $L^2(d\nu_{n+1})$, and we have*

$$\|T_{m,\varepsilon}f\|_{2,\nu_{n+1}}\leqslant\frac{1}{\varepsilon^{\frac{2n+1}{2}}}\|m\|_{2,\nu_{n+1}}\|f\|_{1,\nu_{n+1}}.$$

(ii) For every $m\in L^2(d\nu_{n+1})$, and $f\in L^2(d\nu_{n+1})$, then $T_{m,\varepsilon}f\in L^\infty(d\nu_{n+1})$, and we have

$$T_{m,\varepsilon}f(r,x)=\int\int_{\Upsilon_+}(m_\varepsilon\circ\theta)(\mu,\lambda)\mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}d\gamma_{n+1}(\mu,\lambda),$$

and

$$\|T_{m,\varepsilon}f\|_{\infty,\nu_{n+1}}\leqslant\frac{1}{\varepsilon^{\frac{2n+1}{2}}}\|m\|_{2,\nu_{n+1}}\|f\|_{2,\nu_{n+1}}.$$

(iii) For every $m\in L^\infty(d\nu_{n+1})$, and $f\in L^2(d\nu_{n+1})$, the function $T_{m,\varepsilon}f$ belongs to $L^2(d\nu_{n+1})$, and we have

$$\|T_{m,\varepsilon}f\|_{2,\nu_{n+1}}\leqslant\|m\|_{\infty,\nu_{n+1}}\|f\|_{2,\nu_{n+1}}.$$

Proof. (i) From relations (2.5), (2.8), (3.1), and Theorem 2.4, the function $T_{m,\varepsilon}$ belongs to $L^2(d\nu_{n+1})$, and we have

$$\begin{aligned}\|\mathscr{F}(T_{m,\varepsilon}f)\|_{2,\gamma_{n+1}} &= \|(m_\varepsilon\circ\theta)\mathscr{F}(f)\|_{2,\gamma_{n+1}}\\ &\leqslant \|(m_\varepsilon\circ\theta)\|_{2,\gamma_{n+1}}\|\mathscr{F}(f)\|_{\infty,\gamma_{n+1}}\\ &\leqslant \|m_\varepsilon\|_{2,\nu_{n+1}}\|f\|_{1,\nu_{n+1}}.\end{aligned}$$

Then, the result follows from (2.9), and Theorem 2.4.
(ii) Using (2.5), (3.1), and Remark 2.6 (iv), for every $m\in L^2(d\nu_{n+1})$, and $f\in L^2(d\nu_{n+1})$, the function $T_{m,\varepsilon}f\in L^\infty(d\nu_{n+1})$, and we have

$$\|T_{m,\varepsilon}f\|_{\infty,\nu_{n+1}}\quad\leqslant\quad\|(m_\varepsilon\circ\theta)\mathscr{F}(f)\|_{1,\gamma_{n+1}}.$$

From Hölder's inequality, relation (2.9), and Theorem 2.4, we obtain

$$\begin{aligned} \|T_{m,\varepsilon}f\|_{\infty,\nu_{n+1}} &\leqslant \|(m_\varepsilon\circ\theta)\|_{2,\gamma_{n+1}}\|\mathscr{F}(f)\|_{2,\gamma_{n+1}} \\ &= \|m_\varepsilon\|_{2,\nu_{n+1}}\|f\|_{2,\nu_{n+1}}, \\ &= \frac{1}{\varepsilon^{\frac{2n+1}{2}}}\|m\|_{2,\nu_{n+1}}\|f\|_{2,\nu_{n+1}}. \end{aligned}$$

Part (iii) follows from (2.5), (3.1), and Theorem 2.4. □

Remark 3.3. According to relation (2.10), for every $m\in L^2(d\nu_{n+1})$ and $f\in L^2(d\nu_{n+1})$, we can write the spherical mean L^2- multiplier as

$$\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n,\quad T_{m,\varepsilon}f(r,x)=\mathscr{F}^{-1}(m_\varepsilon\circ\theta)*f(r,x). \tag{3.2}$$

Theorem 3.4. *Let m be a function in $L^2(d\nu_{n+1})$, satisfying the admissibility condition*

$$\int_0^{+\infty}|m_\varepsilon\circ\theta(\mu,\lambda)|^2\frac{d\varepsilon}{\varepsilon}=1,\quad(\mu,\lambda)\in\Upsilon. \tag{3.3}$$

*(i)***Plancherel formula:** *For every $f\in L^2(d\nu_{n+1})$, we have*

$$\|f\|^2_{2,\nu_{n+1}}=\int_0^{+\infty}\|T_{m,\varepsilon}f\|^2_{2,\nu_{n+1}}\frac{d\varepsilon}{\varepsilon}.$$

(ii) **First Calderón's formula:** *Let f be a function in $L^1(d\nu_{n+1})$, such that $\mathscr{F}(f)$ in $L^1(d\gamma_{n+1})$, we have*

$$f(r,x)=\int_0^{+\infty}\left(T_{m,\varepsilon}f*\mathscr{F}^{-1}(\overline{m_\varepsilon\circ\theta})\right)(r,x)\frac{d\varepsilon}{\varepsilon},\quad a.e.\,(r,x)\in[0,+\infty[\times\mathbb{R}^n.$$

Proof. (i) From relations (2.11) and (3.2), we have

$$\begin{aligned} \int_0^{+\infty}\|T_{m,\varepsilon}f\|^2_{2,\nu_{n+1}}\frac{d\varepsilon}{\varepsilon} &= \int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}|(\mathscr{F}^{-1}(m_\varepsilon\circ\theta)*f)(r,x)|^2d\nu_{n+1}(r,x)\frac{d\varepsilon}{\varepsilon} \\ &= \int_0^{+\infty}\iint_{\Upsilon_+}|m_\varepsilon\circ\theta(r,x)\mathscr{F}(f)(r,x)|^2d\gamma_{n+1}(r,x)\frac{d\varepsilon}{\varepsilon} \\ &= \iint_{\Upsilon_+}|\mathscr{F}(f)(r,x)|^2\left(\int_0^{+\infty}|m_\varepsilon\circ\theta(r,x)|^2\frac{d\varepsilon}{\varepsilon}\right)d\gamma_{n+1}(r,x). \end{aligned}$$

The result follows from Theorem 2.4, and (3.3).
(ii) Let f in $L^1(d\nu_{n+1})$. According to Proposition 3.2 (i), relation (2.6), and Corollary 2.5, we have

$$\begin{aligned} &\int_0^{+\infty}\left(T_{m,\varepsilon}f*\mathscr{F}^{-1}(\overline{m_\varepsilon\circ\theta})\right)(r,x)\frac{d\varepsilon}{\varepsilon} \\ &=\int_0^{+\infty}\left[\int_0^{+\infty}\int_{\mathbb{R}^n}T_{m,\varepsilon}f(s,y)\overline{\mathcal{T}_{(r,-x)}(\mathscr{F}^{-1}(m_\varepsilon\circ\theta))(s,y)}d\nu_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon} \\ &=\int_0^{+\infty}\left[\iint_{\Upsilon_+}\mathscr{F}(T_{m,\varepsilon}f)(s,y)\overline{\mathscr{F}(\mathcal{T}_{(r,-x)}(\mathscr{F}^{-1}(m_\varepsilon\circ\theta)))(s,y)}d\gamma_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon}. \end{aligned}$$

Using (2.7), we obtain

$$\begin{aligned}&\int_0^{+\infty}\left(T_{m,\varepsilon}f*\mathscr{F}^{-1}(\overline{m_\varepsilon\circ\theta})\right)(r,x)\frac{d\varepsilon}{\varepsilon}\\&\quad=\int_0^{+\infty}\left[\iint_{\Upsilon_+}\mathscr{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}\,|m_\varepsilon\circ\theta(s,y)|^2\,d\gamma_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon}.\end{aligned}$$

Since,

$$\begin{aligned}&\int_0^{+\infty}\left[\iint_{\Upsilon_+}\left|\mathscr{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}\right|\,|m_\varepsilon\circ\theta(s,y)|^2\,d\gamma_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon}\\&\quad\leqslant\iint_{\Upsilon_+}|\mathscr{F}(f)(s,y)|\,d\gamma_{n+1}(s,y).\end{aligned}$$

Then, the result follows from Fubini's theorem, relation (3.3), and Theoren 2.3. □

Lemma 3.5. Let $m\in L^2(d\nu_{n+1})\cap L^\infty(d\nu_{n+1})$, satisfy the admissibility condition (3.3). For every $0<\xi<\delta<\infty$, the function

$$\mathcal{K}_{\xi,\delta}(\mu,\lambda)=\int_\xi^\delta|m_\varepsilon\circ\theta(\mu,\lambda)|^2\frac{d\varepsilon}{\varepsilon},$$

belongs to $L^2(d\gamma_{n+1})$, and we have

$$\|\mathcal{K}_{\xi,\delta}\|^2_{2,\gamma_{n+1}}\leqslant\ln(\frac{\delta}{\xi})\frac{\xi^{-(2n+1)}-\delta^{-(2n+1)}}{2n+1}\|m\|^2_{2,\nu_{n+1}}\|m\|^2_{\infty,\nu_{n+1}}.$$

Proof. Using Hölder's inequality for the measure $\dfrac{d\varepsilon}{\varepsilon}$, we get for every $(\mu,\lambda)\in\Upsilon$

$$|\mathcal{K}_{\xi,\delta}(\mu,\lambda)|^2\leqslant\ln(\frac{\delta}{\xi})\int_\xi^\delta|m_\varepsilon\circ\theta(\mu,\lambda)|^4\frac{d\varepsilon}{\varepsilon}.$$

From (2.5), and (2.9), we obtain

$$\begin{aligned}\|\mathcal{K}_{\xi,\delta}\|^2_{2,\gamma_{n+1}}&\leqslant\ln(\frac{\delta}{\xi})\int_\xi^\delta\left[\iint_{\Upsilon_+}|m_\varepsilon\circ\theta(\mu,\lambda)|^4d\gamma_{n+1}(\mu,\lambda)\right]\frac{d\varepsilon}{\varepsilon}\\&\leqslant\ln(\frac{\delta}{\xi})\frac{\xi^{-(2n+1)}-\delta^{-(2n+1)}}{2n+1}\|m\|^2_{2,\nu_{n+1}}\|m\|^2_{\infty,\nu_{n+1}}<\infty.\end{aligned}$$

□

Theorem 3.6. Second Calderón's formula. *Let $m \in L^2(d\nu_{n+1}) \cap L^\infty(d\nu_{n+1})$, satisfy the admissibility condition (3.3).Then for every $f \in L^2(d\nu_{n+1})$ and $0 < \xi < \delta < \infty$, the function*

$$f^{\xi,\delta}(r,x) = \int_\xi^\delta \left(T_{m,\varepsilon} f * \mathscr{F}^{-1}(\overline{m_\varepsilon \circ \theta})\right)(r,x)\frac{d\varepsilon}{\varepsilon},$$

belongs to $L^2(d\nu_{n+1})$ and satisfies

$$\lim_{(\xi,\delta)\longrightarrow(0^+,+\infty)} \|f^{\xi,\delta} - f\|_{2,\nu_{n+1}} = 0. \tag{3.4}$$

Proof. From Proposition 3.2 (iii), (2.6), (2.7) and Corollary 2.5, we have

$$\begin{aligned} f^{\xi,\delta}(r,x) =& \int_\xi^\delta \left[\int_0^{+\infty}\int_{\mathbb{R}^n} T_{m,\varepsilon} f(s,y)\overline{\mathcal{T}_{(r,-x)}(\mathscr{F}^{-1}(m_\varepsilon\circ\theta))(s,y)}d\nu_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon} \\ =& \int_\xi^\delta \left[\iint_{\Upsilon_+} \mathscr{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}\,|m_\varepsilon\circ\theta(s,y)|^2\,d\gamma_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon}. \end{aligned}$$

By Fubini-Tonnelli's theorem, Hölder's inequality, relation (2.4) and Lemma 3.5, we get

$$\begin{aligned} &\int_\xi^\delta \left[\iint_{\Upsilon_+} \left|\mathscr{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}\right| |m_\varepsilon\circ\theta(s,y)|^2\,d\gamma_{n+1}(s,y)\right]\frac{d\varepsilon}{\varepsilon} \\ &\quad\leqslant \iint_{\Upsilon_+} |\mathscr{F}(f)(s,y)|\,\mathcal{K}_{\xi,\delta}(s,y)d\gamma_{n+1}(s,y) \\ &\quad\leqslant \sqrt{\ln(\frac{\delta}{\xi})\frac{\xi^{-(2n+1)}-\delta^{-(2n+1)}}{2n+1}}\|f\|_{2,\nu_{n+1}}\|m\|_{2,\nu_{n+1}}\|m\|_{\infty,\nu_{n+1}} < \infty. \end{aligned}$$

Then, from Fubini's theorem and Theorem 2.3, we obtain

$$\begin{aligned} f^{\xi,\delta}(r,x) =& \iint_{\Upsilon_+} \mathscr{F}(f)(s,y)\overline{\varphi_{(s,y)}(r,x)}\mathcal{K}_{\xi,\delta}(s,y)d\gamma_{n+1}(s,y) \\ =& \mathscr{F}^{-1}(\mathscr{F}(f)\mathcal{K}_{\xi,\delta})(r,x). \end{aligned}$$

On the other hand, from relation (3.3), the function $\mathcal{K}_{\xi,\delta}$ belongs to $L^\infty(d\gamma_{n+1})$, from this fact and Theorem 2.4, the function $f^{\xi,\delta} \in L^2(d\nu_{n+1})$, and we have

$$\mathscr{F}(f^{\xi,\delta}) = \mathscr{F}(f)\mathcal{K}_{\xi,\delta}.$$

Using the previous result and Theorem 2.4, we get

$$\|f^{\xi,\delta} - f\|^2_{2,\nu_{n+1}} = \iint_{\Upsilon_+} |\mathscr{F}(f)(\mu,\lambda)|^2(\mathcal{K}_{\xi,\delta}(\mu,\lambda)-1)^2 d\gamma_{n+1}(\mu,\lambda).$$

The relation (3.4) follows from $\lim_{(\xi,\delta)\longrightarrow(0^+,+\infty)} \mathcal{K}_{\xi,\delta}(\mu,\lambda) = 1$, and the dominated convergence theorem. □

4. The extremal function related to spherical mean L^2-multiplier operators

In this section, by using the theory of extremal function and reproducing Kernel of Hilbert space [19–22], we study the extremal function associated to the spherical mean L^2-multiplier operators. The main result of this section can be stated as follows.

Definition 4.1. Let σ be a positive function on Υ satisfying :

$$\sigma(\mu,\lambda) \geqslant 1, \quad (\mu,\lambda) \in \Upsilon, \tag{4.1}$$

and

$$\frac{1}{\sigma} \in L^1(d\gamma_{n+1}). \tag{4.2}$$

We define the space $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, by

$$\Omega_\sigma([0,+\infty[\times\mathbb{R}^n) = \{f \in L^2(d\nu_{n+1}), \sqrt{\sigma}\mathscr{F}(f) \in L^2(d\gamma_{n+1})\}.$$

The space $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$ provided with inner product

$$\langle f,g\rangle_\sigma = \int\int_{\Upsilon_+} \sigma(\mu,\lambda)\mathscr{F}(f)(\mu,\lambda)\overline{\mathscr{F}(g)(\mu,\lambda)}d\gamma_{n+1}(\mu,\lambda),$$

and the norm $\|f\|_\sigma = \sqrt{\langle f,f\rangle_\sigma}$ is a Hilbert space.

Proposition 4.2. *Let $m \in L^\infty(d\nu_{n+1})$. For every $f \in \Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, the operators $T_{m,\varepsilon}$ are bounded linear operators from $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$ into $L^2(d\nu_{n+1})$, and we have*

$$\|T_{m,\varepsilon}f\|_{2,\nu_{n+1}} \leqslant \|m\|_{\infty,\nu_{n+1}}\|f\|_\sigma.$$

Proof. Let $f \in \Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$. According to Proposition 3.2(iii), the operator $T_{m,\varepsilon}f$ belongs to $L^2(d\nu_{n+1})$, and

$$\|T_{m,\varepsilon}f\|_{2,\nu_{n+1}} \leqslant \|m\|_{\infty,\nu_{n+1}}\|f\|_{2,\nu_{n+1}}.$$

By relation (4.1), we have $\|f\|_\sigma^2 \geqslant \int\int_{\Upsilon_+} |\mathscr{F}(f)(\mu,\lambda)|^2 d\gamma_{n+1}(\mu,\lambda)$, which gives the result. □

Definition 4.3. Let $\rho > 0$, and let $m \in L^\infty(d\nu_{n+1})$, we denote by $\langle,\rangle_{\sigma,\rho}$ the inner product defined on the space $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$ by

$$\langle f,g\rangle_{\sigma,\rho} = \int\int_{\Upsilon_+} \left(\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2\right)\mathscr{F}(f)(\mu,\lambda)\overline{\mathscr{F}(g)(\mu,\lambda)}d\gamma_{n+1}(\mu,\lambda), \tag{4.3}$$

and the norm $\|f\|_{\sigma,\rho} = \sqrt{\langle f,f\rangle_{\sigma,\rho}}$.

Lemma 4.4. Let $(s,y) \in [0,+\infty[\times\mathbb{R}^n$. Then
(i) The function

$$\Lambda_{(s,y)} : (\mu,\lambda) \longmapsto \frac{\varphi_{(\mu,\lambda)}(s,y)}{\rho\sigma(\mu,\lambda)+|m_\varepsilon \circ \theta(\mu,\lambda)|^2},$$

belongs to $L^1(d\gamma_{n+1}) \cap L^2(d\gamma_{n+1})$.
(ii) The function

$$\Phi_{(s,y)} : (\mu,\lambda) \longmapsto \frac{m_\varepsilon \circ \theta(\mu,\lambda)\varphi_{(\mu,\lambda)}(s,y)}{\rho\sigma(\mu,\lambda)+|m_\varepsilon \circ \theta(\mu,\lambda)|^2},$$

belongs to $L^1(d\gamma_{n+1}) \cap L^2(d\gamma_{n+1})$.
Where $\varphi_{(\mu,\lambda)}$ is the function given by relation (2.2).

Proof. The proof of the Lemma follows from relations (2.4), (4.1) and (4.2). □

Proposition 4.5. *Let $m \in L^\infty(d\nu_{n+1})$. Then the Hilbert space $(\Omega_\sigma([0,+\infty[\times\mathbb{R}^n), \langle .,. \rangle_{\sigma,\rho})$ has the following reproducing Kernel*

$$K_{\sigma,\rho}((r,x),(s,y)) = \iint_{\Upsilon_+} \frac{\overline{\varphi_{(\mu,\lambda)}(r,x)}\varphi_{(\mu,\lambda)}(s,y)}{\rho\sigma(\mu,\lambda)+|m_\varepsilon \circ \theta(\mu,\lambda)|^2} d\gamma_{n+1}(\mu,\lambda), \qquad (4.4)$$

that is
(i) For every $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, the function $(r,x) \longmapsto K_{\sigma,\rho}((r,x),(s,y))$ belongs to $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$.
(ii) For every $f \in \Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, and $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, we have the reproducing property,

$$\langle f, K_{\sigma,\rho}((.,.),(s,y))\rangle_{\sigma,\rho} = f(s,y).$$

Proof. From Lemma 4.4 (i), the function $K_{\sigma,\rho}$ is well defined and by Theorem 2.3, we have

$$K_{\sigma,\rho}((r,x),(s,y)) = \mathscr{F}^{-1}(\Lambda_{(s,y)})(r,x), \quad (r,x) \in [0,+\infty[\times\mathbb{R}^n.$$

By Theorem 2.4, it follows that the function $K_{\sigma,\rho}((.,.),(s,y)$, belongs to $L^2(d\nu_{n+1})$, and we have

$$\mathscr{F}(K_{\sigma,\rho}((.,.),(s,y))(\mu,\lambda) = \Lambda_{(s,y)}(\mu,\lambda), \quad (\mu,\lambda) \in \Upsilon. \qquad (4.5)$$

Then by relations (2.4), (4.2) and (4.5), we obtain

$$\|K_{\sigma,\rho}((.,.),(s,y)\|_\sigma^2 \leqslant \frac{1}{\rho^2}\|\frac{1}{\sigma}\|_{1,\gamma_{n+1}}.$$

This proves that for every $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function $K_{\sigma,\rho}((.,.),(s,y))$ belongs to $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$.
(ii) From (4.3) and (4.5), we obtain

$$\langle f,K_{\sigma,\rho}((.,.),(s,y))\rangle_{\sigma,\rho}=\iint_{\Upsilon_+}\mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(s,y)}d\gamma_{n+1}(\mu,\lambda).$$

On the other hand, from relation (4.2) the function $\frac{1}{\sqrt{\sigma}}$ belongs to $L^2(d\gamma_{n+1})$, hence for every $f\in\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, the function $\mathscr{F}(f)$ belongs to $L^1(d\gamma_{n+1})$. From this result and Theorem 2.3, we obtain

$$\langle f,K_{\sigma,\rho}((.,.),(s,y))\rangle_{\sigma,\rho}=f(s,y).$$

This completes the proof of the Proposition. □

Theorem 4.6. *Let $m\in L^\infty(d\nu_{n+1})$ and $\varepsilon>0$, for every $h\in L^2(d\nu_{n+1})$ and $\rho>0$, there exists a unique function $f^*_{\rho,h,\varepsilon}$, where the infimum*

$$\inf_{f\in\Omega_\sigma}\{\rho\|f\|^2_\sigma+\|h-T_{m,\varepsilon}f\|^2_{2,\nu_{n+1}}\},\tag{4.6}$$

is attained. Moreover the extremal function $f^*_{\rho,h,\varepsilon}$ is given by

$$f^*_{\rho,h,\varepsilon}(s,y)=\int_0^{+\infty}\int_{\mathbb{R}^n}h(r,x)\overline{V_{\sigma,\rho}((r,x),(s,y))}d\nu_{n+1}(r,x),\tag{4.7}$$

where $V_{\sigma,\rho}((r,x),(s,y))=\iint_{\Upsilon_+}\dfrac{m_\varepsilon\circ\theta(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}\varphi_{(\mu,\lambda)}(s,y)}{\rho\sigma(\mu,\lambda)+|m_\varepsilon\circ\theta(\mu,\lambda)|^2}d\gamma_{n+1}(\mu,\lambda).$

Proof. The existence and unicity of the extremal function $f^*_{\rho,h,\varepsilon}$ satisfying relation (4.6) is given by [10, 12, 21]. On the other hand From Proposition 4.2 and 4.5, we have

$$f^*_{\rho,h,\varepsilon}(s,y)=\langle h,T_{m,\varepsilon}(K_{\sigma,\rho})((.,.)(s,y))\rangle_{\nu_{n+1}},\tag{4.8}$$

where $\langle,\rangle_{\nu_{n+1}}$ denoted the inner product of $L^2(d\nu_{n+1})$, and $K_{\sigma,\rho}$ is the Kernel given by relation (4.4). According to Proposition 3.2 (ii), (4.5) and (4.8), we obtain

$$V_{\sigma,\rho}((r,x),(s,y))=\iint_{\Upsilon_+}\frac{m_\varepsilon\circ\theta(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(r,x)}\varphi_{(\mu,\lambda)}(s,y)}{\rho\sigma(\mu,\lambda)+|m_\varepsilon\circ\theta(\mu,\lambda)|^2}d\gamma_{n+1}(\mu,\lambda).$$ □

Theorem 4.7. *Let $m\in L^\infty(d\nu_{n+1})$ and $h\in L^2(d\nu_{n+1})$. The extremal function $f^*_{\rho,h,\varepsilon}$ belongs to $\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, and we have*

$$\|f^*_{\rho,h,\varepsilon}\|^2_\sigma\leqslant\frac{1}{4\rho}\|h\|^2_{2,\nu_{n+1}}.$$

Proof. Let $(s,y) \in [0,+\infty[\times\mathbb{R}^n$. From Lemma 4.4 (ii) and Theorem 2.3, we have

$$V_{\sigma,\rho}((r,x),(s,y)) = \mathscr{F}^{-1}(\Phi_{(s,y)})(r,x).$$

By Theorem 2.4, it follows that the function $V_{\sigma,\rho}((.,.),(s,y))$ belongs to $L^2(d\nu_{n+1})$ and using Corollary 2.5, we get

$$\begin{aligned} f^*_{\rho,h,\varepsilon}(s,y) &= \iint_{\Upsilon_+} \mathscr{F}(h)(\mu,\lambda)\overline{\Phi_{(s,y)}(\mu,\lambda)}d\gamma_{n+1}(\mu,\lambda) \\ &= \iint_{\Upsilon_+} \mathscr{F}(h)(\mu,\lambda)\frac{\overline{m_\varepsilon\circ\theta(\mu,\lambda)\varphi_{(\mu,\lambda)}(s,y)}}{\rho\sigma(\mu,\lambda)+|m_\varepsilon\circ\theta(\mu,\lambda)|^2}d\gamma_{n+1}(\mu,\lambda). \end{aligned}$$

On the other hand, the function $(\mu,\lambda)\longmapsto \mathscr{F}(h)(\mu,\lambda)\dfrac{\overline{m_\varepsilon\circ\theta(\mu,\lambda)}}{\rho\sigma(\mu,\lambda)+|m_\varepsilon\circ\theta(\mu,\lambda)|^2}$ belongs to $L^1(d\gamma_{n+1})\cap L^2(d\gamma_{n+1})$, then by Theorem 2.3, we have

$$f^*_{\rho,h,\varepsilon}(s,y) = \mathscr{F}^{-1}\left(\mathscr{F}(h)\frac{\overline{m_\varepsilon\circ\theta(.,.)}}{\rho\sigma(.,.)+|m_\varepsilon\circ\theta(.,.)|^2}\right)(s,y).$$

From Theorem 2.4, it follows that, the function $f^*_{\rho,h,\varepsilon}$ belongs to $L^2(d\nu_{n+1})$, and we have for every $(\mu,\lambda)\in\Upsilon$,

$$\begin{aligned} \left|\mathscr{F}(f^*_{\rho,h,\varepsilon})(\mu,\lambda)\right|^2 &= \left|\mathscr{F}(h)(\mu,\lambda)\frac{\overline{m_\varepsilon\circ\theta(\mu,\lambda)}}{\rho\sigma(\mu,\lambda)+|m_\varepsilon\circ\theta(\mu,\lambda)|^2}\right|^2, \quad (4.9) \\ &\leqslant \frac{1}{4\rho\sigma(\mu,\lambda)}|\mathscr{F}(h)(\mu,\lambda)|^2, \end{aligned}$$

thus, from Theorem 2.4 and Definition 4.1, we obtain

$$\|f^*_{\rho,h,\varepsilon}\|^2_\sigma \leqslant \frac{1}{4\rho}\|h\|^2_{2,\nu_{n+1}}.$$

□

Theorem 4.8. Third Calderón's formula *Let $m\in L^\infty(d\nu_{n+1})$, and $f\in\Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$. The extremal function $f^*_{\rho,\varepsilon}$ given by*

$$f^*_{\rho,\varepsilon}(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} T_{m,\varepsilon}f(r,x)\overline{V_{\sigma,\rho}((r,x),(s,y))}d\nu_{n+1}(r,x),$$

satisfies

(i)

$$\lim_{\rho\longrightarrow 0^+}\|f^*_{\rho,\varepsilon}-f\|_\sigma = 0. \quad (4.10)$$

(ii)

$$\lim_{\rho\longrightarrow 0^+} f^*_{\rho,\varepsilon} = f, \quad \textit{uniformly}.$$

Proof. (i) Let $f \in \Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, $h = T_{m,\varepsilon}f$, and $f^*_{\rho,\varepsilon} = f^*_{\rho,h,\varepsilon}$. From Proposition 4.2, the function h belongs to $L^2(d\nu_{n+1})$. Applying Definition 3.1, and relation (4.9), we obtain

$$\mathscr{F}(f^*_{\rho,\varepsilon})(\mu,\lambda) = \mathscr{F}(f)(\mu,\lambda)\frac{|m_\varepsilon \circ \theta(\mu,\lambda)|^2}{\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2}$$

Thus, it follows that for every $(\mu,\lambda) \in \Upsilon$

$$\mathscr{F}(f^*_{\rho,\varepsilon} - f)(\mu,\lambda) = \frac{-\rho\sigma(\mu,\lambda)\mathscr{F}(f)(\mu,\lambda)}{\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2}. \tag{4.11}$$

Consequently,

$$\|f^*_{\rho,\varepsilon} - f\|^2_\sigma = \iint_{\Upsilon_+} \frac{\rho^2\sigma^3(\mu,\lambda)|\mathscr{F}(f)(\mu,\lambda)|^2}{\left(\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2\right)^2} d\gamma_{n+1}(\mu,\lambda).$$

Then, the result follows from the fact

$$\frac{\rho^2\sigma^3(\mu,\lambda)|\mathscr{F}(f)(\mu,\lambda)|^2}{\left(\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2\right)^2} \leqslant \sigma(\mu,\lambda)|\mathscr{F}(f)(\mu,\lambda)|^2,$$

and the dominated convergence theorem.
(ii) By relation (4.2), the function $\frac{1}{\sqrt{\sigma}}$ belongs to $L^2(d\gamma_{n+1})$, hence for $f \in \Omega_\sigma([0,+\infty[\times\mathbb{R}^n)$, the function $\mathscr{F}(f)$ belongs to $L^1(d\gamma_{n+1})$. Then, from (4.11) and Theorem 2.3, we get

$$f^*_{\rho,\varepsilon}(s,y) - f(s,y) = \iint_{\Upsilon_+} \frac{-\rho\sigma(\mu,\lambda)\mathscr{F}(f)(\mu,\lambda)}{\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2} \overline{\varphi_{(\mu,\lambda)}(s,y)} d\gamma_{n+1}(\mu,\lambda).$$

By using the dominated convergence theorem and the fact

$$\frac{\left|-\rho\sigma(\mu,\lambda)\mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{(\mu,\lambda)}(s,y)}\right|}{\rho\sigma(\mu,\lambda) + |m_\varepsilon \circ \theta(\mu,\lambda)|^2} \leqslant |\mathscr{F}(f)(\mu,\lambda)|,$$

we deduce that

$$\lim_{\rho \longrightarrow 0^+} \sup_{(s,y)\in[0,+\infty[\times\mathbb{R}^n} |f^*_{\rho,\varepsilon}(s,y) - f(s,y)| = 0.$$

Which completes the proof of the Theorem. □

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