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ON SPECTRUM OF I-GRAPHS AND ITS ORDERING WITH RESPECT TO SPECTRAL MOMENTS

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Suppose G is a graph, $A(G)$ its adjacency matrix, and $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ are eigenvalues of $A(G)$. The numbers $S_k(G) = \sum_{i=1}^{n} \mu_i^k(G)$, $0 \leq k \leq n-1$ are said to be the $k-$th spectral moment of G and the sequence $S(G) = (S_0(G), S_1(G), \cdots, S_{n-1}(G))$ is called the spectral moments sequence of G. For two graphs G_1 and G_2, we define $G_1 \prec_S G_2$, if there exists an integer k, $1 \leq k \leq n-1$, such that for each i, $0 \leq i \leq k-1$, $S_i(G_1) = S_i(G_2)$ and $S_k(G_1) < S_k(G_2)$.

The $I-$graph $I(n, j, k)$ is a graph of order $2n$ with the vertex and edge sets

$$\begin{array}{rcl} V(I(n,j,k)) & = & \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}, \\ E(I(n,j,k)) & = & \{u_iu_{u+j}, u_iv_i, v_iv_{i+k} \ ; \ 0 \leq i \leq n-1\}, \end{array}$$

respectively. The aim of this paper is to compute the spectrum of an arbitrary $I-$graph and the extremal $I-$graphs with respect to the $S-$order.

1. Introduction

In this section we recall some definitions that will be used in the paper. All graphs considered here are simple and undirected, without loops or multiple

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edges. The symbols P_n, C_n, S_n and U_n stand for the path of length n, the cycle of size n, the star graph on n vertices and a graph obtained from C_{n-1} by attaching a leaf to one of its vertices, respectively. Our undefined terminology and notation can be found in [3].

Suppose G is a graph with adjacency matrix $A(G)$ and $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ are eigenvalues of $A(G)$. The numbers $S_k(G) = \sum_{i=1}^{n} \mu_i^k(G)$, $0 \leq k \leq n-1$, are said to be the $k-$th spectral moment of G. The sequence $S(G) = (S_0(G), S_1(G), \cdots, S_{n-1}(G))$ is called the spectral moments sequence of G. It is well-known that $S_0 = n$, $S_1 = 0$, $S_2 = 2m$ and $S_3 = 6t$, where n, m and t denote the number of vertices, edges and triangles, respectively [3]. Suppose G_1 and G_2 are graphs. If there exists an integer k, $1 \leq k \leq n-1$, such that for each i, $0 \leq i \leq k-1$, $S_i(G_1) = S_i(G_2)$ and $S_k(G_1) < S_k(G_2)$ then we write $G_1 \prec_S G_2$. If for each $i, 0 \leq i \leq n-1$, $S_i(G_1) = S_i(G_2)$ then we shall write $G_1 =_S G_2$. We shall also write $G_1 \preceq_S G_2$ if $G_1 \prec_S G_2$ or $G_1 =_S G_2$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two sets of graphs. We write $\mathcal{G}_1 \prec_S \mathcal{G}_2$ if $G_1 \prec_S G_2$ for each $G_1 \in \mathcal{G}_1$ and each $G_2 \in \mathcal{G}_2$. Suppose F and G are graphs. An $F-$subgraph of G is a subgraph isomorphic to the graph F. The number of all F-subgraphs of G is denoted by $\phi_G(F)$ or $\phi(F)$ for short.

The generalized Petersen graph $GP(n,k)$ is a graph with vertices and edges given by

$$\begin{aligned} V(GP(n,k)) &= \{a_i, b_i \mid 1 \leq i \leq n\}, \\ E(GP(n,k)) &= \{a_ib_i, a_ia_{i+1}, b_ib_{i+k} \mid 1 \leq i \leq n\} \end{aligned}$$

respectively, where $i+k$ are integers modulo n, $n > 6$. Since $GP(n,k) \cong GP(n,n-k)$, we can assume that $k \leq \lfloor \frac{n-1}{2} \rfloor$. The $I-$graph $I(n,j,k)$ has

$$\begin{aligned} V(I(n,j,k)) &= \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}, \\ E(I(n,j,k)) &= \{u_iu_{i+j}, u_iv_i, v_iv_{i+k} \; ; \; 0 \leq i \leq n-1\}. \end{aligned}$$

as vertices and edges, respectively. Since $I(n,j,k) = I(n,k,j)$, we have $j \leq k$ or $k \leq j$. Since $I(n,1,k) = GP(n,k)$, the class of $I-$graphs contains the class of generalized Petersen graphs.

Following Boben et al. [1], we draw the vertices of an $I-$graph on two concentric circles, vertices u_i on one circle and vertices v_i on another circle. We call the vertices on these two circles the vertices on the outer rim and the vertices on the inner rim. Edges between the two rims are called spokes. A proper $I-$graph is a connected $I-$graph which is not isomorphic to a generalized Petersen graph. The smallest proper $I-$graphs are $I(12,2,3)$ and $I(12,3,4)$ that are depicted in [1, Figure 1].

In [4], Cvetković and Rowlinson obtained the first and the last graphs in an $S-$order, in the classes of trees and unicyclic graphs with a given girth, respectively. In [2], the authors continued the pioneering work of Cvetković and Rowlinson to compute the last $d+\lfloor \frac{d}{2} \rfloor - 2$ in the $S-$order, among all unicyclic graphs of order n and diameter d. In [9], the present authors computed the last third graphs in the $S-$order, among all generalized Petersen graphs. The aim of this paper is to generalize our earlier results to $I-$graphs. It is merit to mention here that a result of Gera and Stănică regarding the eigenvalues of generalized Petersen graphs [5] and another result of Hu et al. [6, Lemma 2.3] on computing the $k-th$, $4 \leq k \leq 8$, spectral moment of an arbitrary graph G are critical in our work. We encourage the interested readers to also consult [7, 8, 10] for more information on this topic.

2. Main Results

In this section, we find our description for the spectrum of I-graphs $I(n, j, k)$. Let $A(I(n, j, k))$ be the adjacency matrix of $I(n, j, k)$. An $n \times n$ matrix is called circulant, if it has the following form:

$$circ(s_1, s_2, \ldots, s_n) = \begin{pmatrix} s_1 & s_2 & \ldots & s_n \\ s_n & s_1 & \ldots & s_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ s_2 & s_3 & \ldots & s_1 \end{pmatrix}$$

For the sake of completeness, we mention here two results of [1] which are used later.

Theorem 2.1. *([1, Proposition 1]) The I-graph $I(n, j, k)$ is connected if and only if $gcd(n, j, k) = 1$. If $gcd(n, j, k) = d > 1$, then this graph consist of d copies of $I(\frac{n}{d}, \frac{j}{d}, \frac{k}{d})$.*

Theorem 2.2. *([1, Theorem 2]) A connected graph $I(n, j, k)$ is bipartite, if and only if n is even and j and k are odd.*

Lemma 2.3. *The adjacency matrix of the $I(n, j, k)$ has the block form*

$$A(I(n, j, k)) = \begin{bmatrix} C_j^n & I_n \\ I_n & C_k^n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix, C_k^n and C_j^n are circulant matrices, with

$$\begin{aligned} C_k^n &= circ(\underbrace{0, 0, \ldots, 0}_{k\ times}, 1, 0, 0, \ldots, 0, 1, \underbrace{0, 0, \ldots, 0}_{k-1\ times}), \\ C_j^n &= circ(\underbrace{0, 0, \ldots, 0}_{j\ times}, 1, 0, 0, \ldots, 0, 1, \underbrace{0, 0, \ldots, 0}_{j-1\ times}) \end{aligned}$$

being the adjacency matrix for inner rim and outer rim, respectively.

Proof. Let $gcd(n, j) = d$ and $gcd(n,k) = d'$. If $d,d' = 1$, then there is a cycle with n vertices in outer and inner rims. If $d,d' > 1$, then the outer rim whose adjacency matrix is C_j^n has d connected components each of which is isomorphic to a cycle graph with $\frac{n}{d}$ vertices and the inner rim whose adjacency matrix is C_k^n has d' connected components each of which is isomorphic to a cycle graph with $\frac{n}{d'}$ vertices. It is clear that the adjacency matrix depends on the labeling of the graph. We label the graph $I(n, j,k)$ in the following manner: the vertices of the outer rim are consecutively labeled by i, $i+k$, $i+2k$, $\ldots$ and the vertices of the inner rim are labeled by i, $i+j$, $i+2j$, $\ldots$. All of labels are computed modulo n. Notice that the vertex u_0 is adjacent to the vertices u_j, u_{n-j} and v_0 in inner and outer rims, respectively. On the other hand, the vertex v_0 is adjacent to the vertices v_k, v_{n-k} and u_0 in outer and inner rims, respectively. Therefore,

$$\begin{aligned} C_k^n &= circ(\underbrace{0,0,\ldots,0}_{k\ times},1,0,0,\ldots,0,1,\underbrace{0,0,\ldots,0}_{k-1\ times}),\\ C_j^n &= circ(\underbrace{0,0,\ldots,0}_{j\ times},1,0,0,\ldots,0,1,\underbrace{0,0,\ldots.,0}_{j-1\ times}),\end{aligned}$$

which complete the proof.

□

Lemma 2.4. *The eigenvalues of circulant matrix C_j^n associated to the inner rim and the circulant matrix C_k^n associated to the outer rim can be calculated as $\lambda_r = 2\cos(\frac{2\pi jr}{n})$ and $\mu_r = 2\cos(\frac{2\pi kr}{n})$, respectively.*

Proof. The proof follows from this fact that the eigenvalues and eigenvectors of the cycle graph C_n are $\alpha_r = 2\cos(\frac{2\pi r}{n})$ and $v_r = (1, \varepsilon_n^r, ..., \varepsilon_n^{(n-1)r})^t$, respectively. □

We are now ready to state one of our main theorem as follows:

Theorem 2.5. *The eigenvalues of $I(n, j,k)$ are all roots of the quadratic equation*

$$\rho^2 - (\lambda_r + \mu_r)\rho + \mu_r\lambda_r - 1 = 0,$$

where $\lambda_r = 2\cos(\frac{2\pi jr}{n})$ and $\mu_r = 2\cos(\frac{2\pi kr}{n})$, $0 \le r \le n-1$, are eigenvalues of C_j and C_k, respectively.

Proof. Suppose $d = gcd(n,k)$ and $d' = gcd(n, j)$. We first consider the case that $d = d' = 1$. In this case, C_k^n and C_j^n are the adjacency matrices of a cycle with n vertices, and so they are similar to C_n. This means that there are two permutation matrices P and Q such that $P^{-1}C_k^nP = C_n$ and $Q^{-1}C_j^nQ = C_n$. Hence, C_k^n and C_j^n are containing the same eigenvalues and eigenvectors. Define λ_r and μ_r to be the eigenvalues corresponding to the eigenvector $v_r = (1, \varepsilon_n^r, \cdots, \varepsilon_n^{(n-1)r})^t$. We are looking for an eigenvector for $A(I(n, j, k))$ of the form $w_r = (a_rv_r, b_rv_r)^t$, $0 \le r \le n-1$, where a_r and b_r are appropriate multipliers. To do this, we have to find a number ρ depending on r such that

$$\begin{bmatrix} C_j^n & I_n \\ I_n & C_k^n \end{bmatrix} \begin{bmatrix} a_rv_r \\ b_rv_r \end{bmatrix} = \rho \begin{bmatrix} a_rv_r \\ b_rv_r \end{bmatrix}$$

Therefore, $a_rv_rC_j^n + b_rv_r = \rho a_rv_r$ and $a_rv_r + b_rC_k^nv_r = \rho b_rv_r$. This implies that

$$\begin{cases} a_r\lambda_rv_r + b_rv_r = \rho a_rv_r \\ a_rv_r + \mu_rb_rv_r = \rho b_rv_r \end{cases} \text{and so} \quad \begin{cases} a_r(\rho-\lambda_r)v_r = b_rv_r \\ b_r(\rho-\mu_r)v_r = a_rv_r \end{cases}.$$

Thus, $(\rho-\lambda_r)(\rho-\mu_r) = 1$. We now consider the case of $d, d' > 1$. The eigenvectors w_r must be the form $w_r = (a_1v'_r, \ldots, a_dv'_r, b_1v''_r, \ldots, b_{d'}v''_r)^t$, where $v'_r = (1, \varepsilon_n^r, \ldots, \varepsilon_n^{(n'-1)r})^t$, $v''_r = (1, \varepsilon_n^r, \ldots, \varepsilon_n^{(n''-1)r})^t$, $n' = \frac{n}{d}$ and $n'' = \frac{n}{d'}$. So, we have:

$$\begin{bmatrix} C_j^n & I_n \\ I_n & C_k^n \end{bmatrix} \begin{bmatrix} a_1v'_r \\ \vdots \\ a_dv'_r \\ b_1v''_r \\ \vdots \\ b_{d'}v''_r \end{bmatrix} = \rho \begin{bmatrix} a_1v'_r \\ \vdots \\ a_dv'_r \\ b_1v''_r \\ \vdots \\ b_{d'}v''_r \end{bmatrix}.$$

Therefore,

$$\begin{cases} C_j^n(a_1+\ldots+a_d)v'_r + (b_1+\ldots+b_{d'})v''_r = \rho(a_1+\ldots+a_d)v'_r \\ (a_1+\ldots+a_d)v'_r + C_k^n(b_1+\ldots+b_{d'})v''_r = \rho(b_1+\ldots+b_{d'})v''_r \end{cases},$$

which concludes that $(\rho-\lambda_r)(\rho-\mu_r) = 1$.

Next suppose that $d > 1$ and $d' = 1$. Then w_r must have the form $w_r = (a_1v'_r, \ldots, a_dv'_r, v_r)^t$, where $v_r = (1, \varepsilon_n^r, \ldots, \varepsilon_n^{(n-1)r})^t$, $v'_r = (1, \varepsilon_n^r, \ldots, \varepsilon_n^{(n'-1)r})^t$ and $n' = \frac{n}{d}$. Then a similar system to the two above cases will be obtained and finally we have $\rho^2 - (\lambda_r+\mu_r)\rho + \mu_r\lambda_r - 1 = 0$.

□

Corollary 2.6. *The eigenvalues of $I(n, j, k)$ are given by*

$$\cos\left(\frac{2\pi jr}{n}\right) + \cos\left(\frac{2\pi kr}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi jr}{n}\right) - \cos\left(\frac{2\pi kr}{n}\right)\right)^2 + 1},$$

$0 \le r \le n-1$.

We now represent the spectral moments of $I(n,j,k)$ by using of their eigenvalues. It is well-known that the $k-th$ spectral moment of G is equal to the number of closed walks of length k. The following result will be used in computing the spectral order of $I-$graphs.

Lemma 2.7. *(See [6, 8]) For every graph G, we have:*

$$\begin{array}{rcl}
(1)\ S_4(G) & = & 2\phi(P_2)+4\phi(P_3)+8\phi(C_4),\\
(2)\ S_5(G) & = & 30\phi(C_3)+10\phi(H_1)+10\phi(C_5),\\
(3)\ S_6(G) & = & 2\phi(P_2)+12\phi(P_3)+6\phi(P_4)+12\phi(S_4)+12\phi(H_2)\\
 & + & 36\phi(H_3)+24\phi(H_4)+24\phi(C_3)+48\phi(C_4)+12\phi(C_6),\\
(4)\ S_7(G) & = & 126\phi(C_3)+84\phi(H_1)+28\phi(H_7)+14\phi(H_5)\\
 & + & 14\phi(H_6)+112\phi(H_3)+42\phi(H_{15})+28\phi(H_8)\\
 & + & 70\phi(C_5)+14\phi(H_{18})+14\phi(C_7).
\end{array}$$

Lemma 2.8. *The spectral moments $S_2(I(n,j,k))$ and $S_3(I(n,j,k))$ can be computed by the following formulas:*

$$S_2(I(n,j,k)) = 6n \quad \text{and} \quad S_3(I(n,j,k)) = \begin{cases} 4n & 3 \mid n,\ k=j=\frac{n}{3},\\ 2n & 3 \mid n,\ j=\frac{n}{3}\ or\ k=\frac{n}{3},\\ 0 & Otherwise \end{cases}$$

Proof. It is easy to see that $|E(I(n,j,k))| = 3n$. Therefore, $S_2(I(n,j,k)) = 6n$. Suppose $3 \mid n$ and $k = j = \frac{n}{3}$. Then $u_iu_{i+\frac{n}{3}}u_{i+\frac{2n}{3}}u_i$ is a triangle in inner rim. Also $v_iv_{i+\frac{n}{3}}v_{i+\frac{2n}{3}}v_i$ is a triangle in outer rim. Thus $t = \frac{2n}{3}$ and so, $S_3(I(n,\frac{n}{3},\frac{n}{3})) = 4n$. If $j = \frac{n}{3}$ and $k \neq \frac{n}{3}$, then there is only one of the triangle for any vertex and so $t = \frac{n}{3}$. Thus, $S_3(I(n,j,\frac{n}{3})) = 2n$. Otherwise, there are not any cycle of length 3 in $I(n,j,k)$ and so $S_3(I(n,j,k)) = 0$.

□

Lemma 2.9. *The spectral moment $S_4(I(n,j,k))$ is computed by the following formula:*

$$S_4(I(n,j,k)) = \begin{cases} 38n & k=j\\ 42n & 4 \mid n,\ k=j=\frac{n}{4}\\ 32n & 4 \mid n,\ j=\frac{n}{4}\ or\ k=\frac{n}{4}\\ 30n & Otherwise \end{cases}.$$

Proof. We first assume that $j = k$. Choose vertices u_i in the inner rim and v_i in the outer rim. Then $u_iv_iv_{i+k}u_{i+k}u_i$ is a quadrangle and so $\phi(C_4) = n$. On the other hand, there exists a path of length 3 corresponding to each pair of edges attached to a vertex. Hence, $\phi(P_3) = 6n$, $\phi(P_2) = 3n$. Thus, $S_4(I(n,k,k)) = 6n+24n+8n$

= $38n$. We now assume that $k=j=\frac{n}{4}$. Then, $u_iu_{i+\frac{n}{4}}u_{i+\frac{n}{2}}u_{i+\frac{3n}{4}}u_i$ is a quadrangle in the inner rim and $v_iv_{i+\frac{n}{4}}v_{i+\frac{n}{2}}v_{i+\frac{3n}{4}}v_i$ is a quadrangle in the outer rim. Since $j=k$, $u_iv_iv_{i+k}u_{i+k}u_i$ is a quadrangle in $I(n,j,k)$. Therefore, $\phi(C_4) = \frac{n}{4}+\frac{n}{4}+n = \frac{3n}{2}$, and so $S_4(I(n,\frac{n}{4},\frac{n}{4})) = 6n+24n+12n = 42n$. Let $j=\frac{n}{4}$ and $k\neq\frac{n}{4}$. Then, $\phi(C_4) = \frac{n}{4}$, and so $S_4(I(n,\frac{n}{4},k)) = 6n+24n+2n = 32n$. Otherwise, $\phi(C_4) =0$, $\phi(P_3) = 6n$ and $\phi(P_2) = 3n$. Therefore $S_4(I(n,j,k)) = 30n$.

□

Lemma 2.10. *The spectral moment $S_5(I(n,j,k))$ is computed by the following formula:*

$$S_5(I(n,j,k)) = \begin{cases} 4n & (5\mid n,\ k=j=\frac{n}{5})\ or\ (5\mid n,\ k=j=\frac{2n}{5}) \\ 2n & (5\mid n,\ j=\frac{n}{5}, k\notin\{\frac{n}{5},\frac{2n}{5}\})\ or\ (5\mid n, j=\frac{2n}{5}, k\notin\{\frac{n}{5},\frac{2n}{5}\}) \\ 10n & (k=2j, j\notin\{\frac{n}{5},\frac{n}{3}\})\ or\ (2\mid n, j=2r, k=\frac{n}{2}-r) \\ & or\ (2\nmid n, j=2r+1, k=\frac{n-1}{2}-r;\ r\geqslant 1) \\ 60n & 3\mid n,\ k=j=\frac{n}{3} \\ 20n & 3\mid n,\ k=\frac{n}{3},\ j\notin\{\frac{n}{3},\frac{n}{6},\frac{n}{5},\frac{2n}{5}\} \\ 22n & (15\mid n,\ j=\frac{n}{3}\ and\ k=\frac{n}{5}\ or\ k=\frac{n}{3}\ and\ j=\frac{2n}{5}) \\ 24n & 5\mid n, k=\frac{n}{5}, j=\frac{2n}{5} \\ 30n & 6\mid n, k=\frac{n}{3}, j=\frac{n}{6}\ and\ j,k\notin\{\frac{n}{5},\frac{2n}{5}\} \\ 12n & 10\mid n, j=\frac{n}{10},\ k=\frac{n}{5} \\ 0 & Otherwise \end{cases}.$$

Proof. We first consider the case that $5\mid n$ and $j=k=\frac{n}{5}$. Then for each vertex u_i in the inner rim and each vertex v_i in the outer rim, $u_iu_{i+\frac{n}{5}}u_{i+\frac{2n}{5}}u_{i+\frac{3n}{5}}u_{i+\frac{4n}{5}}u_i$ and $v_iv_{i+\frac{n}{5}}v_{i+\frac{2n}{5}}v_{i+\frac{3n}{5}}v_{i+\frac{4n}{5}}v_i$ are two pentagons in the inner and outer rims, respectively. Thus, $\phi(C_5) = \frac{n}{5}+\frac{n}{5} = \frac{2n}{5}$. Since $I(n,j,k)$ does not have a triangle, $\phi(C_3) = \phi(H_1) = 0$, and so $S_5(I(n,\frac{n}{5},\frac{n}{5})) = 4n$. Similarly, if $j=k=\frac{2n}{5}$, then $u_iu_{i+\frac{2n}{5}}u_{i+\frac{4n}{5}}u_{i+\frac{6n}{5}}u_{i+\frac{8n}{5}}u_i$ and $v_iv_{i+\frac{2n}{5}}v_{i+\frac{4n}{5}}v_{i+\frac{6n}{5}}v_{i+\frac{8n}{5}}v_i$ are pentagons in $I(n,j,k)$. Thus, $\phi(C_5) = \frac{n}{5}+\frac{n}{5} = \frac{2n}{5}$ and $S_5(I(n,\frac{2n}{5},\frac{2n}{5})) = 4n$.

Suppose $j=\frac{n}{5}$ and $k\notin\{\frac{n}{5},\frac{2n}{5}\}$. Then $\phi(C_5) = \frac{n}{5}$ and so $S_5(I(n,\frac{n}{5},k)) = 2n$. Similarly, $S_5(I(n,\frac{2n}{5},k)) = 2n$. We now assume that $k=2j$ and $j\notin\{\frac{n}{3},\frac{n}{5}\}$. Hence for each vertex v_i, $v_iv_{i+j}v_{i+2j}u_{i+2j}u_iv_i$ is a pentagon in $I(n,j,k)$, and so $\phi(C_5) = n$. Thus, $S_5(I(n,j,2j)) = 10n$. Let $r\geq 1$ be an integer. Choose an arbitrary vertex v_i. If $(2|n,\ j=2r,\ k=\frac{n}{2}-r)$ or $(2\nmid n,\ j=2r+1,\ k=\frac{n-1}{2}-r)$, then $v_iv_{i+j}u_{i+j}u_{i+j+k}u_{i+j+2k}v_i$ is a pentagon in $I(n,j,k)$. Thus, $\phi(C_5) = n$ and $S_5(I(n,j,k)) = 10n$.

Suppose that $3|n$ and $j=k=\frac{n}{3}$. Then $\phi(C_3) = \frac{2n}{3}$ and $\phi(H_1) = 2n$. Also for each vertex v_i, $v_iv_{i+\frac{n}{3}}u_{i+\frac{n}{3}}u_{i+\frac{2n}{3}}u_iv_i$ and $v_iv_{i+\frac{n}{3}}u_{i+\frac{n}{3}}u_{i+\frac{2n}{3}}v_{i+\frac{2n}{3}}v_i$ are pentagons in $I(n,j,k)$. So, $\phi(C_5) = 2n$ and therefore, $S_5(I(n,\frac{n}{3},\frac{n}{3})) = 20n+20n+20n = 60n$.

Let $3|n$, $k=\frac{n}{3}$ and $j \notin \{\frac{n}{3}, \frac{n}{6}, \frac{n}{5}, \frac{2n}{5}\}$. Then it is easy to see that $\phi(C_3) = \frac{n}{3}$, $\phi(H_1) = n$ and $\phi(C_5) = 0$. Thus $S_5(I(n, j, \frac{n}{3})) = 20n$.

If $15|n$, ($j=\frac{n}{3}$ and $k=\frac{n}{5}$) or ($j=\frac{n}{3}$ and $k=\frac{2n}{5}$), then $\phi(C_3) = \frac{n}{3}$, $\phi(H_1) = n$ and $\phi(C_5) = \frac{n}{5}$. Therefore, $S_5(I(n, \frac{n}{3}, \frac{n}{5})) = 10n+10n+2n = 22n$. Let $5 \mid n$, $k=\frac{n}{5}$ and $j=\frac{2n}{5}$. Then $\phi(C_5) = \frac{12n}{5}$ and so $S_5(I(n, \frac{n}{5}, \frac{2n}{5})) = 24n$.

If $6 \mid n$, $k=\frac{n}{3}$ and $j=\frac{n}{6}$, then $\phi(C_5) = n$, $\phi(C_3) = \frac{n}{3}$ and $\phi(H_1) = n$. This implies that $S_5(I(n, \frac{n}{3}, \frac{n}{6})) = 30n$. Let $10 \mid n$, $k=\frac{n}{10}$ and $j=\frac{n}{5}$, then $\phi(C_5) = \frac{6n}{5}$ and so $S_5(I(n, \frac{n}{5}, \frac{n}{10})) = 12n$. Otherwise, $\phi(C_5) = 0$, $\phi(C_3) = 0$ and $\phi(H_1) = 0$. Therefore $S_5(I(n, j, k)) = 0$. This completes the proof.

□

Lemma 2.11. *The spectral moment $S_6(I(n, j, k))$ is computed by the following formula:*

$$S_6(I(n,j,k)) = \begin{cases} 282n & j=k,\ k \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{6}\} \\ 286n & (6 \mid n, k=j=\frac{n}{6})\ or\ (3 \mid n, k=j=\frac{n}{3}) \\ 366n & 4 \mid n,\ k=j=\frac{n}{4} \\ 200n & (12 \mid n, k=\frac{n}{3},\ j=\frac{n}{4})\ or\ (12 \mid n, k=\frac{n}{6},\ j=\frac{n}{4}) \\ 190n & 6 \mid n,\ k=\frac{n}{3},\ j=\frac{n}{6} \\ 186n & (k=3j, j \notin \{\frac{n}{3}, \frac{n}{6}, \frac{n}{4}\})\ or\ (2 \mid n, k=\frac{n}{2}-j) \\ 176n & (6 \mid n, j=\frac{n}{6}, k \notin \{\frac{n}{6}, \frac{n}{3}, \frac{n}{4}\}, n \neq 18) \\ & or (3 \mid n, j=\frac{n}{3}, k \notin \{\frac{n}{3}, \frac{n}{6}, \frac{n}{4}\}) \\ 198n & (4 \mid n,\ j=\frac{n}{4}, k \notin \{\frac{n}{4}, \frac{n}{6}\}, n \neq 12) \\ & or\ (5 \mid n, j=\frac{n}{5}, k=\frac{2n}{5})\ or\ (10 \mid n, k=3j, j=\frac{n}{10}) \\ 188n & (6 \mid n, k=\frac{n}{3}, j=\frac{n}{9})\ or\ (6 \mid n, k=\frac{n}{6}, j=\frac{n}{18}) \\ 210n & 4 \mid n, k=\frac{n}{4}, j=\frac{n}{12} \\ 174n & Otherwise \end{cases}.$$

Proof. It is easy to see that $\phi(P_4) = 12n$, $\phi(S_4) = 2n$ and $\phi(H_3) = \phi(H_4) = 0$. We first assume that $j=k$. Then $\phi(C_4) = n$ and $\phi(H_2) = 4n$. Also for each vertex v_i, $v_iu_iu_{i+j}u_{i+2j}v_{i+2j}v_{i+j}v_i$ is a hexagon in $I(n, j, k)$. Thus $\phi(C_6) = n$ and $S_6(I(n, j, j)) = 6n+72n+72n+24n+48n+48n+12n = 282n$.

Suppose $6 \mid n$ and $j=k=\frac{n}{6}$, then $\phi(C_4) = n$ and $\phi(H_2) = 4n$. Choose vertices u_i and v_i in the inner and outer rims, respectively. Then $v_iv_{i+\frac{n}{6}}v_{i+\frac{2n}{6}}v_{i+\frac{3n}{6}}v_{i+\frac{4n}{6}}v_{i+\frac{5n}{6}}v_i$ is a hexagon in the inner rim and $u_iu_{i+\frac{n}{6}}u_{i+\frac{2n}{6}}u_{i+\frac{3n}{6}}u_{i+\frac{4n}{6}}u_{i+\frac{5n}{6}}u_i$ is a hexagon in the outer rim. Also there is a hexagon of the form $v_iv_{i+\frac{n}{6}}v_{i+\frac{2n}{6}}u_{i+\frac{2n}{6}}u_{i+\frac{n}{6}}u_iv_i$. Thus, $\phi(C_6) = \frac{4n}{3}$ and so, $S_6(I(n, \frac{n}{6}, \frac{n}{6})) = 6n+72n+72n+24n+48n+48n+16n = 286n$. Let $3 \mid n$ and $j=k=\frac{n}{3}$. Then, $\phi(P_4) = 10n$, $\phi(C_3) = \frac{2n}{3}$, $\phi(H_1) = 2n$, $\phi(C_4) = n$ and $\phi(H_2) = 4n$. On the other hand, we can correspond a hexagon $v_iv_{i+\frac{n}{3}}v_{i+\frac{2n}{3}}u_{i+\frac{2n}{3}}u_{i+\frac{n}{3}}u_iv_i$ to each vertex of $I(n, j, k)$. So,

$\phi(C_6) = n$ and $S_6(I(n, \frac{n}{3}, \frac{n}{3})) = 6n + 72n + 60n + 24n + 48n + 16n + 48n + 12n =$ $286n$.

Next we suppose that $4|n$ and $j = k = \frac{n}{4}$. Then, $\phi(C_4) = \frac{3n}{2}$, $\phi(H_2) = 6n$ and $\phi(C_6) = 4n$. Thus, $S_6(I(n, \frac{n}{4}, \frac{n}{4})) = 6n + 72n + 72n + 24n + 72n + 72n + 48n = 366n$. If $12|n$, $j = \frac{n}{4}$ and $k = \frac{n}{3}$, then $\phi(C_4) = \frac{n}{4}$, $\phi(H_2) = n$, $\phi(C_3) = \frac{n}{3}$, $\phi(P_4) = 11n$. Thus, $S_6(I(n, \frac{n}{4}, \frac{n}{3})) = 6n + 72n + 66n + 24n + 12n + 8n + 12n = 200n$. In the case that $12|n$, $j = \frac{n}{4}$ and $k = \frac{n}{6}$, we have $\phi(C_4) = \frac{n}{4}$, $\phi(H_2) = n$, $\phi(C_6) = \frac{n}{6}$ and so $S_6(I(n, \frac{n}{4}, \frac{n}{6})) = 6n + 72n + 72n + 24n + 12n + 12n + 2n = 200n$. If $6|n$, $j = \frac{n}{6}$ and $k = \frac{n}{3}$, then there are $\frac{n}{6}$ hexagons in the inner rim and for each vertex v_i, $v_iv_{i+\frac{n}{6}}v_{i+\frac{2n}{6}}u_{i+\frac{n}{3}}u_{i+\frac{2n}{3}}u_iv_i$ is a hexagon. This implies that $\phi(C_6) = \frac{n}{6} + n = \frac{7n}{6}$. Since $\phi(C_4) = 0$, $\phi(H_2) = 0$, $\phi(P_4) = 11n$ and $\phi(C_3) = \frac{n}{3}$, $S_6(I(n, \frac{n}{6}, \frac{n}{3})) = 6n + 72n + 66n + 24n + 8n + 14n = 190n$.

Suppose $k = 3j$. Then for each vertex v_i, $v_iv_{i+j}v_{i+2j}v_{i+3j}u_{i+3j}u_iv_i$ is a hexagon in $I(n, j, k)$, and so $\phi(C_6) = n$ and $\phi(C_3) = 0$. Hence $S_6(I(n, j, 3j)) = 6n + 72n + 72n + 24n + 12n = 186n$. Also, if $2 \mid n$ and $k = \frac{n}{2} - j$, then $u_iu_{i+\frac{n}{2}-j}v_{i+\frac{n}{2}-j}v_{i+\frac{n}{2}}v_{i+\frac{n}{2}+j}u_{i+\frac{n}{2}+j}u_i$ is a hexagon in $I(n, j, k)$ and so $\phi(C_6) = n$ and $S_6(I(n, j, \frac{n}{2} - j)) = 6n + 72n + 72n + 24n + 12n = 186n$. If $3|n$, $j = \frac{n}{3}$ and $k \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{6}\}$, then $\phi(C_3) = \frac{n}{3}$ and so $S_6(I(n, \frac{n}{3}, k)) = 6n + 72n + 66n + 24n + 8n = 176n$. If $6|n$, $j = \frac{n}{6}$ and $k \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{6}\}$, then $\phi(C_3) = 0$ and $\phi(C_6) = \frac{n}{6}$. So, $S_6(I(n, \frac{n}{6}, k)) = 6n + 72n + 72n + 24n + 2n = 176n$.

We now assume that $4|n$, $j = \frac{n}{4}$ and $k \notin \{\frac{n}{4}, \frac{n}{6}\}$. Then, $\phi(C_4) = \frac{n}{4}$, $\phi(C_6) = 0$, $\phi(C_3) = 0$ and $\phi(H_2) = n$. So, $S_6(I(n, \frac{n}{4}, k)) = 6n + 72n + 72n + 24n + 12n + 12n = 198n$. If $j = \frac{n}{5}$ and $k = \frac{2n}{5}$, then for each vertex v_i, there are two hexagons of the forms $v_iv_{i+\frac{n}{5}}v_{i+\frac{2n}{5}}u_{i+\frac{2n}{5}}u_{i+\frac{4n}{5}}v_{i+\frac{4n}{5}}u_i$ and $v_iv_{i+\frac{n}{5}}u_{i+\frac{n}{5}}u_{i+\frac{3n}{5}}v_{i+\frac{3n}{5}}v_{i+\frac{4n}{5}}v_i$. Thus $\phi(C_4) = 0$, $\phi(C_6) = 2n$ and so $S_6(I(n, \frac{n}{5}, \frac{2n}{5})) = 198n$. If $j = \frac{n}{10}$ and $k = \frac{3n}{10}$, then it is easy to see that $\phi(C_6) = 2n$ and so $S_6(I(n, \frac{n}{10}, \frac{3n}{10})) = 198n$. If $j = \frac{n}{3}$ and $k = \frac{n}{9}$ then, $\phi(C_6) = n$ and $\phi(C_3) = \frac{n}{3}$. Thus, $S_6(I(n, \frac{n}{3}, \frac{n}{9})) = 6n + 66n + 72n + 24n + 8n + 12n = 188n$. If $j = \frac{n}{18}$ and $k = \frac{n}{6}$ then $\phi(C_6) = n + \frac{n}{6} = \frac{7n}{6}$ and so $S_6(I(n, \frac{n}{6}, \frac{n}{18})) = 6n + 72n + 72n + 24n + 14n = 188n$. If $j = \frac{n}{12}$ and $k = \frac{n}{4}$ then $\phi(C_6) = n$, $\phi(C_4) = \frac{n}{4}$ and $\phi(U_5) = n$. Thus, $S_6(I(n, \frac{n}{4}, \frac{n}{12})) = 6n + 72n + 72n + 24n + 12n + 12n + 12n = 210n$. In other cases, $\phi(C_6) = 0$, $\phi(C_3) = 0$, $\phi(H_2) = n$ and $\phi(C_4) = 0$. Therefore, $S_6(I(n, j, k)) = 6n + 72n + 72n + 24n = 174n$. This completes the proof.

□

Lemma 2.12. *The spectral moment $S_7(I(n, j, k))$ is computed by the following* *formula:*

$$S_7(I(n,j,k)) = \begin{cases} 4n & (k=j=\frac{n}{7})\ or\ (k=j=\frac{2n}{7})\ or\ (k=j=\frac{3n}{7}) \\ 2n & (k=\frac{n}{7}\ or\ k=\frac{2n}{7}\ or\ k=\frac{3n}{7}\ and\ j\notin\{\frac{n}{3},\frac{n}{5}\}) \\ 172n & (k=\frac{n}{7},j=\frac{2n}{7})\ or\ (k=\frac{n}{7},j=\frac{3n}{7})\ or\ (k=\frac{2n}{7},j=\frac{3n}{7}) \\ 644n & 3\mid n,k=j=\frac{n}{3} \\ 112n & (5\mid n,k=j=\frac{n}{5})\ or\ (5\mid n,k=j=\frac{2n}{5}) \\ 322n & k=\frac{n}{3},j=\frac{n}{6} \\ 168n & (k=\frac{n}{3},j\notin\{\frac{n}{3},\frac{n}{6},\frac{n}{5}\})\ or\ (k=\frac{n}{5},j=\frac{n}{10}) \\ 196n & (k=\frac{n}{3},j=\frac{n}{5}) or (k=\frac{n}{3},j=\frac{2n}{5}) \\ 28n & (k=\frac{n}{5},j\notin\{\frac{n}{3},\frac{n}{6},\frac{n}{5},\frac{n}{7}\})\ or\ (k=\frac{2n}{5},j\notin\{\frac{n}{3},\frac{n}{6},\frac{n}{5},\frac{n}{7}\}) \\ & or\ (k=\frac{n}{4},j=\frac{n}{6}) \\ 336n & k=\frac{n}{5},j=\frac{2n}{5} \\ 140n & (k=2j)\ or\ (2\mid n,j=2r,k=\frac{n}{2}-r) \\ & or\ (2\nmid n,j=2r+1,k=\frac{n-1}{2}-r;\ r\geqslant 1) \\ 154n & (k=\frac{n}{4},j=\frac{n}{8})\ or\ (k=\frac{n}{4},j=\frac{3n}{8}) \\ 170n & (k=\frac{n}{7},j=\frac{n}{3})\ or\ (k=\frac{2n}{7},j=\frac{n}{3}) or (k=\frac{3n}{7},j=\frac{n}{3}) \\ 30n & (k=\frac{n}{5},j=\frac{n}{7})\ or\ (k=\frac{2n}{5},j=\frac{n}{7})\ or\ (k=\frac{n}{5},j=\frac{2n}{7}) or \\ & (k=\frac{2n}{5},j=\frac{2n}{7}) or\ (k=\frac{n}{7},j=\frac{3n}{14})\ or (k=\frac{2n}{7},j=\frac{n}{14}) \\ 16n & (k=\frac{n}{7},j=\frac{n}{28})\ or\ (k=\frac{n}{7},j=\frac{2n}{21})\ or\ (k=\frac{3n}{7},j=\frac{3n}{28}) \\ 42n & (k=\frac{n}{5},j=\frac{n}{20})\ or\ (k=\frac{2n}{5},j=\frac{n}{10}) \\ 14n & (k=4j,j\notin\{\frac{n}{7},\frac{2n}{7},\frac{3n}{7},\frac{n}{3},\frac{n}{5}\})\ or\ (2k=3j,n\neq 22) \\ 182n & (k=\frac{n}{3},j=\frac{n}{12})\ or\ (k=\frac{n}{3},j=\frac{n}{9})\ or\ (k=\frac{n}{3},j=\frac{2n}{9}) \\ 56n & k=\frac{2n}{5},j=\frac{n}{10} \\ 142n & (k=\frac{n}{7},j=\frac{n}{14})\ or (k=\frac{3n}{7},j=\frac{3n}{14}) \\ 0 & Otherwise \end{cases}$$

Proof. We first assume that $7\mid n$ and $k=j=\frac{n}{7}$ or $k=j=\frac{2n}{7}$ or $k=j=\frac{3n}{7}$. Then $\phi(C_7)=\frac{2n}{7}$ and so $S_7(I(n,j,k))=4n$. Suppose $k=\frac{n}{7}$ or $k=\frac{2n}{7}$ or $k=\frac{3n}{7}$ and $j\notin\{\frac{n}{3},\frac{n}{5}\}$, then $\phi(C_7)=\frac{n}{7}$ and so $S_7(I(n,j,k))=2n$. If $k=\frac{n}{7}$ and $j=\frac{2n}{7}$, then it is easy to see that $\phi(C_7)=\frac{16n}{7}$, $\phi(C_5)=n$ and $\phi(H_{18})=5n$. So, $S_7(I(n,j,k))$ $=70n+70n+32n$=$172n$.

If $3|n$ and $k=j=\frac{n}{3}$, then $\phi(C_3)=\frac{2n}{3}$, $\phi(C_5)=2n$, $\phi(H_{18})=10n$, $\phi(H_1)=2n$, $\phi(H_5)=2n$, $\phi(H_6)=4n$ and $\phi(H_{15})=\frac{2n}{3}$. So, $S_7(I(n,j,k))=84n+168n+56n+28n+28n+140n+140n=644n$. If $5|n$ and $k=j=\frac{n}{5}$, then $\phi(C_5)=\frac{2n}{5}$, $\phi(H_{18})=2n$ and $\phi(C_7)=4n$ and hence $S_7(I(n,j,k))=28n+28n+56n$ $=112n$. Moreover, if $k=\frac{n}{3}$ and $j=\frac{n}{6}$ then $\phi(C_3)$=$\frac{n}{3}$, $\phi(H_1)=n$, $\phi(H_5)=n$, $\phi(H_6)=2n$, $\phi(C_5)$=n, $\phi(H_{18})=5n$ and $\phi(C_7)=n$. Thus, $S_7(I(n,j,k))=42n+84n+14n+28n+70n+70n+14n=322n$. If $k=\frac{n}{3}$ and $j\notin\{\frac{n}{3},\frac{n}{5},\frac{n}{6}\}$ then $\phi(C_3)=\frac{n}{3}$, $\phi(H_1)=n$, $\phi(H_5)=n$ and $\phi(H_6)=2n$ and so $S_7(I(n,j,k))=42n+84n+14n+28n=168n$. We now assume that $k=\frac{n}{5}$ and $j=\frac{n}{10}$. Then

$\phi(C_5) = \frac{6n}{5}$, $\phi(H_{18}) = 6n$ and so $S_7(I(n,j,k)) = 84n + 84n = 168n$.

If $k = \frac{n}{3}$ and $j = \frac{n}{5}$ or $j = \frac{2n}{5}$, then $\phi(C_3) = \frac{n}{3}$, $\phi(C_5) = \frac{n}{5}$, $\phi(H_1) = n$, $\phi(H_{18}) = n$, $\phi(H_5) = n$, $\phi(H_6) = 2n$ and so $S_7(I(n,j,k)) = 42n + 84n + 14n + 28n + 14n + 14n = 196n$. If $k = \frac{n}{5}$ and $j \notin \{\frac{n}{3}, \frac{n}{5}, \frac{n}{6}, \frac{n}{7}\}$, then $\phi(C_5) = \frac{n}{5}$ and $\phi(H_{18}) = n$. Hence, $S_7(I(n,j,k)) = 14n + 14n = 28n$. If $k = \frac{n}{5}$ and $j = \frac{2n}{5}$ then $\phi(C_5) = \frac{12n}{5}$, $\phi(H_{18}) = 12n$ and so $S_7(I(n,j,k)) = 168n + 168n = 336n$.

Suppose $k = 2j$. Then $\phi(C_5) = n$ and so $S_7(I(n,j,k)) = 140n$. Similarly if $(2 \mid n, j = 2r, k = \frac{n}{2} - r)$ or $(2 \nmid n, j = 2r+1, k = \frac{n-1}{2} - r)$, for $r \geqslant 1$, $S_7(I(n,j,k)) = 140n$. If $k = \frac{n}{4}$ and $j = \frac{n}{8}$, then $\phi(C_5) = n$, $\phi(H_{18})$=$5n$, and $\phi(C_7) = n$, which implies that $S_7(I(n,j,k)) = 70n + 70n + 14n = 154n$. In a similarly way, if $k = \frac{n}{4}$ and $j = \frac{3n}{8}$ then it is easy to see that $S_7(I(n,j,k))$= $154n$. If $k = \frac{n}{7}$ and $j = \frac{n}{3}$, then $\phi(C_3) = \frac{n}{3}$, $\phi(H_1) = n$, $\phi(H_5) = n$, $\phi(H_6) = 2n$ and $\phi(C_7) = \frac{n}{7}$. Thus, $S_7(I(n,j,k)) = 42n + 84n + 14n + 28n + 2n = 170n$. Similarly in the case that ($k = \frac{2n}{7}$ and $j = \frac{n}{3}$) or ($k = \frac{3n}{7}$ and $j = \frac{n}{3}$) one can see that $S_7(I(n,j,k)) = 170n$.

We now assume that $k = \frac{n}{5}$ and $j = \frac{n}{7}$. Then $\phi(C_5) = \frac{n}{5}$, $\phi(H_{18}) = n$ and $\phi(C_7) = \frac{n}{7}$. This shows that $S_7(I(n,j,k)) = 14 + 14n + 2n = 30n$. In a similar way, if $k = \frac{2n}{5}$ and $j = \frac{n}{7}$, $k = \frac{n}{5}$ and $j = \frac{2n}{7}$ or $k = \frac{2n}{5}$ and $j = \frac{2n}{7}$, then we have $S_7(I(n,j,k)) = 30n$. If $k = \frac{n}{7}$ and $j = \frac{n}{28}$ or $k = \frac{n}{7}$ and $j = \frac{2n}{21}$ or $k = \frac{3n}{7}$ and $j = \frac{3n}{28}$, then it is easy to see that $\phi(C_7) = \frac{8n}{7}$ and we have $S_7(I(n,j,k)) = 16n$.

Next we assume that $j = \frac{n}{20}$ and $k = \frac{n}{5}$. Then $\phi(C_7) = n$, $\phi(C_5)$=$\frac{n}{5}$ and $\phi(H_{18})$=n. This implies that $S_7(I(n,j,k)) = 14n + 14n + 14n = 42n$. If $j = \frac{n}{10}$ and $k = \frac{2n}{5}$, then by a simple check we have $S_7(I(n,j,k)) = 42n$. If $k = 4j$ and $j \notin \{\frac{n}{7}, \frac{2n}{7}, \frac{3n}{7}, \frac{n}{3}, \frac{n}{5}\}$, then one can see that $\phi(C_7) = n$ and so $S_7(I(n,j,k)) = 14n$. Similarly if $2k = 3j$, then $\phi(C_7) = n$ and so $S_7(I(n,j,k)) = 14n$. If $j = \frac{n}{12}$ and $k = \frac{n}{3}$ then $\phi(C_7) = n$, $\phi(C_3) = \frac{n}{3}$, $\phi(H_1) = n$, $\phi(H_5) = n$ and $\phi(H_6) = 2n$. So, $S_7(I(n,j,k)) = 42n + 84n + 14n + 28n + 14n = 182n$. One can easily prove that if $j = \frac{n}{9}$ and $k = \frac{n}{3}$ or $j = \frac{2n}{9}$ and $k = \frac{n}{3}$, then we have $S_7(I(n,j,k)) = 182n$. If $k = \frac{n}{10}$ and $j = \frac{2n}{5}$, then $\phi(C_7) = 2n$, $\phi(C_5) = \frac{n}{5}$ and $\phi(H_{18}) = n$, which implies that $S_7(I(n,j,k)) = 14n + 14n + 28n = 56n$. In the case that $k = \frac{n}{7}$ and $j = \frac{n}{14}$, we have $\phi(C_5) = n$, $\phi(H_{18}) = 5n$ and $\phi(C_7) = \frac{n}{7}$. Thus $S_7(I(n,j,k)) = 70n + 2n + 70n = 142n$. Similarly if $k = \frac{3n}{7}$ and $j = \frac{3n}{14}$, then $S_7(I(n,j,k)) = 142n$. Otherwise, $\phi(C_5) = 0$, $\phi(C_7) = 0$, $\phi(C_3) = 0$ and so $S_7(I(n,j,k)) = 0$.

□

We are now ready to order all of the $I-$graphs with respect to spectral moments in an $S-$order.

Theorem 2.13. *Let* $3 \mid n$ *and* $\mathcal{G} = \{I(n,j,k) : j,k \neq \frac{n}{3}\}$. *Then* $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S I(n, \frac{n}{3}, \frac{n}{3})$. *where* $\mathcal{G}_1 = \{I(n, \frac{n}{3}, k) : k \neq \frac{n}{3}\}$.

Proof. Apply Lemmas 2.9−2.12. Since number of edges in $I(n,j,k)$ is $3n$, we have to consider the number of triangles. Suppose $G \in \mathcal{G}$ and $G_1 \in \mathcal{G}_1$. Then

$S_i(G) = S_i(G_1) = S_i(I(n,\frac{n}{3},\frac{n}{3}))$, $i = 0,1,2$, $S_3(G) < S_3(G_1) < S_3(I(n,\frac{n}{3},\frac{n}{3}))$. Thus, $G \prec_S G_1 \prec_S I(n,\frac{n}{3},\frac{n}{3})$. This shows that $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S I(n,\frac{n}{3},\frac{n}{3})$, as desired. □

Theorem 2.14. *Suppose $3 \nmid n$. If n is odd and $\mathcal{G} = \{I(n,j,k) : j \neq k, j,k \leq \frac{n-1}{2}\}$, then $\mathcal{G} \prec_S \mathcal{G}_2$, where $\mathcal{G}_2 = \{I(n,k,k) : k \leq \frac{n-1}{2}\}$. If $4 \mid n$ and $\mathcal{G} = \{I(n,j,k) : j \neq k$ and $j,k \neq \frac{n}{4}\}$, then $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S \mathcal{G}_2 \prec_S I(n,\frac{n}{4},\frac{n}{4})$, where $\mathcal{G}_1 = \{I(n,k,\frac{n}{4}) : k \neq \frac{n}{4}\}$ and $\mathcal{G}_2 = \{I(n,k,k) : k \neq \frac{n}{4}\}$.*

Proof. We first assume that n is odd. Let $G \in \mathcal{G}$ and $G_2 \in \mathcal{G}_2$. Then $S_4(G) = 30n$ and $S_4(G_2) = 38n$. So, $S_i(G) = S_i(G_2)$, $i = 0,1,2,3$, and $S_4(G) < S_4(G_2)$. Thus, $G \prec_S G_2$ and so $\mathcal{G} \prec_S \mathcal{G}_2$. Now we let $4 \mid n$, $G \in \mathcal{G}$, $G_2 \in \mathcal{G}_2$ and $G_1 \in \mathcal{G}_1$. Then $S_4(G) = 30n$, $S_4(G_1) = 32n$ and $S_4(G_2) = 38n$. Hence, $S_i(G) = S_i(G_1) = S_i(G_2) = S_i(I(n,\frac{n}{4},\frac{n}{4})$, $i = 0,1,2,3$, and $S_4(G) < S_4(G_1) < S_4(G_2) < S_4(I(n,\frac{n}{4},\frac{n}{4})$. This shows that $G \prec_S G_1 \prec_S G_2 \prec_S I(n,\frac{n}{4},\frac{n}{4})$ and hence $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S \mathcal{G}_2 \prec_S I(n,\frac{n}{4},\frac{n}{4})$, proving the theorem. □

Theorem 2.15. *Let $3 \mid n$. If n is odd and $\mathcal{G} = \{I(n,j,k) : j \neq k$ and $j,k \neq \frac{n}{3}\}$ then $\mathcal{G} \prec_S \mathcal{G}_1$, where $\mathcal{G}_1 = \{I(n,k,k) : k \neq \frac{n}{3}\}$. If $4|n$ and $\mathcal{G} = \{I(n,j,k) : j \neq k$ and $j,k \notin \{\frac{n}{3},\frac{n}{4}\}\}$, then $\mathcal{G} \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_1 \prec_S I(n,\frac{n}{3},\frac{n}{4})$, where $\mathcal{G}_2 = \{I(n,k,\frac{n}{4}) : k \notin \{\frac{n}{3},\frac{n}{4}\}\}$ and $\mathcal{G}_1 = \{I(n,k,k) : k \notin \{\frac{n}{3},\frac{n}{4}\}\}$.*

Proof. Suppose n is odd, $G \in \mathcal{G}$ and $G_1 \in \mathcal{G}_1$. Then $S_3(G) = S_3(G_1) = 0$, $S_4(G) = 30n$ and $S_4(G_1) = 38n$. So, $S_i(G) = S_i(G_1)$, $i = 0,1,2,3$, and $S_4(G) < S_4(G_1)$. Thus, $G \prec_S G_1$ and so $\mathcal{G} \prec_S \mathcal{G}_1$. If $4|n$ and $G_2 \in \mathcal{G}_2$, then $S_i(G) = S_i(G_1) = S_i(G_2) = S_i(I(n,\frac{n}{4},\frac{n}{4}))$ for $i = 0,1,2,3$ and $S_4(G) < S_4(G_2) < S_4(G_1) < S_4(I(n,\frac{n}{4},\frac{n}{4}))$. Thus, $G \prec_S G_2 \prec_S G_1 \prec_S I(n,\frac{n}{4},\frac{n}{4})$ and so $\mathcal{G} \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_1 \prec_S I(n,\frac{n}{4},\frac{n}{4})$. □

Theorem 2.16. *Suppose n is even such that $3 \nmid n$ and $4|n$. Then,*

1. *If $5|n$ and $\mathcal{G} = \{I(n,j,k) : k \notin \{j,2j\}$ and $j,k \notin \{\frac{n}{5},\frac{2n}{5},\frac{n}{4}\}\}$, then*

$$\mathcal{G} \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_1 \prec_S I(n,\frac{n}{5},\frac{n}{10}) \prec_S I(n,\frac{n}{5},\frac{2n}{5}),$$

   where $\mathcal{G}_1 = \{I(n,j,2j), I(n,2r,\frac{n}{2}-r), r \geq 1$ and $j \neq \{\frac{n}{5},\frac{n}{4}\}\}$ and $\mathcal{G}_2 = \{I(n,k,\frac{n}{5}), I(n,k,\frac{2n}{5}) : k \notin \{\frac{n}{5},\frac{2n}{5},\frac{n}{4}\}\}$.

2. *If $5 \nmid n$ and $\mathcal{G} = \{I(n,j,k) : k \notin \{j,2j\}$ and $j,k \neq \frac{n}{4}\}$ then $\mathcal{G} \prec_S \mathcal{G}_1$, where $\mathcal{G}_1 = \{I(n,j,2j), I(n,2r,\frac{n}{2}-r)\ j \neq \frac{n}{4}, r \geq 1\}$.*

Proof. Let $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $S_5(G) = 0$, $S_5(G_1) = 10n$ and $S_5(G_2) = 2n$. So, $S_i(G) = S_i(G_1) = S_i(G_2) = S_i(I(n, \frac{n}{5}, \frac{n}{10})) = S_i(I(n, \frac{n}{5}, \frac{2n}{5}))$, $i = 0,1,2,3,4$, and $S_5(G) < S_5(G_2) < S_5(G_1) < S_5(I(n, \frac{n}{5}, \frac{n}{10})) < S_5(I(n, \frac{n}{5}, \frac{2n}{5}))$. Thus, $G \prec_S G_2 \prec_S G_1 \prec_S I(n, \frac{n}{5}, \frac{n}{10}) \prec_S I(n, \frac{n}{5}, \frac{2n}{5})$. We now assume that $5 \nmid n$, $G \in \mathcal{G}$ and $G_1 \in \mathcal{G}_1$. Then $S_i(G) = S_i(G_1)$, $i = 0,1,2,3,4$, and $S_5(G) < S_5(G_1)$. So, $G \prec_S G_1$ which implies that $\mathcal{G} \prec_S \mathcal{G}_1$.

□

Theorem 2.17. *Suppose* $12|n$. *Then the following are satisfied:*

1. *If* $5|n$ *and* $\mathcal{G} = \{I(n, j, k) : j \notin \{k, 2k\}$ *and* $j, k \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{5}, \frac{2n}{5}\}\}$, *then*
$$\mathcal{G} \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_1 \prec_S I(n, \frac{n}{5}, \frac{n}{10}) \prec_S I(n, \frac{n}{5}, \frac{2n}{5}),$$
where $\mathcal{G}_1 = \{I(n, j, 2j), I(n, \frac{n}{2} - r, 2r) : r \geq 1, j \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{5}, \frac{2n}{5}\}\}$ *and* $\mathcal{G}_2 = \{I(n, j, \frac{n}{5}), I(n, j, \frac{2n}{5}) : j \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{5}, \frac{2n}{5}\}\}$.
2. *If* $5 \nmid n$ *and* $\mathcal{G} = \{I(n, j, k) : k \notin \{j, 2j\}$ *and* $j, k \neq \{\frac{n}{3}, \frac{n}{4}\}\}$, *then* $\mathcal{G} \prec_S \mathcal{G}_1$, *where* $\mathcal{G}_1 = \{I(n, j, 2j), I(n, \frac{n}{2} - r, 2r) : j \neq \frac{n}{3}, r \geq 1\}$.
3. *Suppose* $2|n$, $3 \nmid n$ *and* $\mathcal{G} = \{I(n, j, k) : j \notin \{k, 2k, 3k\}, k \neq \frac{n}{2} - j\}$. *Also, when* $4|n$ *we assume that* $\mathcal{G} \not\supseteq \{I(n, k, \frac{n}{4}) : k \neq \frac{n}{4}\}$ *and when* $10 \mid n$ *we assume that* $\mathcal{G} \not\supseteq \{I(n, k, \frac{n}{5}), I(n, k, \frac{2n}{5}), I(n, k, \frac{n}{10}) : k \neq \{\frac{n}{5}, \frac{2n}{5}\}\}$. *Then,* $\mathcal{G} \prec_S \mathcal{G}_1$, *where* $\mathcal{G}_1 = \{I(n, k, 3k), I(n, j, \frac{n}{2} - j) : k, j \notin \{\frac{n}{4}, \frac{n}{5}, \frac{2n}{5}\}\}$.
4. *Suppose* $6 \mid n$ *and* $\mathcal{G} = \{I(n, j, k) : j, k \notin \{\frac{n}{3}, \frac{n}{6}\}, k \notin \{j, 2j, 3j, \frac{n}{2} - j\}\}$. *When* $4|n$ *we assume that* $\mathcal{G} \not\supseteq \{I(n, k, \frac{n}{4}) : k \neq \frac{n}{4}\}$ *and when* $10 \mid n$ *we assume that* $\mathcal{G} \not\supseteq \{I(n, j, \frac{n}{5}), I(n, j, \frac{2n}{5}), I(n, j, \frac{n}{10}) : j \neq \{\frac{n}{5}, \frac{2n}{5}\}\}$. *Then* $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S \mathcal{G}_2$, *where* $\mathcal{G}_1 = \{I(n, j, \frac{n}{6}) : j \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{5}, \frac{2n}{5} \frac{n}{6}\}\}$ *and* $\mathcal{G}_2 = \{I(n, j, 3j), I(n, j, \frac{n}{2} - j) : j \notin \{\frac{n}{3}, \frac{n}{4}, \frac{n}{5}, \frac{2n}{5}, \frac{n}{6}\}\}$.

Proof. Suppose $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. If $0 \leq i \leq 4$ then $S_i(G) = S_i(G_1) = S_i(G_2) = S_i(I(n, \frac{n}{5}, \frac{n}{10})) = S_i(I(n, \frac{n}{5}, \frac{2n}{5}))$ and $S_5(G) < S_5(G_2) < S_5(G_1) < S_5(I(n, \frac{n}{5}, \frac{n}{10})) < S_5(I(n, \frac{n}{5}, \frac{2n}{5}))$. So, $G \prec_S G_2 \prec_S G_1 \prec_S I(n, \frac{n}{5}, \frac{n}{10})) \prec_S I(n, \frac{n}{5}, \frac{2n}{5})$, which proves that $\mathcal{G} \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_1 \prec_S I(n, \frac{n}{5}, \frac{n}{10}) \prec_S I(n, \frac{n}{5}, \frac{2n}{5})$. A similar argument as Theorem 2.16 will prove the second part of this theorem.

We now assume that $G \in \mathcal{G}$ and $G_1 \in \mathcal{G}_1$. Then $S_4(G) = 30n$ and $S_5(G) = 0$. We have to count the number of closed walk of length 6. Suppose that $G_1 \in \mathcal{G}_1$. Then $S_6(G) = 174n$ and $S_6(G_1) = 186n$. If $i = 0,1,2,3,4,5$, then $S_i(G) = S_i(G_1)$ and $S_6(G) < S_6(G_1)$. Thus, $G \prec_S G_1$ and therefore $\mathcal{G} \prec_S \mathcal{G}_1$, proving the result.

Finally, we assume that $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $S_6(G) = 174n$, $S_6(G_1) = 176n$ and $S_6(G_2) = 186n$. Hence, $S_i(G) = S_i(G_1) = S_i(G_2)$, $0 \leq i \leq 5$, and $S_6(G) < S_6(G_1) < S_6(G_2)$. Thus, $G \prec_S G_1 \prec_S G_2$ and therefore $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S \mathcal{G}_2$. This completes our argument.

□

Theorem 2.18. *Suppose* $6|n$ *and* $\mathcal{G} = \{I(n,j,k) : k \notin \{j,2j,3j,4j,\frac{3j}{2}\}$ *and* $j,k \notin \{\frac{n}{3},\frac{n}{6},\frac{v}{7},\frac{2n}{7},\frac{3n}{7}\}\}$. *We also assume that when* $4|n$, $\mathcal{G} \not\supseteq \{I(n,k,\frac{n}{4}) : k \neq \frac{n}{4}\}$ *and when* $10 \mid n$, $\mathcal{G} \not\supseteq \{I(n,j,\frac{n}{5}),I(n,j,\frac{2n}{5}),I(n,j,\frac{n}{10}) : j \neq \{\frac{n}{5},\frac{2n}{5}\}\}$. *Then* $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_3 \prec_S \mathcal{G}_4$, *where*

$$\begin{aligned}
\mathcal{G}_1 &= \{I(n,j,\frac{n}{7}),I(n,j,\frac{2n}{7}),I(n,j,\frac{3n}{7}) : j \notin \{\frac{n}{3},\frac{n}{4},\frac{n}{5},\frac{2n}{5},\frac{n}{7},\frac{2n}{7},\frac{3n}{7}\}\},\\
\mathcal{G}_2 &= \{I(n,j,4j),I(n,j,\frac{3j}{2}) : j \notin \{\frac{n}{3},\frac{n}{5},\frac{n}{6},\frac{n}{4},\frac{n}{7}\}\},\\
\mathcal{G}_3 &= \{I(n,\frac{n}{7},\frac{n}{28}),I(n,\frac{n}{7},\frac{2n}{21}),I(n,\frac{3n}{7},\frac{3n}{28})\},\\
\mathcal{G}_4 &= \{I(n,\frac{n}{7},\frac{3n}{14}),I(n,\frac{n}{14},\frac{2n}{7})\}.
\end{aligned}$$

Proof. Let $G \in \mathcal{G}$, $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$, $G_3 \in \mathcal{G}_3$ and $G_4 \in \mathcal{G}_4$. If $0 \leq i \leq 6$, then $S_i(G) = S_i(G_1) = S_i(G_2) = S_i(G_3) = S_i(G_4)$, and $S_7(G) < S_7(G_1) < S_7(G_2) < S_7(G_3) < S_7(G_4)$. Therefore, $G \prec_S G_1 \prec_S G_2 \prec_S G_3 \prec_S G_4$. This implies that $\mathcal{G} \prec_S \mathcal{G}_1 \prec_S \mathcal{G}_2 \prec_S \mathcal{G}_3 \prec_S \mathcal{G}_4$, as desired.

□

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