LE MATEMATICHE
Vol. LXXII (2017) – Fasc. I, pp. 97–102
doi: 10.4418/2017.72.1.7

FUNCTIONAL CONTINUITY OF UNITAL B_0-ALGEBRAS WITH ORTHOGONAL BASES

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Let A be a unital B_0-algebra with an orthogonal basis, then every multiplicative linear functional on A is continuous. This gives an answer to a problem posed by Z. Sawon and Z. Wronski.

1. Preliminaries

A topological algebra is a complex associative algebra which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra is a topological algebra whose topology is determined by a family of seminorms. A complete metrizable locally convex algebra is called a B_0-algebra. The topology of a B_0-algebra A may be given by a countable family $(\|.\|_i)_{i\geq 1}$ of seminorms such that $\|x\|_i \leq \|x\|_{i+1}$ and $\|xy\|_i \leq \|x\|_{i+1}\|y\|_{i+1}$ for all $i \geq 1$ and $x, y \in A$. A multiplicative linear functional on a complex algebra A is an algebra homomorphism from A to the complex field. Let A be a topological algebra. $M^*(A)$ denotes the set of all nonzero multiplicative linear functionals on A. $M(A)$ denotes the set of all nonzero continuous multiplicative linear functionals on A. A seminorm p on A is lower semicontinuous if the set $\{x \in A : p(x) \leq 1\}$ is closed in A.
Let A be a topological algebra. A sequence $(e_n)_{n\geq 1}$ in A is a basis if for each $x \in A$ there is a unique sequence $(\alpha_n)_{n\geq 1}$ of complex numbers such that $x =$

Entrato in redazione: 29 gennaio 2016
AMS 2010 Subject Classification: 46H40
Keywords: B_0-algebra, orthogonal basis, multiplicative linear functional

$\Sigma_{n=1}^{\infty}\alpha_n e_n$. Each linear functional $e_n^*: A \to \mathbb{C}$, $e_n^*(x) = \alpha_n$, is called a coefficient functional. If each e_n^* is continuous, the basis $(e_n)_{n\geq 1}$ is called a Schauder basis. A basis $(e_n)_{n\geq 1}$ is orthogonal if $e_i e_j = \delta_{ij} e_i$ where δ_{ij} is the Kronecker symbol. If $(e_n)_{n\geq 1}$ is an orthogonal basis, then each e_n^* is a multiplicative linear functional on A. Let A be a topological algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If A has a unity e, then $e = \Sigma_{n=1}^{\infty} e_n$. Let $(x_k)_k$ be a net in A converging to 0, since the multiplication is separately continuous, $e_n x_k = e_n^*(x_k)e_n \to_k 0$ and so $e_n^*(x_k) \to_k 0$. Then each orthogonal basis in a topological algebra is a Schauder basis. Let f be a multiplicative linear functional on A. If $f(e_{n_0}) \neq 0$ for some $n_0 \geq 1$, then $f(x)f(e_{n_0}) = f(xe_{n_0}) = f(e_{n_0}^*(x)e_{n_0}) = e_{n_0}^*(x)f(e_{n_0})$ for all $x \in A$ and therefore $f = e_{n_0}^* \in M(A)$. This shows that $M(A) = \{e_n^* : n \geq 1\}$.
Here we consider unital B_0-algebras with orthogonal bases. These algebras were investigated in [3] where we can find examples of such algebras.

2. Results

Proposition 2.1. *Let $(A, (\|.\|_i)_{i\geq 1})$ be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then there exists $x \in A$ such that the sequence $(e_n^*(x))_{n\geq 1}$ is not bounded.*

Proof. Suppose that $\sup_{n\geq 1}|e_n^*(x)| < \infty$ for all $x \in A$. For each $x \in A$, let $\|x\| = \sup_{n\geq 1}|e_n^*(x)|$, $\|.\|$ is a lower semicontinuous norm on A since $(e_n)_{n\geq 1}$ is a Schauder basis. Let τ_A be the topology on A determined by the family $(\|.\|_i)_{i\geq 1}$ of seminorms. We define a new topology τ on A described by the norm $\|.\|$ and the family $(\|.\|_i)_{i\geq 1}$ of seminorms. The topology τ is stronger than the topology τ_A. By Garling's completeness theorem [1, Theorem 1], (A, τ) is complete. The topologies τ_A and τ are homeomorphic by the open mapping theorem. Then there exist $i_0 \geq 1$ and $M > 0$ such that $\|x\| \leq M\|x\|_{i_0}$ for all $x \in A$, hence $1 = |e_n^*(e_n)| \leq \|e_n\| \leq M\|e_n\|_{i_0}$ for all $n \geq 1$. This contradicts the fact that $e_n \to_n 0$. □

Proposition 2.2. *Let A be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If $x = \Sigma_{n=1}^{\infty} t_n e_n \in A$ such that $t_n \in \mathbb{R}$, $t_n \leq t_{n+1}$ for $n \geq 1$ and $t_n \to \infty$, then $f(x) \in \mathbb{R}$ for all $f \in M^*(A)$.*

Proof. If $f \in M(A)$, $f = e_n^*$ for some $n \geq 1$, then $f(x) = t_n \in \mathbb{R}$. If $f \in M^*(A) \setminus M(A)$, then $f(e_n) = 0$ for all $n \geq 1$. Suppose that $f(x) \notin \mathbb{R}$, $f(x) = \alpha + i\beta$ with $\beta \neq 0$. Since $t_n \to \infty$, there exists $n_0 \geq 1$ such that $t_n \geq \alpha + |\beta|$ for all $n \geq n_0$. We define the sequence $(s_n)_{n\geq 1}$ by $s_n = t_{n_0}$ for $1 \leq n \leq n_0$ and $s_n = t_n$ for $n \geq n_0 + 1$. It is clear that $y = \Sigma_{n=1}^{\infty} s_n e_n \in A$ such that $s_n \geq \alpha + |\beta|$, $s_n \leq s_{n+1}$ for $n \geq 1$ and $s_n \to \infty$. Since $f(e_n) = 0$ for all $n \geq 1$, $f(y) = f(x) = \alpha + i\beta$. We have $f(|\beta|^{-1}y) = |\beta|^{-1}\alpha + i|\beta|^{-1}\beta$, then $f(|\beta|^{-1}y - |\beta|^{-1}\alpha e) = i|\beta|^{-1}\beta$. Set

$z = |\beta|^{-1}y - |\beta|^{-1}\alpha e = \Sigma_{n=1}^{\infty}\frac{s_n-\alpha}{|\beta|}e_n$. The real sequence $(\frac{s_n-\alpha}{|\beta|})_{n\geq 1}$ is positive increasing and $\frac{s_n-\alpha}{|\beta|} \to \infty$, then $z^{-1} = \Sigma_{n=1}^{\infty}\frac{|\beta|}{s_n-\alpha}e_n \in A$ by [3, Theorem 0.1] and $f(z^{-1}) = -i|\beta|\beta^{-1}$, so $f(z+z^{-1}) = 0$. Set $v = z+z^{-1} = \Sigma_{n=1}^{\infty}(\frac{s_n-\alpha}{|\beta|}+\frac{|\beta|}{s_n-\alpha})e_n$ and $v_n = \frac{s_n-\alpha}{|\beta|}+\frac{|\beta|}{s_n-\alpha}$ for all $n \geq 1$. Since the map $g : [1,\infty) \to \mathbb{R}$, $g(x) = x+\frac{1}{x}$, is increasing and the sequence $(\frac{s_n-\alpha}{|\beta|})_{n\geq 1} \subset [1,\infty)$ is increasing, it follows that $(v_n)_{n\geq 1}$ is a positive increasing sequence and $v_n \to \infty$. By [3, Theorem 0.1], $v^{-1} = \Sigma_{n=1}^{\infty}\frac{1}{v_n}e_n \in A$ and therefore $f(e) = f(v)f(v^{-1}) = 0$. This contradicts the fact that f is nonzero. □

The following two results are due to Sawon and Wronski [3], the proofs are given for completeness.

Theorem 2.3. *([3, Theorem 2.1]). Let A be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If $x = \Sigma_{n=1}^{\infty}t_ne_n \in A$ such that $t_n \in \mathbb{R}$, $t_n \leq t_{n+1}$ for $n \geq 1$ and $t_n \to \infty$, then every multiplicative linear functional on A is continuous.*

Proof. Suppose that $M^*(A) \setminus M(A)$ is nonempty. Let $f \in M^*(A) \setminus M(A)$, then $f(e_n) = 0$ for all $n \geq 1$. Put $f(x) = \alpha$, then $\alpha \in \mathbb{R}$ by Proposition 2.2. Since $t_n \to \infty$, there exists $n_0 \geq 1$ such that $t_n > \alpha$ for $n \geq n_0$. Consider $y = \Sigma_{n=1}^{\infty}\lambda_ne_n \in A$ such that $\lambda_n = t_{n_0}$ for $1 \leq n \leq n_0$ and $\lambda_n = t_n$ for $n \geq n_0+1$. Since $f(e_n) = 0$ for all $n \geq 1$, it follows that $f(x) = f(y) = \alpha$. We have $y-\alpha e = \Sigma_{n=1}^{\infty}v_ne_n \in A$ where $v_n = \lambda_n - \alpha$ for all $n \geq 1$. It is clear that $v_n > 0$, $v_n \leq v_{n+1}$ for $n \geq 1$ and $v_n \to \infty$. By [3, Theorem 0.1], $(y-\alpha e)^{-1} = \Sigma_{n=1}^{\infty}\frac{1}{v_n}e_n \in A$, so $y-\alpha e$ is invertible and $f(y-\alpha e) = 0$, a contradiction. □

Proposition 2.4. *([3, p.109]). Let $(A,(\|.\|_i)_{i\geq 1})$ be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then the set N of all positive integers can be split into two disjoint subsets N_1 and N_2 such that by putting $A_1 = \overline{span}(e_n)_{n\in N_1}$ and $A_2 = \overline{span}(e_n)_{n\in N_2}$, we have*
(1) $A = A_1 \oplus A_2$;
(2) if f is a multiplicative linear functional on A such that $f \notin M(A)$, then $f_{/A_1} = 0$.

Proof. By Proposition 2.1, there is $x = \Sigma_{n=1}^{\infty}t_ne_n \in A$ such that the sequence $(t_n)_{n\geq 1}$ is not bounded. Then there exists a subsequence $(t_{k_n})_{n\geq 1}$ of $(t_n)_{n\geq 1}$ such that $|t_{k_n}| \geq n^2$ for all $n \geq 1$. For each $i \geq 1$, there is $M_i > 0$ such that $\|t_ne_n\|_i \leq M_i$ for all $n \geq 1$. Let $i \geq 1$ and $n \geq 1$, $n^2\|e_{k_n}\|_i \leq |t_{k_n}|\|e_{k_n}\|_i = \|t_{k_n}e_{k_n}\|_i \leq M_i$, then $\|e_{k_n}\|_i \leq n^{-2}M_i$. This implies that $\Sigma_{n=1}^{\infty}e_{k_n}$ is absolutely convergent. Let $A_1 = \overline{span}\{e_{k_n} : n \geq 1\}$, A_1 is a unital B_0-algebra with an orthogonal basis $(e_{k_n})_{n\geq 1}$ and $\Sigma_{n=1}^{\infty}n^{\frac{1}{2}}e_{k_n} \in A_1$ since $n^{\frac{1}{2}}\|e_{k_n}\|_i \leq n^{-\frac{3}{2}}M_i$ for all $i \geq 1$ and $n \geq 1$. Set $N_1 = \{k_n : n \geq 1\}$, $N_2 = N \setminus N_1$ and $A_2 = \overline{span}\{e_n : n \in N_2\}$. A_2 is a B_0-algebra with an orthogonal basis $(e_n)_{n\in N_2}$ and the unity $u_2 = e - u_1$ where $u_1 = \Sigma_{n=1}^{\infty} e_{k_n}$ is the unity of A_1. Let $x \in A$, $x = xe = x(u_1 + u_2) = xu_1 + xu_2 \in A_1 + A_2$, then $A = A_1 \oplus A_2$. If f is a multiplicative linear functional on A such that $f \notin M(A)$, $f_{/A_1}$ is a multiplicative linear functional on A_1 such that $f_{/A_1}(e_{k_n}) = 0$ for all $n \geq 1$. Since $\Sigma_{n=1}^{\infty} n^{\frac{1}{2}} e_{k_n} \in A_1$, $f_{/A_1}$ is continuous on A_1 by Theorem 2.3 and therefore $f_{/A_1} = 0$. □

Sawon and Wronski [3, p.109] posed the following problem:

Problem 2.5. Let A be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Does there exist a maximal subalgebra $A_1^{'} = \overline{span}\{e_n : n \in N_1^{'}\}(N_1^{'} \subset N)$ of A for which (1) and (2) hold?

Proposition 2.6. *Let A be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then the following assertions are equivalent:*
$(i)\, A = \overline{span}\{e_n : n \in N\}$ is a maximal subalgebra of itself for which (1) and (2) hold;
(ii) every multiplicative linear functional on A is continuous.

Proof. $(i) \Rightarrow (ii)$: Let f be a multiplicative linear functional on A such that $f \notin M(A)$, f is zero on A by (i). Then every multiplicative linear functional on A is continuous.
$(ii) \Rightarrow (i)$: It is clear that A satisfies (1). Let f be a multiplicative linear functional on A such that $f \notin M(A)$, then f is zero on A since f is continuous, hence A satisfies (2). □

Proposition 2.7. *Let $(t_n)_{n\geq n_0}$ be a complex sequence, the following assertions are equivalent:*
$(i)\, \Sigma_{n=n_0}^{\infty} |t_n - t_{n+1}| < \infty$;
(ii) *there exists $M > 0$ such that $|t_q| + \Sigma_{n=p}^{q-1} |t_n - t_{n+1}| \leq M$ for all $q > p \geq n_0$.*

Proof. $(i) \Rightarrow (ii)$: Let $\varepsilon > 0$, there exists $n_1 \geq n_0$ such that $\Sigma_{k=n}^{\infty} |t_k - t_{k+1}| \leq \varepsilon$ for every $n \geq n_1$. let $m > n \geq n_1$, $|t_n - t_m| \leq |t_n - t_{n+1}| + ... + |t_{m-1} - t_m| \leq \varepsilon$. Then the sequence $(t_n)_{n\geq n_0}$ converges, so there is $M_0 > 0$ such that $|t_n| \leq M_0$ for all $n \geq n_0$. Let $q > p \geq n_0$, $|t_q| + \Sigma_{n=p}^{q-1} |t_n - t_{n+1}| \leq M_0 + \Sigma_{n=n_0}^{\infty} |t_n - t_{n+1}|$.
$(ii) \Rightarrow (i)$: Let $p \geq n_0$ and $q = p + 1$, $\Sigma_{n=n_0}^{p} |t_n - t_{n+1}| \leq |t_q| + \Sigma_{n=n_0}^{q-1} |t_n - t_{n+1}| \leq M$. Then the sequence $(\Sigma_{n=n_0}^{p} |t_n - t_{n+1}|)_{p\geq n_0}$ is positive increasing and bounded, so it is convergent i.e. $\Sigma_{n=n_0}^{\infty} |t_n - t_{n+1}| < \infty$. □

Proposition 2.8. *Let $(A, (\|.\|_i)_{i\geq 1})$ be a unital B_0-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If $(t_n)_{n\geq n_0}$ is a complex sequence such that $\Sigma_{n=n_0}^{\infty} |t_n - t_{n+1}| < \infty$, then $\Sigma_{n=n_0}^{\infty} t_n e_n \in A$.*

Proof. Let $q > p \geq n_0$, by using the equality $t_n = t_q + \Sigma_{k=n}^{q-1}(t_k - t_{k+1})$ for every $p \leq n < q$, we obtain that $\Sigma_{n=p}^{q-1} t_n e_n = t_q(e_p + ... + e_{q-1}) + \Sigma_{k=p}^{q-1}(t_k - t_{k+1})(e_p + ... + e_k)$. Let $i \geq 1$, $\|\Sigma_{n=p}^{q-1} t_n e_n\|_i \leq |t_q| \|e_p + ... + e_{q-1}\|_i + \Sigma_{k=p}^{q-1}|t_k - t_{k+1}| \|e_p + ... + e_k\|_i \leq (|t_q| + \Sigma_{k=p}^{q-1}|t_k - t_{k+1}|) \sup_{p \leq k \leq q} \|e_p + ... + e_k\|_i \leq M \sup_{p \leq k \leq q} \|e_p + ... + e_k\|_i$ by Proposition 2.7. Let $\varepsilon > 0$, since $e = \Sigma_{n=1}^{\infty} e_n \in A$, there is $n_1 \geq n_0$ such that $\|e_p + ... + e_k\|_i \leq \varepsilon M^{-1}$ for $n_1 \leq p \leq k$, hence $\sup_{p \leq k \leq q} \|e_p + ... + e_k\|_i \leq \varepsilon M^{-1}$ for $n_1 \leq p < q$. Then $\|\Sigma_{n=p}^{q-1} t_n e_n\|_i \leq \varepsilon$ for $n_1 \leq p < q$. This shows that $\Sigma_{n=n_0}^{\infty} t_n e_n$ is convergent in A. □

Theorem 2.9. *Let $(A, (\|.\|_i)_{i \geq 1})$ be a unital B_0-algebra with an orthogonal basis $(e_n)_{n \geq 1}$. Then every multiplicative linear functional on A is continuous.*

Proof. By Proposition 2.8, $x = \Sigma_{n=1}^{\infty} \frac{1}{n} e_n \in A$. Let $f \in M^*(A)$ and $\alpha = f(x)$, then $f(\alpha e - x) = 0$. We have $\alpha e - x = \Sigma_{n=1}^{\infty}(\alpha - \frac{1}{n}) e_n = \Sigma_{n=1}^{\infty} \frac{\alpha n - 1}{n} e_n$. Let $\mathbb{N}$ be the set of all positive integers. Put $I_\alpha = \{n \in \mathbb{N} : n \neq \alpha^{-1}\}$, $I_\alpha = \mathbb{N}$ if $\alpha^{-1} \notin \mathbb{N}$ and $I_\alpha = \mathbb{N} \setminus \{\alpha^{-1}\}$ if $\alpha^{-1} \in \mathbb{N}$. Let $m_\alpha = \inf\{n \in \mathbb{N} : |\alpha| n - 1 > 0\}$ and consider the complex sequence $(\frac{n}{\alpha n - 1})_{n \geq m_\alpha}$. Let $n \geq m_\alpha$, $\frac{n}{\alpha n - 1} - \frac{n+1}{\alpha(n+1) - 1} = \frac{1}{(\alpha n - 1)(\alpha(n+1) - 1)}$. We have $|\alpha n - 1| \geq |\alpha| n - 1$ and $|\alpha(n+1) - 1| \geq |\alpha|(n+1) - 1 = |\alpha| n + |\alpha| - 1 \geq |\alpha| n - 1$. Let $n \geq m_\alpha$, $|\alpha| n - 1 > 0$, hence $\frac{1}{|\alpha n - 1|} \leq \frac{1}{|\alpha| n - 1}$ and $\frac{1}{|\alpha(n+1) - 1|} \leq \frac{1}{|\alpha| n - 1}$. Consequently $|\frac{n}{\alpha n - 1} - \frac{n+1}{\alpha(n+1) - 1}| = \frac{1}{|\alpha n - 1||\alpha(n+1) - 1|} \leq \frac{1}{(|\alpha| n - 1)^2}$.

Then $\Sigma_{n=m_\alpha}^{\infty} \frac{n}{\alpha n - 1} e_n \in A$ by Proposition 2.8 and therefore $\Sigma_{n \in I_\alpha} \frac{n}{\alpha n - 1} e_n \in A$. If $\alpha^{-1} \notin \mathbb{N}$ i.e. $I_\alpha = \mathbb{N}$, $(\alpha e - x) \Sigma_{n=1}^{\infty} \frac{n}{\alpha n - 1} e_n = e$, then $\alpha e - x$ is invertible, a contradiction. If $\alpha^{-1} \in \mathbb{N}$, put $n_\alpha = \alpha^{-1}$, then $(\alpha e - x) \Sigma_{n \in I_\alpha} \frac{n}{\alpha n - 1} e_n = \Sigma_{n \in I_\alpha} e_n$, hence $\Sigma_{n \in I_\alpha} e_n \in Ker(f)$ since $\alpha e - x \in Ker(f)$. If $e_{n_\alpha} \in Ker(f)$, then $e = e_{n_\alpha} + \Sigma_{n \in I_\alpha} e_n \in Ker(f)$, a contradiction. Finally $f(e_{n_\alpha}) \neq 0$ and therefore $f = e_{n_\alpha}^*$. □

Remark 2.10. Proposition 2.6 and Theorem 2.9 give an answer to Sawon and Wronski's problem.

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