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$\mathcal{R}$-PARTS AND MODULES DERIVED FROM STRONGLY $\mathcal{U}$-REGULAR RELATIONS ON HYPERMODULES

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This paper concerns a new relationship between hypermodules and modules. We generalize the notion of complete parts and θ-parts by the notion of $\mathfrak{R}$-parts on hypermodules and then $\mathfrak{R}$-closures of hypermodules as a generalization of θ-closures are defined. In addition, we give the notion of a strongly $\mathcal{U}$-regular relation on hypermodules and investigate some properties of it.

1. Introduction

If M is an R-hypermodule [1] and $\rho \subseteq M \times M$ is an equivalence relation, then for all pairs (A,B) of non-empty subsets of M, we set $A\overline{\overline{\rho}}B$ if and only if $a\rho b$ for all $a \in A$, $b \in B$. The relation ρ is said to be *strongly regular to the right* if $x\rho y$ implies $x+a \ \overline{\overline{\rho}} \ y+a$ and $r \cdot x \ \rho \ r \cdot y$ for all $x,y,a \in H$ and $r \in R$. Analogously, we can define *strongly regular to the left*. Moreover ρ is called *strongly regular* if it is strongly regular to the right and to the left. Let M be a hypermodule and ρ an equivalence relation on M. Let $\rho(a)$ be the equivalence class of a with respect to ρ and set $M/\rho = \{\rho(a) \mid a \in M\}$. The hyperoperations $\oplus$ are $\odot$ are defined on M/ρ by $\rho(a) \oplus \rho(b) = \{\rho(x) \mid x \in \rho(a)+\rho(b)\}$ and $r \odot \rho(a) = \{\rho(z)|z \in r \cdot \rho(a)\}$. If ρ is strongly regular then it readily follows that $\rho(a) \oplus \rho(b) = \{\rho(x) \mid x \in a+b\}$ and $r \odot \rho(a) = \{\rho(x)|x \in r \cdot a\}$ It is well

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known for ρ strongly regular that $(M/\rho, \oplus, \odot)$ is an R-hypermodule. That is $\rho(a) \oplus \rho(b) = \rho(c)$ for all $c \in a+b$ and $r \odot \rho(a) = \rho(x)$ for all $x \in r \cdot a$ [1].

Several relations have been studied in hypergroups, hyperrings and hypermodules such $\beta, \gamma, \varepsilon, \theta$ etc., for example see Anvariyeh et al. [1–4], Corsini and Leoreanu [6], Davvaz et al. [8, 9], Freni [10, 11], Koskas [12] and Vougiouklis [15–17]. Complete parts were introduced by Koskas [12] and studied then by Corsini [5], Davvaz and Karimian [7], Miglirato [13], Mousavi et al. [14], and others.

Let M be an R-hypermodule. We consider the relation ε on M as follows [16]:

$$x \varepsilon y \Leftrightarrow \ x, y \in \sum_{i=1}^{n} m_i'; \quad m_i' = m_i \ \text{ or } \ m_i' = \sum_{j=1}^{n_i} (\prod_{k=1}^{k_{ij}} x_{ijk}) z_i,$$

$$m_i \in M, \ x_{ijk} \in R, \ z_i \in M.$$

The fundamental relation ε^* on M can be considered as the smallest equivalence relation such that the quotient M/ε^* be a module over the corresponding fundamental ring such that M/ε^* as a group is not abelian [1, 16]. Now, we recall the following definition from [1].

Definition 1.1. [1]. Let M be an R-hypermodule. We define the relation θ as follows:

$$x \theta y \iff \exists\, n \in \mathbb{N}, \ \exists (m_1', \ldots, m_n'), \ \exists (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n, \ \exists \sigma \in \mathbb{S}_n,$$

$$\exists (x_{i1}, x_{i2}, \ldots, x_{ik_i}) \in R^{k_i}, \ \exists \sigma_i \in \mathbb{S}_{n_i}, \ \exists \sigma_{ij} \in \mathbb{S}_{k_{ij}},$$

such that

$$x \in \sum_{i=1}^{n} m_i'; \quad m_i' = m_i \ \text{ or } \ m_i' = \sum_{j=1}^{n_i} (\prod_{k=1}^{k_{ij}} x_{ijk}) m_i$$

and

$$y \in \sum_{i=1}^{n} m'_{\sigma(i)},$$

where

$$m'_{\sigma(i)} = m_{\sigma(i)} \ \text{ if } \ m_i' = m_i \qquad \text{and}$$

$$m'_{\sigma(i)} = B_{\sigma(i)} m_{\sigma(i)} \ \text{ if } \ m_i' = \sum_{j=1}^{n_i} (\prod_{k=1}^{k_{ij}} x_{ijk}) m_i,$$

with

$$B_i = \sum_{j=1}^{n_i} A_{i\sigma_i(j)}, \quad A_{ij} = \prod_{k=1}^{k_{ij}} x_{ij\sigma_{ij}(k)}.$$

If θ^* is the transitive closure of θ, then θ^* is a strongly regular relation on M as an R-hypermodule [1]. The fundamental relation θ is not transitive in general [2]. The following theorem gives the sufficient conditions, that the relation θ is transitive.

Theorem 1.2. *[3]. Let R be a commutative hyperring. If M is an R-hypermodule and for every $m \in M$, $R \cdot m = M$, then the fundamental relation θ is transitive on hypermodules.*

2. θ-parts and $\mathfrak{R}$-parts of hypermodules

In this section, we begin with the definition of $\theta-$parts of hypermodules which are valid in every hypermodule [3]. In the following m'_i is the notation that defined in Definition 1.1.

Definition 2.1. [3]. Let M be an $R-$hypermodule and H be a non-empty subset of M. We say that H is a *θ-part* of M if for every $n \in \mathbb{N}$, for every $\sigma \in \mathbb{S}_n$ and for every $(m'_1, \ldots, m'_p)$

$$\sum_{i=1}^{p} m'_i \cap H \neq \emptyset \Rightarrow \sum_{i=1}^{p} m'_{\sigma(i)} \subseteq H.$$

H is said to be a *complete part* of M, if σ is identity.

Now, we generalize the notion of complete parts and θ-parts and by the notion of $\mathfrak{R}$-parts and then we study $\mathfrak{R}$-closures in hypermodules. Recently, $\mathcal{R}$-parts in (semi)-hypergroups introduced by Mousavi, Leoreanu-Fotea and Jafarpour [14].

Let M be an R-hypermodule and $\mathcal{U}$ be the set of finite sums of $\sum_{i=1}^{p} m'_i$ and $\mathfrak{R}$ be a relation on M.

Definition 2.2. For a nonempty subset A of M, we say that A is a *left $\mathfrak{R}$-part of M with respect to $\mathcal{U}$* (or briefly in $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part) if for all $\sum_{i=1}^{p} m'_i$ and $\sum_{i=1}^{q} z'_i$ in $\mathcal{U}$ the following implication is valid

$$\left(\sum_{i=1}^{p} m'_i \cap A \neq \emptyset \text{ and } \sum_{i=1}^{q} z'_i \;\mathfrak{R}\; \sum_{i=1}^{p} m'_i\right) \Rightarrow \sum_{i=1}^{q} z'_i \subseteq A.$$

Similarly, we can define a right $\mathfrak{R}$-part of M with respect to $\mathcal{U}$ (or briefly in $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part). A is an *$\mathfrak{R}$-part on M with respect to $\mathcal{U}$* (or briefly in $\mathfrak{R}_{\mathcal{U}}$-part) if it is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part and an $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part.

Remark 2.3. By Definition 2.2, it is straightforward for any nonempty subset A of a hypermodule M, A is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}^{-1}$-part ($\mathcal{R}\mathfrak{R}_{\mathcal{U}}^{-1}$-part) if and only if A is an $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part ($\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part).

Now, we recall that a K_M-semihypergroup is the semihypergroup constructed from a semihypergroup $(M,+)$ and a family $\{A(x)\}_{x\in M}$ of nonempty and mutually disjoint subsets of M. Set $K_M = \bigcup_{x\in M} A(x)$ and consider the hyperoperation $*$ on K_M as follows:

$$\forall (a,b) \in K_M^2;\ a \in A(x),\ b \in A(y),\ a*b = \bigcup_{z\in x+y} A(z).$$

Then, $(M,+)$ is a hypergroup if and and only if $(K_M,*)$ is a hypergroup (see Theorem 375 [5]).

Theorem 2.4. *Let $(M,+,\cdot)$ be an R-hypermodule. Then, the $(K_M,*,\circ)$ is an R-hypermodule.*

Proof. We define the scalar hyperoperation $\circ$ as follows:

$$r \in R,\ a \in A(x); \quad r\circ a := \bigcup_{z\in r\cdot x} A(z).$$

Suppose that $r,s \in R$ and $a \in A(x)$, $b \in A(y)$. Then,
(1)

$$\begin{aligned}(r+s)\circ a &= \bigcup_{z\in(r+s)\cdot x} A(z) = \bigcup_{z\in r\cdot x+s\cdot x} A(z)\\ &= \bigcup_{m_1\in r\cdot x, m_2\in s\cdot x}\ \bigcup_{z\in m_1+m_2} A(z)\end{aligned}$$

and

$$\begin{aligned}(r\circ a)*(s\circ a) &= (\bigcup_{k\in r\cdot x} A(k)) * (\bigcup_{t\in s\cdot x} A(t))\\ &= \bigcup_{k\in r\cdot x, t\in s\cdot x}\ \bigcup_{w\in k+t} A(w).\end{aligned}$$

(2)

$$\begin{aligned}r\circ(a*b) &= r\circ(\bigcup_{z\in x+y} A(z)) = \bigcup_{z\in x+y} r\circ A(z)\\ &= \bigcup_{z\in x+y}\ \bigcup_{u\in r\cdot z} A(u) = \bigcup_{u\in r\cdot(x+y)} A(u)\end{aligned}$$

and

$$\begin{aligned}(r\circ a)*(r\circ b) &= (\bigcup_{k\in r\cdot a} A(k)) * (\bigcup_{t\in r\cdot b} A(t))\\ &= \bigcup_{k\in r\cdot a, t\in r\cdot b}\ \bigcup_{w\in k+t} A(w) = \bigcup_{u\in(r\cdot x+r\cdot y)} A(u).\end{aligned}$$

(3)

$$\begin{aligned}r\circ(s\circ a) &= r\circ(\bigcup_{z\in s\cdot x} A(z)) = \bigcup_{z\in s\cdot x}\ \bigcup_{u\in r\cdot z} A(u)\\ &= \bigcup_{u\in r(s\cdot x)} A(u) = \bigcup_{z\in(rs)\cdot x} A(z) = (rs)\circ a.\end{aligned}$$

Therefore, K_M is an R-hypermodule. $\square$

For all $P \in \wp^*(H)$, set $A(P) = \bigcup_{x \in P} A(x)$.

Theorem 2.5. *If $\Re$ is a relation on $\mathcal{U}$, then P is an $L\Re_{\mathcal{U}}$-part of hypermodule M if and only if $A(P)$ is an $L\widehat{\Re}_{\mathcal{U}}$-part of K_M, where the relation $\widehat{\Re}$ is defined as follows:*

$$\bigcup_{v \in \sum_{i=1}^{p} m_i'} A(v) \;\widehat{\Re} \bigcup_{u \in \sum_{i=1}^{q} z_i'} A(u) \Leftrightarrow \sum_{i=1}^{p} m_i' \;\Re\; \sum_{i=1}^{q} z_i'$$

Proof. Suppose that $A(P)$ is an $L\widehat{\Re}_{\mathcal{U}}$-part of K_M, and $\left(\sum_{i=1}^{p} m_i', \sum_{i=1}^{q} z_i'\right) \in \Re$ is such that $\sum_{i=1}^{q} z_i' \cap P \neq \emptyset$. So,

$$\bigcup_{v \in \sum_{i=1}^{p} m_i'} A(v) \;\widehat{\Re} \bigcup_{u \in \sum_{i=1}^{q} z_i'} A(u)$$

and

$$\begin{aligned}
\sum_{i=1}^{q} z_i' \cap P \neq \emptyset \quad &\Longrightarrow \exists p \in P, \text{ such that } p \in \sum_{i=1}^{q} z_i' \\
&\Longrightarrow \exists p \in P, \text{ such that } A(p) \subseteq \bigcup_{u \in \sum_{i=1}^{q} z_i'} A(u) \\
&\Longrightarrow \bigcup_{u \in \sum_{i=1}^{q} z_i'} A(u) \cap A(P) \neq \emptyset. \\
&\Longrightarrow \bigcup_{u \in \sum_{i=1}^{p} m_i'} A(v) \subseteq A(P), \text{ because } A(P) \text{ is a } L\widehat{\Re_{\mathcal{U}}} - \text{part.}
\end{aligned}$$

For all $t \in \sum_{i=1}^{p} m_i'$, $A(t) \subseteq A(P)$, so there exists $q \in P$ such that $A(t) \cap A(q) \neq \emptyset$. Thus, $t = q$ and hence $t \in P$. Therefore, $\sum_{i=1}^{p} m_i' \subseteq P$.

Conversely, suppose that $*\sum_{i=1}^{p} m_i' \cap A(P) \neq \emptyset$, where $*\sum$ denotes a hypersum of K_M. Suppose that $*\sum_{i=1}^{q} z_i' \;\widehat{\Re}\; *\sum_{i=1}^{p} m_i'$. Then, there exists $(x_1, \ldots, x_p)$ such that for any $t_i \in m_i'$ $(1 \leq i \leq q)$, $t_i \in A(x_i)$. Now, if

$$u \in \bigcup_{y \in \sum_{i=1}^{p} x_i} A(y) \cap A(P),$$

then $u \in A(y_0)$ for some $y_0 \in \sum_{i=1}^{p} x_i$. Since $u \in A(P)$, there exists $y_1 \in P$ such that $u \in A(y_1)$. So, $A(y_0) \cap A(y_1) \neq \emptyset$, which implies that $y_0 = y_1 \in \sum_{i=1}^{p} x_i \cap P$. Since P is an $L\Re_{\mathcal{U}}$-part of M and $\sum_{i=1}^{q} v_i \;\Re\; \sum_{i=1}^{p} x_i$, where $u_i \in z_i'$, $u_i \in A(v_i)$ for all $1 \leq i \leq q$. It follows that $\sum_{i=1}^{q} v_i \subseteq P$. Therefore,

$$\sum_{i=1}^{q} z_i' = \bigcup_{w \in \sum_{i=1}^{q} v_i} A(w) \subseteq \bigcup_{l \in P} A(l) = A(P).$$

$\square$

3. $\mathfrak{R}$-closure and $\mathfrak{R}$-parts of hypermodules

Let M be an R-hypermodule and $\mathcal{U}$ be the set of finite sums of $\sum_{i=1}^{p} m_i'$ and $\mathcal{R}$ be the relation on M. The intersection of all $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-parts (or $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-parts, $\mathfrak{R}$-parts) which contain A is called $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$*-closure* (or $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$*-closure,* $\mathfrak{R}$*-closure*) of A in M and it is denoted by $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A)$ (or $\overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A), \overline{\mathfrak{R}_{\mathcal{U}}}(A)$).

Remark 3.1. By Remark 2.3, for any nonempty subset A of a hypermodule M, A is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}^{-1}$-part ($\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part) if and only if A is an $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part ($\mathcal{R}\mathfrak{R}_{\mathcal{U}}^{-1}$-part). So, immediately, we obtain

$$\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A) \ \left(\overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A)\right).$$

For a nonempty subset A of M, we define:

$$_{A}\sum^{\mathcal{U}} := \left\{\mathfrak{R} \subseteq \mathcal{U}\times\mathcal{U} \mid \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A) = A\right\}$$

and

$$\sum_{A}^{\mathcal{U}} := \left\{\mathfrak{R} \subseteq \mathcal{U}\times\mathcal{U} \mid \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A) = A\right\}.$$

Lemma 3.2. *If* $_{A}\sum^{\mathcal{U}} \neq \emptyset$ *(or* $\sum_{A}^{\mathcal{U}} \neq \emptyset$*), then* $(_{A}\sum^{\mathcal{U}} \neq \emptyset, \circ)$ *(or* $(\sum_{A}^{\mathcal{U}} \neq \emptyset, \circ)$*) is closed under the composition* $\circ$ *of relations.*

Proof. Suppose that $\mathfrak{R}, \mathfrak{R}' \in {_{A}\sum^{\mathcal{U}}}$ and $\left(\sum_{i=1}^{p} m_i', \sum_{i=1}^{q} z_i'\right) \in \mathcal{U}\times\mathcal{U}$ are given. Also, let $\sum_{i=1}^{p} m_i' \cap A \neq \emptyset$ and $\sum_{i=1}^{p} z_i' \ \mathfrak{R}\circ\mathfrak{R}' \ \sum_{i=1}^{p} m_i'$. So, there exists $(y_1', \ldots, y_k')$ such that $\sum_{i=1}^{k} y_i' \ \mathfrak{R} \ \sum_{i=1}^{p} m_i'$ and $\sum_{i=1}^{q} z_i' \ \mathfrak{R}' \ \sum_{i=1}^{k} y_i'$. From $\sum_{i=1}^{k} y_i' \ \mathfrak{R} \ \sum_{i=1}^{p} m_i'$ and $\mathfrak{R} \in {_{A}\sum^{\mathcal{U}}}$ it follows that $\sum_{i=1}^{k} y_i' \subseteq A$. Since $\mathfrak{R}' \in {_{A}\sum^{\mathcal{U}}}$ and $\sum_{i=1}^{q} z_i' \ \mathfrak{R} \ \sum_{i=1}^{k} y_i'$, we obtain that $\sum_{i=1}^{q} z_i' \subseteq A$. Hence, $_{A}\sum^{\mathcal{U}}$ and so $(\sum_{A}^{\mathcal{U}} \neq \emptyset, \circ)$ is a semigroup. □

Theorem 3.3. *Let* $\mathfrak{R}$ *be a permutation of finite order in* $\mathbb{S}_{\mathcal{U}}$*. If* A *is* $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$*-part, then* A *is* $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$*-part.*

Proof. Since A is $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part, $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A) = A$ and hence $\mathfrak{R} \in {_{A}\sum^{\mathcal{U}}}$. Since $\mathfrak{R}$ is a permutation of finite order in $\mathbb{S}_{\mathcal{U}}$, $<\mathfrak{R}>= \{\mathfrak{R}^n \mid n \in \mathbb{N}\}$ is a subgroup of $_{A}\sum^{\mathcal{U}}$ and so $\mathfrak{R}^{-1} \in {_{A}\sum^{\mathcal{U}}}$. By Remark 3.1, $A = \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A)$. Thus, $\mathfrak{R} \in \sum_{A}^{\mathcal{U}}$ and hence A is a $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part. □

In the following, we determine the sets $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A)$, $\overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A)$ and $\overline{\mathfrak{R}_{\mathcal{U}}}(A)$), where $\mathfrak{R} \subseteq \mathcal{U}\times\mathcal{U}$ and A is a nonempty subset of M. Set $K_{1,\mathfrak{R}}^{\mathcal{L}}(A) = A$ and

$$K_{t+1,\mathfrak{R}}^{\mathcal{L}}(A) = \left\{x \in M \mid \exists \left(\sum_{i=1}^{p} m_i', \sum_{i=1}^{q} z_i'\right) \in \mathfrak{R}, \ x \in \sum_{i=1}^{p} m_i', \ \sum_{i=1}^{q} z_i' \cap K_{t,\mathfrak{R}}^{\mathcal{L}}(A) \neq \emptyset\right\}.$$

Denote $K^{\mathcal{L}}_{\mathfrak{R}}(A) = \bigcup_{n\geq 1} K^{\mathcal{L}}_{n,\mathfrak{R}}(A)$. Similarly, set $K^{\mathcal{R}}_{1,\mathfrak{R}}(A) = A$ and

$$K^{\mathcal{R}}_{t+1,\mathfrak{R}}(A) = \left\{ x \in M \mid \exists \left(\sum_{i=1}^{q} z'_i, \sum_{i=1}^{p} m'_i \right) \in \mathfrak{R},\ x \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{q} z'_i \cap K^{\mathcal{R}}_{t,\mathfrak{R}}(A) \neq \emptyset \right\}.$$

Denote $K^{\mathcal{R}}_{\mathfrak{R}}(A) = \bigcup_{n\geq 1} K^{\mathcal{R}}_{n,\mathfrak{R}}(A)$. Finally, set $K_{1,\mathfrak{R}}(A) = A$ and

$$K_{t+1,\mathfrak{R}}(A) = \left\{ x \in M \mid \exists \left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i \right) \in \mathfrak{R} \cup \mathfrak{R}^{-1}, x \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{q} z'_i \cap K_{t,\mathfrak{R}}(A) \neq \emptyset \right\}.$$

Denote $K_{\mathfrak{R}}(A) = \bigcup_{n\geq 1} K_{n,\mathfrak{R}}(A)$.

Theorem 3.4. *Let A be a nonempty subset of hypermodule M. Then, $K_{\mathfrak{R}}(A) = \overline{\mathfrak{R}_{\mathcal{U}}}(A)$.*

Proof. It is necessary to prove:

(i) $K^{\mathcal{L}}_{\mathfrak{R}}(A)$ is a $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part;

(ii) if $A \subseteq B$ and B is a $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part, then $K^{\mathcal{L}}_{\mathfrak{R}}(A) \subseteq B$.

In order to prove (i), suppose that $\sum_{i=1}^{p} m'_i \cap K^{\mathcal{L}}_{\mathfrak{R}}(A) \neq \emptyset$ and $\sum_{i=1}^{q} z'_i \ \mathfrak{R} \sum_{i=1}^{p} m'_i$. So, there exists $t \in \mathbb{N}$ such that $\sum_{i=1}^{p} m'_i \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(A) \neq \emptyset$, which it follows that $\sum_{i=1}^{q} z'_i \subseteq K^{\mathcal{L}}_{t+1,\mathfrak{R}}(A) \subseteq K^{\mathcal{L}}_{\mathfrak{R}}(A)$.

Now, we prove (ii) by induction on t. We have $K^{\mathcal{L}}_{1,\mathfrak{R}}(A) = A \subseteq B$. Suppose that $K^{\mathcal{L}}_{t,\mathfrak{R}}(A) \subseteq B$. We prove that $K^{\mathcal{L}}_{t+1,\mathfrak{R}}(A) \subseteq B$. If $z \in K^{\mathcal{L}}_{t+1,\mathfrak{R}}(A)$, then there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathcal{U} \times \mathcal{U}$ such that $z \in \sum_{i=1}^{p} m'_i$, $\sum_{i=1}^{p} m'_i \ \mathfrak{R} \ \sum_{i=1}^{q} z'_i$ and $\sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(A) \neq \emptyset$. Hence, $\sum_{i=1}^{q} z'_i \cap B \neq \emptyset$ and so $z \in \sum_{i=1}^{p} m'_i \subseteq B$. Then, $K^{\mathcal{L}}_{t+1,\mathfrak{R}}(A) \subseteq B$. Hence, $K^{\mathcal{L}}_{\mathfrak{R}}(A) = \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A)$. Also, by Remark 3.1, we have $K^{\mathcal{R}}_{\mathfrak{R}}(A) = K^{\mathcal{R}}_{\mathfrak{R}^{-1}}(A) = \overline{\mathcal{L}\mathfrak{R}^{-1}_{\mathcal{U}}}(A) = \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A)$. Therefore, $K_{\mathfrak{R}}(A) = \overline{\mathfrak{R}_{\mathcal{U}}}(A)$. □

Proposition 3.5. *Let A be a nonempty subset of hypermodule M and $\mathfrak{R}$ be a relation on $\mathcal{U}$. Then, $\overline{\mathfrak{R}_{\mathcal{U}}}(A) = \bigcup_{a\in A} \overline{\mathfrak{R}_{\mathcal{U}}}(a)$.*

Proof. It is clear that for all $a \in A$, $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(a) \subseteq \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A)$. By Theorem 3.4, we have $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A) = \bigcup_{n\geq 1} K^{\mathcal{L}}_{n,\mathfrak{R}}(A)$ and $K^{\mathcal{L}}_{1,\mathfrak{R}}(A) = A = \bigcup_{a\in A}\{a\} = \bigcup_{a\in A} K^{\mathcal{L}}_{1,\mathfrak{R}}(a)$. We prove the proposition by induction on n. Supposing it true for n, we prove that $K^{\mathcal{L}}_{n+1,\mathfrak{R}}(A) \subseteq \bigcup_{a\in A} K^{\mathcal{L}}_{n+1,\mathfrak{R}}(a)$.

If $z \in K^{\mathcal{L}}_{n+1,\mathfrak{R}}(A)$, then there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R}$ such that

$$z \in \textstyle\sum_{i=1}^{p} m'_i \text{ and } \sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{n,\mathfrak{R}}(A) \neq \emptyset.$$

By the hypothesis of induction, $\sum_{i=1}^{q} z'_i \cap \left(\bigcup_{a\in A} K^{\mathcal{L}}_{n,\mathfrak{R}}(a)\right) \neq \emptyset$ and so there exists $a' \in A$ such that $\sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{n,\mathfrak{R}}(a') \neq \emptyset$. Hence, $z \in K^{\mathcal{L}}_{n+1,\mathfrak{R}}(a')$, where $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(A) \subseteq \bigcup_{a\in A} \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(a)$. By the similar way, we can prove that $\overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A) = \bigcup_{a\in A} \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(a)$. Therefore, $\overline{\mathfrak{R}_{\mathcal{U}}}(A) = \bigcup_{a\in A} \overline{\mathfrak{R}_{\mathcal{U}}}(a)$. □

Theorem 3.6. *If $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$, then the following relation $K^{\mathcal{L}}_{\mathfrak{R}}$ $\left(K^{\mathcal{R}}_{\mathfrak{R}}\right)$ on a hypermodule M:*

$$x\, K^{\mathcal{L}}_{\mathfrak{R}}\, y \Leftrightarrow x \in K^{\mathcal{L}}_{\mathfrak{R}}(y) \quad \left(x\, K^{\mathcal{R}}_{\mathfrak{R}}\, y \Leftrightarrow x \in K^{\mathcal{R}}_{\mathfrak{R}}(y)\right).$$

where $K^{\mathcal{L}}_{\mathfrak{R}}(y) = K^{\mathcal{L}}_{\mathfrak{R}}(\{y\})$ (where $K^{\mathcal{R}}_{\mathfrak{R}}(y) = K^{\mathcal{R}}_{\mathfrak{R}}(\{y\})$) is a preorder. Furthermore, if $\mathfrak{R}$ is symmetric, then $K^{\mathcal{L}}_{\mathfrak{R}}$ ($K^{\mathcal{R}}_{\mathfrak{R}}$ respectively) is an equivalence relation.

Proof. It is easy to see that $K^{\mathcal{L}}_{\mathfrak{R}}$ is reflexive. Now, suppose that $xK^{\mathcal{L}}_{\mathfrak{R}}\, y$ and $yK^{\mathcal{L}}_{\mathfrak{R}}\, z$. So, $x \in K^{\mathcal{L}}_{\mathfrak{R}}(y)$ and $y \in K^{\mathcal{L}}_{\mathfrak{R}}(z)$. By Theorem 3.4, $K^{\mathcal{L}}_{\mathfrak{R}}(z)$ is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part. Thus, $K^{\mathcal{L}}_{\mathfrak{R}}(y) \subseteq K^{\mathcal{L}}_{\mathfrak{R}}(z)$ and hence $x \in K^{\mathcal{L}}_{\mathfrak{R}}(z)$. Therefore, $K^{\mathcal{L}}_{\mathfrak{R}}$ is preorder. Now, if $\mathfrak{R}$ is symmetric, then we prove that $K^{\mathcal{L}}_{\mathfrak{R}}$ is symmetric as well. We check that:

(i) for all $n \geq 2$ and $x \in M$, $K^{\mathcal{L}}_{n,\mathfrak{R}}(K^{\mathcal{L}}_{2,\mathfrak{R}}(x)) = K^{\mathcal{L}}_{n+1,\mathfrak{R}}(x)$;

(ii) $x \in K^{\mathcal{L}}_{n,\mathfrak{R}}(y)$ if and only if $y \in K^{\mathcal{L}}_{n,\mathfrak{R}}(x)$.

We prove (i) by induction on n. Suppose that $z \in K^{\mathcal{L}}_{2,\mathfrak{R}}(K^{\mathcal{L}}_{2,\mathfrak{R}}(x))$, so there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R}$ such that $z \in \sum_{i=1}^{p} m'_i$ and $\sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{2,\mathfrak{R}}(x) \neq \emptyset$. Thus, $z \in K^{\mathcal{L}}_{3,\mathfrak{R}}$. If $K^{\mathcal{L}}_{t,\mathfrak{R}}(K^{\mathcal{L}}_{2,\mathfrak{R}}(x)) = K^{\mathcal{L}}_{t+1,\mathfrak{R}}(x)$, then

$$\begin{aligned}
& z \in K^{\mathcal{L}}_{t+1,\mathfrak{R}}\left(K^{\mathcal{L}}_{2,\mathfrak{R}}(x)\right) \\
& \Leftrightarrow \exists \left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R},\ z \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(K^{\mathcal{L}}_{2,\mathfrak{R}}(x)) \neq \emptyset \\
& \Leftrightarrow \exists \left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R},\ z \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{t+1,\mathfrak{R}}(x) \neq \emptyset \\
& \Leftrightarrow z \in K^{\mathcal{L}}_{t+2,\mathfrak{R}}(x).
\end{aligned}$$

Hence, for all $t \geq 2$ and $x \in M$, $K^{\mathcal{L}}_{t,\mathfrak{R}}\left(K^{\mathcal{L}}_{2,\mathfrak{R}}(x)\right) = K^{\mathcal{L}}_{t+1,\mathfrak{R}}(x)$.

We prove (ii) by induction on n, too. It is clear that $x \in K^{\mathcal{L}}_{2,\mathfrak{R}}(y)$ if and only if $y \in K^{\mathcal{L}}_{2,\mathfrak{R}}(x)$. Suppose that $x \in K^{\mathcal{L}}_{t,\mathfrak{R}}(y)$ if and only if $y \in K^{\mathcal{L}}_{t,\mathfrak{R}}(x)$. If $x \in K^{\mathcal{L}}_{t+1,\mathfrak{R}}(y)$, then there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R}$ such that $x \in \sum_{i=1}^{p} m'_i$ and $\sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(y) \neq \emptyset$. Therefore, there exists $b \in \sum_{i=1}^{q} z'_i \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(y) \neq \emptyset$, hence $y \in K^{\mathcal{L}}_{t,\mathfrak{R}}(b)$. Since $\mathfrak{R}$ is symmetric $\left(\sum_{i=1}^{q} z'_i, \sum_{i=1}^{p} m'_i\right) \in \mathfrak{R}$. From $b \in \sum_{i=1}^{q} z'_i$ and $\sum_{i=1}^{p} m'_i \cap K^{\mathcal{L}}_{1,\mathfrak{R}}(x)$, it follows that $b \in K^{\mathcal{L}}_{2,\mathfrak{R}}(x)$ and so $K^{\mathcal{L}}_{t,\mathfrak{R}}(K^{\mathcal{L}}_{2,\mathfrak{R}})(x) = K^{\mathcal{L}}_{t+1,\mathfrak{R}}(x)$. Similarly, we can show that if $y \in K^{\mathcal{L}}_{t+1,\mathfrak{R}}(x)$, then $x \in y \in K^{\mathcal{L}}_{t+1,\mathfrak{R}}(y)$. □

Proposition 3.7. *Let $\mathfrak{R}$ be a relation on $\mathcal{U}$ and A be a nonempty subset of hypermodule M. Then, the following conditions are equivalent:*

(1) A is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part ($\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part) of M;

(2) $x \in A$, $zK^{\mathcal{L}}_{\mathfrak{R}}x \Longrightarrow z \in A$ ($xK^{\mathcal{L}}_{\mathfrak{R}}z \Longrightarrow z \in A$, respectively).

Proof. (1)⇒(2): Let $x \in A$ and $z \in M$ be such that $zK^{\mathcal{L}}_{\mathfrak{R}}x$. Then, there exists $\left(\sum_{i=1}^{p} m_i', \sum_{i=1}^{q} z_i'\right) \in \mathfrak{R}$ such that $z \in \sum_{i=1}^{p} m_i' \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(A) \neq \emptyset$ for some $t \in \mathbb{N}$. Since A is a $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part, according to Theorem 3.4, $K^{\mathcal{L}}_{t,\mathfrak{R}}(A) \subseteq A$, and so $\sum_{i=1}^{q} z_i' \cap A \neq \emptyset$. Therefore, $\sum_{i=1}^{p} m_i' \subseteq A$ and hence $z \in A$.

(2)⇒(1): Let $\sum_{i=1}^{p} m_i' \cap A \neq \emptyset$ and $\sum_{i=1}^{q} z_i' \;\mathfrak{R}\; \sum_{i=1}^{p} m_i'$. So, there exists $x \in \sum_{i=1}^{p} m_i' \cap A$ and so $\sum_{i=1}^{q} z_i' \cap K^{\mathcal{L}}_{1,\mathfrak{R}}(A) \neq \emptyset$. Set $z \in \sum_{i=1}^{q} z_i'$. Hence,

$$\sum_{i=1}^{q} z_i' \;\mathfrak{R}\; \sum_{i=1}^{p} m_i',\ z \in \sum_{i=1}^{p} m_i' \Rightarrow z \in K^{\mathcal{L}}_{2,\mathfrak{R}}(x) \Rightarrow z\ K^{\mathcal{L}}_{\mathfrak{R}}\ x \Rightarrow z \in A, \text{ because } x \in A.$$

Therefore, $\sum_{i=1}^{q} z_i' \subseteq A$ and A is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part of M. □

4. Modules derived from strongly $\mathcal{U}$-regular relations

In this section, we give the notion of a strongly $\mathcal{U}$-regular relation and investigate some properties of it.

Definition 4.1. Let $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$. For all $(x,y) \in M^2$, we define the relation $\rho_{\mathcal{L},\mathfrak{R}}$, as follows:

$$x\ \rho_{\mathcal{L},\mathfrak{R}}\ y \Leftrightarrow x = y \text{ or } \exists \left(\sum_{i=1}^{p} m_i', \sum_{i=1}^{q} z_i'\right) \in \mathfrak{R} \text{ such that } x \in \sum_{i=1}^{p} m_i' \text{ and } y \in \sum_{i=1}^{q} z_i'.$$

We denote $\rho^*_{\mathcal{L},\mathfrak{R}}$ the transitive closure of $\rho_{\mathcal{L},\mathfrak{R}}$. Similarly, we can define the relation $\rho_{\mathcal{R},\mathfrak{R}}$. We denote $\rho^*_{\mathcal{R},\mathfrak{R}}$ the transitive closure of $\rho_{\mathcal{R},\mathfrak{R}}$. For all $(x,y) \in M^2$, we define the relation $\rho_{\mathfrak{R}}$, as follows:

$$x\ \rho_{\mathfrak{R}}\ y \Leftrightarrow x = y \text{ or } \exists \left(\sum_{i=1}^{q} z_i', \sum_{i=1}^{p} m_i'\right) \in \mathfrak{R} \cup \mathfrak{R}^{-1} \text{ such that } x \in \sum_{i=1}^{p} m_i' \text{ and } y \in \sum_{i=1}^{q} z_i'.$$

We denote $\rho^*_{\mathfrak{R}}$ the transitive closure of $\rho_{\mathfrak{R}}$.

Theorem 4.2. *Let $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$. Then, for all $(x,y) \in M^2$, $x\ K^{\mathcal{L}}_{\mathfrak{R}}\ y$ if and only if $x\ \rho^*_{\mathfrak{R}}\ y$.*

Proof. It is easy to see that $\rho^*_{\mathcal{L},\mathfrak{R}} \subseteq K^{\mathcal{L}}_{\mathfrak{R}}$.

Conversely, suppose that $x\, K^{\mathcal{L}}_{\mathfrak{R}}\, y$. Then, so by Theorem 3.6, $x \in K^{\mathcal{L}}_{t+1,\mathfrak{R}}$ for some $t \in \mathbb{N}$. So, there exists $\left(\sum_{i=1}^{p_1} m'_{1,i}, \sum_{i=1}^{q_1} z'_{1,i}\right) \in \mathfrak{R}$ such that $x \in \sum_{i=1}^{p_1} m'_{1,i}$ and $\sum_{i=1}^{q_1} z'_{1,i} \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(y) \neq \emptyset$. Thus, there exists $x_1 \in \sum_{i=1}^{q_1} z'_{1,i} \cap K^{\mathcal{L}}_{t,\mathfrak{R}}(y)$ which implies that $x \rho_{\mathcal{L},\mathfrak{R}}\, x_1$. Since $x_1 \in K^{\mathcal{L}}_{t,\mathfrak{R}}(y)$, there exists $\left(\sum_{i=1}^{p_2} m'_{2,i}, \sum_{i=1}^{q_2} z'_{2,i}\right) \in \mathfrak{R}$ such that $x_1 \in \sum_{i=1}^{p_2} m'_{2,i}$ and $\sum_{i=1}^{q_2} z'_{2,i} \cap K^{\mathcal{L}}_{t-1,\mathfrak{R}}(y) \neq \emptyset$. Therefore, $x_1\ \rho_{\mathcal{L},\mathfrak{R}}\ x_2$, where $x_2 \in \sum_{i=1}^{q_2} z'_{2,i} \cap K^{\mathcal{L}}_{t-1,\mathfrak{R}}(y)$. After t steps, we obtain there exists $x_t \in \sum_{i=1}^{q_t} z'_{t,i} \cap K^{\mathcal{L}}_{t-(t-1),\mathfrak{R}}(y)$ such that $x_{t-1}\ \rho_{\mathcal{L},\mathfrak{R}}\ x_t$. Thus, we have:

$$x\ \rho_{\mathcal{L},\mathfrak{R}}\ x_1\ \rho_{\mathcal{L},\mathfrak{R}}\ x_2 \ldots x_t\ \rho_{\mathcal{L},\mathfrak{R}}\ y$$

and from this it follows that $K^{\mathcal{L}}_{\mathfrak{R}} \subseteq \rho^*_{\mathcal{L},\mathfrak{R}}$. By the similar way, we obtain $x\, K^{\mathcal{R}}_{\mathfrak{R}}\, y$ if and only if $x\ \rho^*_{\mathcal{R},\mathfrak{R}}\ y$. Therefore, $x\, K_{\mathfrak{R}}\, y$ if and only if $x\ \rho^*_{\mathfrak{R}}\ y$. □

Proposition 4.3. *If $\mathfrak{R}$ is a permutation of finite order in $S_{\mathcal{U}}$, then $\rho^*_{\mathcal{L},\mathfrak{R}} = \rho^*_{\mathcal{R},\mathfrak{R}}$.*

Proof. By Theorem 3.4, $K^{\mathcal{L}}_{\mathfrak{R}}(y)$ is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part and so by Theorem 3.3, $K^{\mathcal{L}}_{\mathfrak{R}}(y)$ is an $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part and hence $K^{\mathcal{R}}_{\mathfrak{R}}(y) \subseteq K^{\mathcal{L}}_{\mathfrak{R}}(y)$. Analogously, $K^{\mathcal{L}}_{\mathfrak{R}}(y) \subseteq K^{\mathcal{R}}_{\mathfrak{R}}(y)$ and this completes the proof. □

Definition 4.4. Let $(M,+)$ be an R-hypermodule. A relation $\mathfrak{R}$ on $\mathcal{U}$ is called

(1) *compatible on the left* (*on the right*), if for all $P_1, P_2, P \in \mathcal{U}$ and $r \in R$ from $P_1\ \mathfrak{R}\ P_2$ it follows $P + P_1\ \mathfrak{R}\ P + P_2$ ($P_1 + P\ \mathfrak{R}\ P_2 + P$) and $r \cdot P_1\ \mathfrak{R}\ r \cdot P_2$ ($P_1 \cdot r\ \mathfrak{R}\ P_2 \cdot r$). $\mathfrak{R}$ is *compatible* if it is compatible on the left and on the right;

(2) *regular* if for all $x \in M$, implies $K^{\mathcal{L}}_{\mathfrak{R}}(x) = K^{\mathcal{R}}_{\mathfrak{R}}(x)$;

(3) a regular relation $\mathfrak{R}$ on $\mathcal{U}$ is called *strongly regular on the left* (*on the right*) if $\rho^*_{\mathcal{L},\mathfrak{R}}$ ($\rho^*_{\mathcal{R},\mathfrak{R}}$) is strongly regular on the left (on the right, respectively);

(4) a regular relation $\mathfrak{R}$ on $\mathcal{U}$ is called *strongly regular* if $\rho^*_{\mathfrak{R}}$ is strongly regular.

Proposition 4.5. *Let $\mathfrak{R}$ be a regular relation on $\mathcal{U}$. Then,*

(1) $\mathfrak{R}^{-1}$ is regular;

(2) $\rho^*_{\mathcal{L},\mathfrak{R}} = \rho^*_{\mathcal{R},\mathfrak{R}} = \rho^*_{\mathfrak{R}}$ is an equivalence relation.

Proof. The proof follows from Remark 3.1 and Theorem 4.2. □

Let $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$. For any element x of an R-hypermodule M, set

$$P^n_{\mathcal{L},\mathfrak{R}}(x) = \bigcup \left\{ \sum_{i=1}^{q} z'_i \mid \sum_{i=1}^{q} z'_i \,\mathfrak{R}\, \sum_{i=1}^{p} m'_i \text{ and } x \in \sum_{i=1}^{p} m'_i \right\};$$

$$P_{\mathcal{L},\mathfrak{R}}(x) = \bigcup_{n \geq 1} P^n_{\mathcal{L},\mathfrak{R}}(x) \cup \{x\};$$

$$\rho^*_{\mathcal{L},\mathfrak{R}}(x) = \{y \in M \mid y\, \rho^*_{\mathcal{L},\mathfrak{R}}\, x\}.$$

In the next theorem we find the necessary and sufficient conditions for the transitivity of the relation $\rho_{\mathcal{L},\mathfrak{R}}$.

Theorem 4.6. *Let $\mathfrak{R}$ be a relation on $\mathcal{U}$ and M be an R-hypermodule. Then, the following conditions are equivalent:*

(1) $\rho_{\mathcal{L},\mathfrak{R}}$ is transitive;

(2) for every $x \in M$, $\rho^*_{\mathcal{L},\mathfrak{R}}(x) = P_{\mathcal{L},\mathfrak{R}}(x)$;

(3) for every $x \in M$, $P_{\mathcal{L},\mathfrak{R}}(x)$ is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part of M.

Proof. (1) $\Rightarrow$ (2): For any pair $(x,y) \in M^2$, we have

$$y \in \rho^*_{\mathcal{L},\mathfrak{R}}(x) \Leftrightarrow y\, \rho^*_{\mathcal{L},\mathfrak{R}}\, x \Leftrightarrow y\, \rho_{\mathcal{L},\mathfrak{R}}\, x \Leftrightarrow y \in P_{\mathcal{L},\mathfrak{R}}(x).$$

(2) $\Rightarrow$ (3): Suppose that $\left(\sum_{i=1}^{q} z'_i, \sum_{i=1}^{p} m'_i\right) \in \mathfrak{R}$ such that $\sum_{i=1}^{p} m'_i \cap P_{\mathcal{L},\mathfrak{R}}(x) \neq \emptyset$. Then, $\sum_{i=1}^{p} m'_i \cap \rho^*_{\mathcal{L},\mathfrak{R}}(x) \neq \emptyset$ and so there exists $z \in \sum_{i=1}^{p} m'_i$ and $z \in \rho^*_{\mathcal{L},\mathfrak{R}}(x)$. Thus, by Theorem 4.2, $z \in K^{\mathcal{L}}_{\mathfrak{R}}(x)$. On the other hand, $z \in K^{\mathcal{L}}_{\mathfrak{R}}(x)$, so $\sum_{i=1}^{p} m'_i \cap K^{\mathcal{L}}_{\mathfrak{R}}(x) \neq \emptyset$. Hence, $\sum_{i=1}^{q} z'_i \subseteq K^{\mathcal{L}}_{\mathfrak{R}}(z)$, because $\sum_{i=1}^{q} z'_i \mathfrak{R} \sum_{i=1}^{p} m'_i$ and $K^{\mathcal{L}}_{\mathfrak{R}}(z)$ is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part of M. Now, suppose that $t \in \sum_{i=1}^{q} z'_i$ is an arbitrary element. Thus, $t \in K^{\mathcal{L}}_{\mathfrak{R}}(x)$ and $t\, \rho^*_{\mathcal{L},\mathfrak{R}}\, x$. Therefore, $t \in \rho^*_{\mathcal{L},\mathfrak{R}}(x) = P_{\mathcal{L},\mathfrak{R}}(x)$ and so $\sum_{i=1}^{q} z'_i \subseteq P_{\mathcal{L},\mathfrak{R}}(x)$.

(3) $\Rightarrow$(1): Suppose that $x,y,z \in M$ such that $x\, \rho_{\mathcal{L},\mathfrak{R}}\, y$ and $y\, \rho_{\mathcal{L},\mathfrak{R}}\, z$. Since $x\, \rho_{\mathcal{L},\mathfrak{R}}\, y$, there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R}$ such that $x \in \sum_{i=1}^{p} m'_i$ and $y \in \sum_{i=1}^{q} z'_i$. So, $\sum_{i=1}^{q} z'_i \cap P_{\mathcal{L},\mathfrak{R}}(y) \neq \emptyset$ and since $P_{\mathcal{L},\mathfrak{R}}(y)$ is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part, $\sum_{i=1}^{p} m'_i \subseteq P_{\mathcal{L},\mathfrak{R}}(y)$, whence $x \in P_{\mathcal{L},\mathfrak{R}}(y)$. We can easily check that $P_{\mathcal{L},\mathfrak{R}}(y) \subseteq P_{\mathcal{L},\mathfrak{R}}(z)$. Similarly, from $y\, \rho_{\mathcal{L},\mathfrak{R}}\, z$ we obtain $y \in P_{\mathcal{L},\mathfrak{R}}(z)$, then we use that $P_{\mathcal{L},\mathfrak{R}}(z)$ is a $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part of M. Therefore, $x \in P_{\mathcal{L},\mathfrak{R}}(z)$ and hence $x\, \rho_{\mathcal{L},\mathfrak{R}}\, z$. □

A hypermodule M is said to be *regular*, if as a hypergroup is regular [4].

Theorem 4.7. *Let M be a regular hypermodule and $\mathfrak{R}$ be a compatible relation on $\mathcal{U}$, Then, $\rho_{\mathcal{L},\mathfrak{R}}$ is transitive.*

Proof. According to the previous theorem, it is enough to check that for any $x \in M$, $P_{\mathcal{L},\mathfrak{R}}(x)$ is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part of M. Suppose that $\left(\sum_{i=1}^{q} z_i', \sum_{i=1}^{p} m_i'\right) \in \mathfrak{R}$ such that $\sum_{i=1}^{p} m_i' \cap P_{\mathcal{L},\mathfrak{R}}(x) \neq \emptyset$. We check that $\sum_{i=1}^{q} z_i' \subseteq P_{\mathcal{L},\mathfrak{R}}(x)$. Since M is a regular hypermodule, there exists an identity e in M. Moreover, there exist $u, v \in M$ such that $e \in u + x$ and $x \in t + v$, where $t \in \sum_{i=1}^{p} m_i' \cap P_{\mathcal{L},\mathfrak{R}}(x)$. Hence, there exist $P_1, P_2 \in \mathcal{U}$ such that $t \in P_1, x \in P_2$ and $P_1 \mathfrak{R} P_2$. We obtain

$$\begin{aligned} x \in t + v \quad &\subseteq \sum_{i=1}^{p} m_i' + v \subseteq \sum_{i=1}^{p} m_i' + e + v \\ &\subseteq \sum_{i=1}^{p} m_i' + u + x + v \subseteq \sum_{i=1}^{p} m_i' + u + P_2 + v = P_3, \end{aligned}$$

and

$$\sum_{i=1}^{q} z_i' \subseteq \sum_{i=1}^{q} z_i' + e \subseteq \sum_{i=1}^{q} z_i' + u + t + v \subseteq \sum_{i=1}^{q} z_i' + u + P_1 + v = P_4.$$

Since $\sum_{i=1}^{p} m_i' \; \mathfrak{R} \; \sum_{i=1}^{q} z_i'$, $P_1 \mathfrak{R} P_2$ and $\mathfrak{R}$ is regular, it follows that $P_3 \mathfrak{R} P_4$. Therefore, $\sum\limits_{i=1}^{q} z_i' \subseteq P_{\mathcal{L},\mathfrak{R}}(x)$ and so, $\rho_{\mathcal{L},\mathfrak{R}}$ is transitive. □

Similarly, we can prove that if M is a regular hypermodule and $\mathfrak{R}$ is a compatible relation on $\mathcal{U}$, then $\rho_{\mathfrak{R}}$ is transitive.

Theorem 4.8. *Let M be an R-hypermodule and $K = \bigcup_{n \geq 1} A_n$, where A_n is the alternating subgroup of the symmetric group $\mathbb{S}_n$ of order n or $K = \{I\}$, the identity of $\mathbb{S}_n$. We define the relation $\mathfrak{R}^K$ on $\mathcal{U}$ as follows: for all $\left(\sum_{i=1}^{p} m_i', \sum_{i=1}^{q} z_i'\right) \in \mathcal{U}^2$,*

$$\sum_{i=1}^{p} m_i' \; \mathfrak{R}^K \; \sum_{i=1}^{q} z_i' \Leftrightarrow \exists \tau \in K, \; \sum_{i=1}^{q} z_i' = \sum_{i=1}^{p} m_{\tau(i)}'.$$

Then,

$$\rho_{\mathfrak{R}^K} = \rho_{\mathfrak{R}^K}^* = \begin{cases} \varepsilon^* \text{ if } K = \{I\} \\ \\ \theta^* \text{ if } K = \bigcup_{n \geq 1} A_n. \end{cases}$$

Proof. It is straightforward that $\rho_{\mathfrak{R}^K}$ is a strongly relation on $\mathcal{U}$. If $K = \{I\}$, the proof is obvious (see [3]). Now, suppose that $K = \bigcup_{n \geq 1} A_n$. Then, $\rho_{\mathfrak{R}^K}^* \subseteq \theta^*$. Conversely, we prove that $\frac{M}{\rho_{\mathfrak{R}^K}^*}$ is an abelian group. Let $x_1, x_2 \in M$. From $M = x_2 + M$, it follows that there exists $x_3 \in M$ such that $x_2 \in x_2 + x_3$. Thus, we have $x_1 + x_2 \subseteq x_1 + x_2 + x_3$ and $x_2 + x_1 \subseteq x_2 + x_3 + x_1$. We have $\sum_{i=1}^{3} x_i \; \mathfrak{R}^K \; \sum_{i=1}^{3} x_{\tau(i)}$, where $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 1$ and $\tau \in A_3$. We conclude that $\rho_{\mathfrak{R}^K}^*(x_1) + \rho_{\mathfrak{R}^K}^*(x_2) = \rho_{\mathfrak{R}^K}^*(x_2) + \rho_{\mathfrak{R}^K}^*(x_1)$ and hence $\frac{M}{\rho_{\mathfrak{R}^K}^*}$ is abelian. Suppose that $r \in R$ and $x \in M$. Since $\rho_{\mathfrak{R}^K}^*$ is a strongly regular, $r \circ \rho_{\mathfrak{R}^K}^*(x) = \rho_{\mathfrak{R}^K}^*(z)$ for any $z \in r \cdot x$.

Since M is an R-hypermodule, the properties of M as an R-hypermodule, grantee that the abelian group $\frac{M}{\rho^*_{\Re K}}$ is an R-hypermodule. $\square$

Theorem 4.9. *Let M be an R-hypermodule. The relation $\Re$ on $\mathcal{U}$ is defined as follows: for all $m,n \in \mathbb{N}$ and for all $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathcal{U}^2$,*

$$\sum_{i=1}^{p} m'_i \,\Re\, \sum_{i=1}^{q} z'_i \Leftrightarrow \quad \left(\left|\sum_{i=1}^{p} m'_i - \sum_{i=1}^{q} z'_i\right| < \infty \text{ or } \left|\sum_{i=1}^{q} z'_i - \sum_{i=1}^{p} m'_i\right| < \infty\right)$$
$$\text{and } \sum_{i=1}^{p} m'_i \cap \sum_{i=1}^{q} z'_i \neq \emptyset,$$

where $|A|$ is the cardinal number of the set A. Then, $\rho^*_{\Re} = \varepsilon^*$.

Proof. Since $\Re$ regular, by Proposition 4.5(2), $\rho^*_{\Re}$ is an equivalence relation. Suppose that $(x,y) \in \rho_{\Re}$. Then, there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \Re$ such that $x \in \sum_{i=1}^{p} m'_i$ and $y \in \sum_{i=1}^{q} z'_i$. Without the loss of generality, suppose that

$$\left|\sum_{i=1}^{p} m'_i - \sum_{i=1}^{q} z'_i\right| < \infty,$$

so $\sum_{i=1}^{p} m'_i - \sum_{i=1}^{q} z'_i = \{b_1, b_2, \ldots, b_t\}$. Set $z'_q = \sum_{j=1}^{q_i}\left(\prod_{k=1}^{k_{ij}} r_{ijk}\right) z_q$. Since M is a hypermodule, there exists $(c_1,d_1) \in M^2$ such that $z_q \in c_1 + b_1, b_1 \in a + d_1$, where $a \in \sum_{i=1}^{p} m'_i \cap \sum_{i=1}^{q} z'_i$. Thus, we have

$$\begin{aligned}
\sum_{i=1}^{q} z'_i \quad &\subseteq \sum_{i=1}^{q-1} z'_i + \sum_{j=1}^{q_i}\prod_{k=1}^{k_{ij}} r_{ijk}(c_1 + b_1)\\
&\subseteq \sum_{i=1}^{q-1} z'_i + \sum_{j=1}^{q_i}\prod_{k=1}^{k_{ij}} r_{ijk}(c_1 + a + d_1)\\
&\subseteq \sum_{i=1}^{q-1} z'_i + \sum_{j=1}^{q_i}\prod_{k=1}^{k_{ij}} r_{ijk}(c_1 + \sum_{i=1}^{p} m'_i + d_1) = P.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
b_1 \in a + d_1 \quad &\subseteq \sum_{i=1}^{q-1} z'_i + z'_q + d_1\\
&\subseteq \sum_{i=1}^{q-1} z'_i + \sum_{j=1}^{q_i}\prod_{k=1}^{k_{ij}} r_{ijk}(c_1 + b_1) + d_1\\
&\subseteq \sum_{i=1}^{q-1} z'_i + \sum_{j=1}^{q_i}\prod_{k=1}^{k_{ij}} r_{ijk}(c_1 + \sum_{i=1}^{p} m'_i + d_1) = P.
\end{aligned}$$

Denote $\sum_{i=1}^{k_1} v'_{1,i} := \sum_{i=1}^{q-1} z'_i + \sum_{j=1}^{q_i} \prod_{k=1}^{k_{ij}} r_{ijk}(c_1 + \sum_{i=1}^{p} m'_i) + d_1$. Thus, $\{b_1\} \cup \sum_{i=1}^{q} z'_i \subseteq \sum_{i=1}^{k_1} v'_{1,i}$. Using again that M is a hypermodule, there exists $(c_2, d_2) \in M^2$ such that $v'_{1,k_1} = \sum_{j=1}^{q_i} \prod_{k=1}^{k_{ij}} s_{ijk} v_{1,k_1} \subseteq \sum_{j=1}^{q_i} \prod_{k=1}^{k_{ij}} s_{ijk}(c_2 + b_2)$ and $b_2 \in b_1 + d_2$. Suppose that $\sum_{i=1}^{k_2} v'_{2,i} = \sum_{i=1}^{k_1-1} v'_{1,i} + c_2 + \sum_{i=1}^{p} m'_i + d_2$. Similarly, we obtain $\{b_2\} \cup \sum_{i=1}^{k_1} v'_{1,i} \subseteq \sum_{i=1}^{k_2} v'_{2,i}$ and so $\{b_1, b_2\} \subseteq \sum_{i=1}^{q} z'_i \subseteq \sum_{i=1}^{k_2} v'_{2,i}$. After t steps, we obtain $\sum_{i=1}^{k_t} z'_{t,i}$ such that $\{b_1, b_2, \ldots, b_t\} \cup \sum_{i=1}^{q} z'_i \subseteq \sum_{i=1}^{k_2} v'_{t,i}$. Thus, $\sum_{i=1}^{p} m'_i \cup \sum_{i=1}^{q} z'_i \subseteq \sum_{i=1}^{k_t} v'_{t,i}$, which implies that $(x,y) \in \varepsilon^*$ and $\rho_{\mathfrak{R}} \subseteq \varepsilon^*$. Therefore, $\rho^*_{\mathfrak{R}} \subseteq \varepsilon^*$. Now, suppose that $(x,y) \in \varepsilon^*$. Then, there exists $\sum_{i=1}^{p} m'_i$ such that $x, y \in \sum_{i=1}^{p} m'_i$. Since $\sum_{i=1}^{p} m'_i - \sum_{i=1}^{p} m'_i = \emptyset$, $(x,y) \in \rho_{\mathfrak{R}}$, hence $\varepsilon^* \subseteq \rho^*_{\mathfrak{R}}$. Therefore, $\varepsilon^* = \rho^*_{\mathfrak{R}}$. □

Remark 4.10. The relation $\overset{\hookrightarrow}{R}$ on $\mathcal{U}$ defined by

$$\sum_{i=1}^{p} m'_i \overset{\hookrightarrow}{R} \sum_{i=1}^{q} z'_i \Leftrightarrow \sum_{i=1}^{p} m'_i \subseteq \sum_{i=1}^{q} z'_i$$

is not symmetric and the induced strongly regular relation $\rho^*_{\overset{\hookrightarrow}{R}}$ coincides with th induced strongly regular relation $\rho^*_{\mathfrak{R}}$ of Theorem 4.6.

Theorem 4.11. *Let $(M,+)$ be an R-hypermodule and $\mathfrak{R}$ be a strongly relation on $\mathcal{U}$. Then, an R-hypermodule (with ordinary group) structure can be defined on $\frac{M}{\rho^*_{\mathfrak{R}}}$ with respect to the following two operations*

$$\begin{array}{l} \rho^*_{\mathfrak{R}}(x) \oplus \rho^*_{\mathfrak{R}}(y) = \rho^*_{\mathfrak{R}}(z), \text{ where } z \in x+y, \\ r \circ \rho^*_{\mathfrak{R}}(x) = \rho^*_{\mathfrak{R}}(z), \text{ where } z \in r \cdot x, \ \ r \in R. \end{array}$$

Proof. We prove that the operations $\oplus$ and $\circ$ are well defined. Set $\rho^*_{\mathfrak{R}}(x_0) = \rho^*_{\mathfrak{R}}(x_1)$ and $\rho^*_{\mathfrak{R}}(y_0) = \rho^*_{\mathfrak{R}}(y_1)$. It is enough to verify that $\rho^*_{\mathfrak{R}}(x_0) \oplus \rho^*_{\mathfrak{R}}(y_0) = \rho^*_{\mathfrak{R}}(x_1) \oplus \rho^*_{\mathfrak{R}}(y_1)$.

By hypothesis $m,n \in \mathbb{N}$, $(z_0, z_1, \ldots, z_m) \in H^{m+1}$ and $(t_0, t_1, \ldots, t_n) \in H^{m+1}$ exist such that $z_0 = x_0$, $z_m = x_1$, $t_0 = y_0$ and $t_n = y_1$ for all $1 \leq i \leq m$, $z_{i-1} \ \rho_{\mathfrak{R}} \ z_i$ and for all $1 \leq j \leq n$, $t_{j-1} \ \rho_{\mathfrak{R}} \ t_j$. Since $\mathfrak{R}$ is strongly regular, for all $u \in z_{s-1} + t_{s-1}$ and $v \in z_s + t_s$, where $1 \leq s \leq k$ and $k = min\{m,n\}$, we have $u \rho^*_{\mathfrak{R}} v$. Hence,

$$\begin{array}{rl} \rho^*_{\mathfrak{R}}(x_0) \oplus \rho^*_{\mathfrak{R}}(y_0) & = \rho^*_{\mathfrak{R}}(z_1) \oplus \rho^*_{\mathfrak{R}}(t_1) = \ldots = \rho^*_{\mathfrak{R}}(z_k) \oplus \rho^*_{\mathfrak{R}}(t_k) \\ & = \rho^*_{\mathfrak{R}}(a_{k+i}) \oplus \rho^*_{\mathfrak{R}}(b_{k+i}), \end{array}$$

where $k+1 \leq k+i \leq \max\{m,n\}$ and

$$(a_{k+i}, b_{k+i}) = \begin{cases} (x_1, t_{k+i}) \text{ if } k = m \\ \\ (z_{k+i}, y_1) \ \text{ if } \ k = n. \end{cases}$$

Therefore, $\oplus$ is well defined. Now, suppose that $\rho^*_{\mathfrak{R}}(x_1) = \rho^*_{\mathfrak{R}}(x_2)$. Then, there exists $(z_1,\ldots,z_t)$ such that $x_1 = z_1, x_2 = z_t$ and for all $1 \le j \le t$, $t_{j-1}\ \rho_{\mathfrak{R}}\ t_j$. Since $\mathfrak{R}$ is a strongly relation, $\rho^*_{\mathfrak{R}}$ is a strongly relation and so for any $u \in r \circ \rho^*_{\mathfrak{R}}(x_1)$ and $v \in r \circ \rho^*_{\mathfrak{R}}(x_2)$, we have $\rho^*_{\mathfrak{R}}(u) = \rho^*_{\mathfrak{R}}(v)$. Hence, $r \circ \rho^*_{\mathfrak{R}}(x_1) = r \circ \rho^*_{\mathfrak{R}}(z_2) = \ldots = r \circ \rho^*_{\mathfrak{R}}(z_{t-1}) = r \circ \rho^*_{\mathfrak{R}}(z_t)$. Therefore, $\circ$ is well defined. Since M is an R-hypermodule, the properties of M as an R-hypermodule, grantee that the abelian group $\frac{M}{\rho^*_{\mathfrak{R}}}$ is an R-hypermodule.

□

Theorem 4.12. *Let M be an R-hyperring and p be a prime number. If the relation $\mathfrak{R}_{+,p}$ on $\mathcal{U}$ is defined as follows:*

$$\mathfrak{R}_{+,p} = \left\{ \left(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} r_{ij})(s \cdot m_i), \sum_{i=1}^{n} (\prod_{j=1}^{k_i} r_{ij})(t \cdot m_i) \right) \mid s,t \in \{1, p+1\} \right\}.$$

Then, $M/\rho^*_{\mathfrak{R}_{+,p}}$ is an R-hypermodule such that $(M/\rho^*_{\mathfrak{R}_{+,p}}, \oplus)$ is a p-elementary group.

Proof. It is clear that the relation $\mathfrak{R}_{+,p}$ on $\mathcal{U}$ is strongly regular. Now, by Theorem 4.11, the proof is completed. □

By the similar way, we have the following Theorem.

Theorem 4.13. *Let R be a hyperring and p be a prime number. If the relation $\mathfrak{R}^{\sigma}_{+,p}$ on $\mathcal{U}$ is defined as follows:*

$$\mathfrak{R}^{\sigma}_{+,p} = \left\{ \left(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} r_{ij})(s \cdot m_i), \sum_{i=1}^{n} (\prod_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)j})(t \cdot m_{\sigma(i)}) \right) \mid s,t \in \{1, p+1\} \right\}.$$

Then, $M/\rho^*_{\mathfrak{R}^{\sigma}_{+,p}}$ is an R-hypermodule such that $(M/\rho^*_{\mathfrak{R}^{\sigma}_{+,p}}, \oplus)$ is a p-elementary abelian group.

Example 4.14. Let p be a prime. Consider $M := \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n}$ as a $\mathbb{Z}$-module. Then, the relation $\mathfrak{R}_{+,p}$ in Theorem 4.12 is of the form

$$\mathfrak{R}_{+,p} = \{(\sum_{i=1}^{n} tn_i, \sum_{i=1}^{n} tn_i) |\ s,t \in \{1, p+1\}, n_i \in \mathbb{Z}\}.$$

Therefore, $M/\mathfrak{R}_{+,p}$ is a $\mathbb{Z}$-module such that $M/\mathfrak{R}_{+,p} \cong \underbrace{\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}_{n}$.

Example 4.15. Let $p = 2$ and R be a ring. Set $M := \mathbb{S}_3 \times \mathbb{S}_3$, where $\mathbb{S}_3$ is the permutation group of order 3, i.e., $\mathbb{S}_3 = \{(1),(1\,2),(1\,3),(2\,3),(1\,2\,3),(1\,3\,2)\}$. Let K_1 and K_2 be two subgroups of $\mathbb{S}_3$. Define the scalar hyperoperation $r\cdot(\sigma,\tau) = (K_1,K_2)$ for any $r \in R$ and $\sigma,\tau \in \mathbb{S}_3$. Then, $M/\mathfrak{R}^{\sigma}_{+,p}$ is an R-hypermodule such that $M/\mathfrak{R}^{\sigma}_{+,p} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 4.16. Let M be an R-hypermodule and $\rho : M \longrightarrow \frac{M}{\rho^*_{\mathfrak{R}}}$ be the canonical projection. Denote by 0 the zero element of the group $\frac{M}{\rho^*_{\mathfrak{R}}}$. The set $\rho^{-1}(0)$ is called the $\mathfrak{R}$-*heart* of M and it is denoted by $\omega_{\mathfrak{R},M}$.

Notice that if $\mathfrak{R}$ is the diagonal relation of $\mathcal{U}$, then the $\mathfrak{R}$-heart is just the heart of the hypermodule M.

Theorem 4.17. *Let M be a regular R-hypermodule and $\mathfrak{R}$ be a compatible relation with $+$ and $\cdot$ on $\mathcal{U}$. Then, $\omega_{\mathfrak{R},M}$ is the smallest subhypermodule of M, which is also an $\mathfrak{R}$-part.*

Proof. First, we check that $\omega_{\mathfrak{R},M}$ is a subhypermodule of M. If $x,y \in \omega_{\mathfrak{R},M}$ and $z \in x+y$, then $\rho^*_{\mathfrak{R}}(z) = \rho^*_{\mathfrak{R}}(x) \oplus \rho^*_{\mathfrak{R}}(y) = 0$, the identity of the group $\frac{M}{\rho^*_{\mathfrak{R}}}$. Hence, $z \in \omega_{\mathfrak{R},M}$. On the other hand, there exists $u \in M$, such that $x \in u+y$, whence $\rho^*_{\mathfrak{R}}(x) = \rho^*_{\mathfrak{R}}(u) \oplus \rho^*_{\mathfrak{R}}(y)$, so $\rho^*_{\mathfrak{R}}(u) = 0$ and $u \in \omega_{\mathfrak{R},M}$. This means that $\omega_{\mathfrak{R},M} = \omega_{\mathfrak{R},M} + y$ and similarly we obtain that $\omega_{\mathfrak{R},M} = y + \omega_{\mathfrak{R},M}$.

Now, suppose that $x \in \omega_{\mathfrak{R},M}$ and $r \in R$. Then, for any $z \in r\circ x$, we have $\rho^*_{\mathfrak{R}}(z) \subseteq \rho^*_{\mathfrak{R}}(r\circ x) = r\circ\rho^*_{\mathfrak{R}}(x) = r\circ 0 = 0$ by strongly regularity of $\rho^*_{\mathfrak{R}}$. Since M is an R-hypermodule, the properties of M as an R-hypermodule, follows that $\omega_{\mathfrak{R},M}$ is a subhypermodule of M. By Theorems 4.6, 4.7 and Proposition 4.5, for all $x \in \omega_{\mathfrak{R},M}$, $P_{\mathcal{U},\mathfrak{R}}(x) = \rho^*_{\mathcal{U},\mathfrak{R}}(x) = \rho_{\mathfrak{R}}(x)$, which represents the zero element of $\frac{M}{\rho^*_{\mathfrak{R}}}$. On the other hand, $\rho^*_{\mathfrak{R}}(x)$ represents the $\mathfrak{R}$-heart $\omega_{\mathfrak{R},M}$, as a subset of M. So, for all $x \in \omega_{\mathfrak{R},M}$, according to Theorems 4.6, 4.7, $\omega_{\mathfrak{R},M} = P_{\mathcal{U},\mathfrak{R}}(x)$, which is an $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$-part of M. In fact, by Proposition 4.5, $P_{\mathcal{U},\mathfrak{R}}(x)$ is also an $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$-part of M, hence it is an $\mathfrak{R}$-part of M. Moreover, $\omega_{\mathfrak{R},M}$ is the smallest subhypermodule which is an $\mathfrak{R}$-part of M. Indeed, if K is a subhypermodule and an $\mathfrak{R}$-part of M, then for all $k \in K$, there is $e \in K$ such that $k \in e+k$, whence $\rho^*_{\mathfrak{R}}(k) = \rho^*_{\mathfrak{R}}(e) \oplus \rho^*_{\mathfrak{R}}(k)$, so $e \in \omega_{\mathfrak{R},M}$. Since K is an $\mathfrak{R}$-part of M, hence $P_{\mathcal{U},\mathfrak{R}}(e) = \omega_{\mathfrak{R},M} \subseteq K$. □

Theorem 4.18. *For every non-empty subset A of hypermodule M, if A is an $\mathfrak{R}$-part of M, then $\rho^{-1}(\rho(A)) = A$.*

Proof. It is obvious that $A \subseteq \rho^{-1}(\rho(A))$. Moreover, if $x \in \rho^{-1}(\rho(A))$, then there exists an element $a \in A$ such that $\rho(x) = \rho(a)$. Since A is an $\mathfrak{R}$-part, $x \in \rho^*_{\mathfrak{R}}(x) = \rho^*_{\mathfrak{R}}(a) \subseteq A$. Therefore, $\rho^{-1}(\rho(A)) \subseteq A$. □

Theorem 4.19. *Let A be a non-empty subset of a hypermodule M. The following condition are equivalent:*

(1) A is a $\mathfrak{R}_{\mathcal{U}}$- part of M.

(2) $x \in A, x\rho_{\mathfrak{R}}y \Rightarrow y \in A$.

(3) $x \in A, x\rho^*_{\mathfrak{R}}y \Rightarrow y \in A$.

Proof. (1)⇒(2): If $x, y \in M$ is a pair such that $x \in A$ and $x\rho_{\mathfrak{R}}y$, then there exists $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R}_{\mathcal{U}} \cup \mathfrak{R}_{\mathcal{U}}^{-1}$ such that $x \in \sum_{i=1}^{p} m'_i$ and $y \in \sum_{i=1}^{q} z'_i$. Since A is a $\mathfrak{R}_{\mathcal{U}}$-part of R, we obtain $\sum_{i=1}^{p} m'_i \cap A \neq \emptyset$ and $\sum_{i=1}^{p} m'_i \; \mathfrak{R}_{\mathcal{U}} \; \sum_{i=1}^{q} z'_i$ which implies that $\sum_{i=1}^{q} z'_i \subseteq A$. Then, $y \in A$.

(2)⇒(3) Suppose that $x, y \in R$, such that $x \in A$ and $x \in \rho^*_{\mathfrak{R}}(y)$. Obviously, there exist $s \in \mathbb{N}$ and $(w_0 = x, w_1, \ldots, w_{s-1}, w_s = y) \in R^{s+1}$ such that

$$x = w_0\rho_{\mathfrak{R}}w_1 \ldots \rho_{\mathfrak{R}}w_{s-1}\rho_{\mathfrak{R}}w_s = y.$$

Since $x \in A$, applying (2) s times, we obtain $y \in A$.

(3)⇒(1) Suppose that $\sum_{i=1}^{p} m'_i \cap A \neq \emptyset$ and $x \in \sum_{i=1}^{p} m'_i \cap A$.

If $\left(\sum_{i=1}^{p} m'_i, \sum_{i=1}^{q} z'_i\right) \in \mathfrak{R}_{\mathcal{U}} \cup \mathfrak{R}_{\mathcal{U}}^{-1}$, where $\sum_{i=1}^{q} z'_i \in \mathcal{U}$, then for every $y \in \sum_{i=1}^{q} z'_i$, we obtain $y\rho_{\mathfrak{R}}x$ and by (3) we have $y \in A$. □

Corollary 4.20. *Let R be a hyperring and A be a nonempty subset of M. If $\mathfrak{R}$ is a relation on $\mathcal{U}$ then A is an $\mathfrak{R}_{\mathcal{U}}$- part of R if and only if $A = \bigcup_{x \in A} \rho^*_{\mathfrak{R}}(x)$.*

Theorem 4.21. *Let R be a commutative hyperring, M a regular R-hypermodule and for every $m \in M$, $R.m = M$. Let $\mathfrak{R}\mathcal{C}$ be the set of all reflexive and compatible relations with $+$ and $\cdot$ on $\mathcal{U}$. Then, the heart of the hypermodule M is $\omega_M = \bigcap_{\mathfrak{R} \in \mathcal{RC}} \omega_{\mathfrak{R},M}$.*

Proof. Notice that if $(x, y) \in \varepsilon$, then $x, y \in \sum_{i=1}^{p} m'_i$, where $i \in \{1, 2 \ldots, n\}$. So, $\varepsilon \subseteq \bigcap_{\mathfrak{R} \in \mathcal{RC}} \rho^*_{\mathfrak{R}}$. Conversely, it is enough to remark that $\bigcap_{\mathfrak{R} \in \mathcal{RC}} \rho^*_{\mathfrak{R}} \subseteq \varepsilon$. By Theorem 1.2, $\varepsilon = \varepsilon^*$. So, $\varepsilon = \varepsilon^* = \rho^*_{Id}$, where Id is the diagonal relation on $\mathcal{U}$. Hence, $\varepsilon = \bigcap_{\mathfrak{R} \in \mathcal{RC}} \rho^*_{\mathfrak{R}}$. From here it follows that, $\omega_M = \bigcap_{\mathfrak{R} \in \mathcal{RC}} \omega_{\mathfrak{R},M}$, since for all $x \in M$, $\varepsilon(x) = 0$ if and only if $x \in \omega_M$, while for all $\mathfrak{R} \in \mathcal{RC}$, $\rho^*_{\mathfrak{R}}(x) = 0$ if and only if $x \in \omega_{\mathfrak{R},M}$. □

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