LE MATEMATICHE
Vol. LXXI (2016) – Fasc. I, pp. 3–20
doi: 10.4418/2016.71.1.1

SOME PROPERTIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

M. KAMAL AOUF - A. O. MOSTAFA - E. AHMED ADWAN

In this paper, we introduce a new class $H_T(f,g;\alpha,k)$ of analytic functions in the open unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$ defined by convolution. The object of the present paper is to determine coefficient estimates, extreme points, distortion theorems, partial sums and integral means for functions belonging to the class $H_T(f,g;\alpha,k)$. We also obtain several results for the neighborhood of functions belonging to this class.

1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the open unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$.

For functions f given by (1) and $g\in A$ given by

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n \quad (b_n>0), \tag{2}$$

the Hadamard product (or convolution) of f and g is defined by

Entrato in redazione: 25 gennaio 2013
AMS 2010 Subject Classification: 30C45.
Keywords: Analytic functions, partial sums, integral means, neighborhood, convolution.

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z).$$

Definition 1.1 ([9], [10], [13] and [16])**.** For $k\geq 0, 0\leq\alpha<1$ and $z\in U$, let $S(k,\alpha)$ denote the subclass of functions $f\in A$and satisfying the condition:

$$Re\left(\frac{zf'(z)}{f(z)}+k\frac{z^2f''(z)}{f(z)}\right)>\alpha.$$

Definition 1.2. For $0\leq\alpha<1, k\geq 0$and for all $z\in U$, let $H(f,g;\alpha,k)$ denote the subclass of A consisting of functions $f(z)$, $g(z)\in A$ and satisfying the analytic criterion:

$$Re\left\{\frac{z(f*g)'(z)}{(f*g)(z)}+k\frac{z^2(f*g)''(z)}{(f*g)(z)}\right\}>\alpha. \tag{3}$$

We note that for suitable choice of g, we obtain the following subclasses.
(1) If we take $g(z)=\frac{z}{1-z}$, then the class $H(f,\frac{z}{1-z};\alpha,k)$ reduces to the class $S(k,\alpha)$ (see [13]);
(2) If we take

$$g(z)=z+\sum_{n=2}^{\infty}\sigma_nz^n \tag{4}$$

$(or\ \ b_n=\sigma_n)$, where

$$\sigma_n=\frac{\Theta\Gamma(\alpha_1+A_1(n-1))\dots\Gamma(\alpha_q+A_q(n-1))}{(n-1)!\Gamma(\beta_1+B_1(n-1))\dots\Gamma(\beta_s+B_s(n-1))} \tag{5}$$

$$(\alpha_i,A_i>0, i=1,\dots,q;\beta_j,B_j>0, j=1,\dots,s;q\leq s+1;q,s\in\mathbb{N},\mathbb{N}=\{1,2,\dots\})$$

and

$$\Theta=\frac{\left(\prod_{j=0}^{s}\Gamma(\beta_j)\right)}{\left(\prod_{i=0}^{q}\Gamma(\alpha_i)\right)}, \tag{6}$$

then the class $H(f,z+\sum_{n=2}^{\infty}\sigma_nz^n;\alpha,k)$ reduces to the class $W_s^q(\alpha,k)$ (see [5])

$$=\left\{f\in A: Re\left\{\frac{z\left(W_s^qf(z)\right)'}{W_s^qf(z)}+k\frac{z^2\left(W_s^qf(z)\right)''}{W_s^qf(z)}\right\}>\alpha,\right.$$

$$0 \leq \alpha < 1; k \geq 0; q, s \in \mathbb{N}; z \in U\}, \tag{7}$$

where $W_s^q f(z)$ is the Wright's generalized hypergeometric function (see [6] and [22]) which contains well known operators such as the Dziok-Srivastava operator (see [7]), the Carlson-Shaffer linear operator (see [1]), the Bernardi-Libera-Livingston operator (see [11]), Owa-Srivastava fractional derivative operator (see [15]), the Choi-Saigo-Srivastava operator (see [4]), the Cho-Kwon-Srivastava operator (see [3]), the Ruscheweyh derivative operator (see [17]) and the Noor integral operator of n-th order (see [14);

(3) If we take

$$g(z) = z + \sum_{n=2}^{\infty} \left(\frac{l+1+\mu(n-1)}{l+1} \right)^m z^n \tag{8}$$

(or $b_n = \left(\frac{l+1+\mu(n-1)}{l+1} \right)^m$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, l \geq 0$), then the class $H(f, z + \sum_{n=2}^{\infty} \left(\frac{l+1+\mu(n-1)}{l+1} \right)^m z^n; \alpha, k)$ reduces to the class $\mathfrak{L}_m(\mu, l, \alpha, k)$:

$$= \left\{ f \in A : Re \left\{ \frac{z(I^m(\mu,l)f(z))'}{I^m(\mu,l)f(z)} + k \frac{z^2(I^m(\mu,l)f(z))''}{I^m(\mu,l)f(z)} \right\} > \alpha, \right.$$

$$\left. 0 \leq \alpha < 1; k \geq 0; \mu,\ l \geq 0, m \in \mathbb{N}_0; z \in U \right\}, \tag{9}$$

where the operator $I^m(\mu, l)$ was introduced and studied by Cătaş et al. (see [2]).

Denote by T the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0), \tag{10}$$

which are analytic in U. We define the class $H_T(f, g; \alpha, k)$ by:

$$H_T(f, g; \alpha, k) = H(f, g; \alpha, k) \cap T. \tag{11}$$

Also we note that:

(1) $H_T(f, z + \sum_{n=2}^{\infty} \sigma_n z^n; \alpha, k) = TW_s^q(\alpha, k) \ \ (q, s \in \mathbb{N}, k \geq 0, 0 \leq \alpha < 1)$, where σ_n given by (5) (see [5]);

(2) $H_T(f, \frac{z}{1-z}; \alpha, k) = S_T(k, \alpha) \ (k \geq 0, 0 \leq \alpha < 1)$;

(3) $H_T(f, z + \sum_{n=2}^{\infty} \left(\frac{l+1+\mu(n-1)}{l+1} \right)^m z^n; \alpha, k) = T\mathfrak{L}_m(\mu, l, \alpha, k) \ (0 \leq \alpha < 1, k \geq 0,$ $\mu, l \geq 0, m \in \mathbb{N}_0)$.

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $0 \leq \alpha < 1,\ k \geq 0, b_n > 0, n \geq 2,\ z \in U$ *and* $g(z)$ is defined by (2).

Theorem 2.1. *A function $f(z)$ of the form (1) is in the class $H(f,g;\alpha,k)$ if*

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) b_n |a_n| \leq 1 - \alpha. \tag{12}$$

Proof. Assume that the inequality (12) holds true. Then we have

$$\left| \frac{z(f*g)'(z)}{(f*g)(z)} + k\frac{z^2(f*g)''(z)}{(f*g)(z)} - 1 \right| \leq \frac{\sum\limits_{n=2}^{\infty}[n+kn(n-1)-1]b_n |a_n| |z|^{n-1}}{1+\sum\limits_{n=2}^{\infty} b_n |a_n| |z|^{n-1}}$$

$$\leq \frac{\sum\limits_{n=2}^{\infty}[n+kn(n-1)-1]b_n |a_n|}{1-\sum\limits_{n=2}^{\infty} b_n |a_n|} \leq 1 - \alpha.$$

This shows that the values of the function

$$\Phi(z) = \left(\frac{z(f*g)'(z) + kz^2(f*g)''(z)}{(f*g)(z)} \right) \tag{13}$$

lie in a circle centered at $w = 1$ and whose radius is $1-\alpha$. Hence $f(z)$ satisfies the condition (12). This completes the proof of Theorem 2.1. □

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (10) to be in the class $H_T(f,g;\alpha,k)$ is that*

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) b_n a_n \leq 1 - \alpha. \tag{14}$$

Proof. In view of Theorem 2.1, we need only to show that $f(z) \in H_T(f,g;\alpha,k)$ satisfies the coefficient inequality (12). If $f(z) \in H_T(f,g;\alpha,k)$ then the function $\Phi(z)$ given by (13) satisfies $Re\{\Phi(z)\} > \alpha$. This implies that

$$(f*g)(z) = z - \sum_{n=2}^{\infty} b_n a_n z^n \neq 0 \,(z \in U \setminus \{0\}),$$

Noting that $\frac{(f*g)(r)}{r}$ is the real continuous function in the open interval $(0,1)$ with $f(0) = 1$, we have

$$1 - \sum_{n=2}^{\infty} b_n a_n r^{n-1} > 0 \,(0 < r < 1). \tag{15}$$

Now

$$\Phi(r) = \frac{1 - \sum\limits_{n=2}^{\infty} n b_n a_n r^{n-1} - k \sum\limits_{n=2}^{\infty} n(n-1) b_n a_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty} b_n a_n r^{n-1}} > \alpha,$$

and consequently by (15) we obtain

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) b_n a_n r^{n-1} \leq 1 - \alpha. \tag{16}$$

Letting $r \to 1^-$ in (16), we get (14). This completes the proof of Theorem 2.2. □

Corollary 2.3. *Let the function f defined by (10) be in the class $H_T(f, g; \alpha, k)$, then*

$$a_n \leq \frac{(1-\alpha)}{(kn^2 + n - kn - \alpha) b_n} \ (n \geq 2). \tag{17}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{(kn^2 + n - kn - \alpha) b_n} z^n \ (n \geq 2). \tag{18}$$

3. Distortion theorems

Theorem 3.1. *Let the function $f(z)$ defined by (10) belong to the class $H_T(f, g; \alpha, k)$. Then for $|z| = r < 1$, we have*

$$r - \frac{(1-\alpha)}{(2k+2-\alpha) b_2} r^2 \leq |f(z)| \leq r + \frac{(1-\alpha)}{(2k+2-\alpha) b_2} r^2, \tag{19}$$

provided $b_n \geq b_2$ $(n \geq 2)$. The result is sharp with equality for the function $f(z)$ defined by

$$f(z) = z - \frac{(1-\alpha)}{(2k+2-\alpha) b_2} z^2 \tag{20}$$

at $z = r$ and $z = re^{i(2n+1)\pi}$ $(n \in \mathbb{N})$.

Proof. We have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + r^2 \sum_{n=2}^{\infty} a_n. \tag{21}$$

Since for $n \geq 2$, we have

$$(2k+2-\alpha)b_2 \leq \left(kn^2+n-kn-\alpha\right) b_n,$$

then (14) yields

$$(2k+2-\alpha)b_2 \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(kn^2+n-kn-\alpha\right) b_n a_n \leq (1-\alpha) \tag{22}$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1-\alpha)}{(2k+2-\alpha)b_2}. \tag{23}$$

From (23) and (21) we have

$$|f(z)| \leq r + \frac{(1-\alpha)}{(2k+2-\alpha)b_2} r^2$$

and similarly, we have

$$|f(z)| \geq r - \frac{(1-\alpha)}{(2k+2-\alpha)b_2} r^2.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the function $f(z)$ defined by (10) belong to the class $H_T(f,g;\alpha,k)$. Then for $|z| = r < 1$, we have*

$$1 - \frac{2(1-\alpha)}{(2k+2-\alpha)b_2} r \leq \left|f'(z)\right| \leq 1 + \frac{2(1-\alpha)}{(2k+2-\alpha)b_2} r, \tag{24}$$

provided $b_n \geq b_2$ $(n \geq 2)$. The result is sharp for the function $f(z)$ given by (20) at $z = r$ and $z = re^{i(2n+1)\pi}$ $(n \in \mathbb{N})$.

Proof. For a function $f(z) \in H_T(f,g;\alpha,k)$, it follows from (14) and (23) that

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1-\alpha)}{(2k+2-\alpha)b_2}. \tag{25}$$

Since the remaining part of the proof is similar to the proof of Theorem 3.1, we omit the details. □

4. Extreme points

Theorem 4.1. *The class $H_T(f,g;\alpha,k)$is closed under convex linear combinations.*

Proof. Let $f_j(z) \in H_T(f,g;\alpha,k)$ $(j=1,2)$, where

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0;\ j=1,2). \tag{26}$$

Then it is sufficient to prove that the function $h(z)$ given by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \ \ (0 \leq \mu \leq 1)$$

is also in the class $H_T(f,g;\alpha,k)$. For $0 \leq \mu \leq 1$

$$h(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

and with the aid of Theorem 2.2, we have

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) b_n \cdot [\mu a_{n,1} + (1-\mu) a_{n,2}]$$
$$\leq \mu(1-\alpha) + (1-\mu)(1-\alpha) = 1-\alpha,$$

which implies that $h(z) \in H_T(f,g;\alpha,k)$. This completes the proof of Theorem 4.1. □

As a consequence of Theorem 4.1, there exist extreme points of the class $H_T(f,g;\alpha,k)$, which are given by:

Theorem 4.2. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(1-\alpha)}{(kn^2 + n - kn - \alpha) b_n} z^n.$$

Then $f(z)$ is in the class $H_T(f,g;\alpha,k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{27}$$

where $\mu_n \geq 0$ $(n \geq 1)$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z)=z-\sum_{n=2}^{\infty}\frac{(1-\alpha)}{(kn^2+n-kn-\alpha)\,b_n}\mu_n z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty}\frac{\left(kn^2+n-kn-\alpha\right)b_n}{(1-\alpha)}\frac{(1-\alpha)}{(kn^2+n-kn-\alpha)\,b_n}\mu_n$$
$$=\sum_{n=2}^{\infty}\mu_n=(1-\mu_1)\le 1. \quad (28)$$

So, by Theorem 2.2, we have $f(z)\in H_T(f,g;\alpha,k)$.
Conversely, assume that the function $f(z)$ defined by (4) belongs to the class $H_T(f,g;\alpha,k)$. Then a_n are given by (14). Setting

$$\mu_n=\frac{\left(kn^2+n-kn-\alpha\right)b_n}{(1-\alpha)}a_n \quad (29)$$

and

$$\mu_1=1-\sum_{n=2}^{\infty}\mu_n,$$

we can see that $f(z)$ can be expressed in the form (27). This completes the proof of Theorem 4.2. □

Corollary 4.3. *The extreme points of the class $H_T(f,g;\alpha,k)$ are the functions $f_1(z)=z$ and*

$$f_n(z)=z-\frac{(1-\alpha)}{(kn^2+n-kn-\alpha)\,b_n}z^n\,(n\ge 2).$$

5. Partial sums

In this section, applying methods used by Silverman [21], we investigate the ratio of a function of the form (1) to its sequence of partial sums $f_m(z)=z+\sum_{n=2}^{m}a_n z^n$. More precisely, we will determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_m(z)}\right\}$, $Re\left\{\frac{f_m(z)}{f(z)}\right\}$, $Re\left\{\frac{f'(z)}{f'_m(z)}\right\}$ and $Re\left\{\frac{f'_m(z)}{f'(z)}\right\}$. In the sequel, we will make use of the well-known result that $Re\left\{\frac{1+w(z)}{1-w(z)}\right\}>0\,(z\in U)$ if and only if $w(z)=\sum_{n=1}^{\infty}c_n z^n$ satisfies the inequality $|w(z)|\le|z|$.

Theorem 5.1. *If $f(z)$ is of the form (1) and satisfies the condition (12) and $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, then*

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{C_{m+1}} \tag{30}$$

and

$$C_n \geq \left\{\begin{array}{cc} 1 & n = 2,3,\ldots,m \\ C_{m+1} & n = m+1, m+2,\ldots \end{array}\right. , \tag{31}$$

where

$$C_n = \frac{\left(kn^2 + n - kn - \alpha\right) b_n}{(1-\alpha)}. \tag{32}$$

The result in (30) is sharp for every m, with the extremely function

$$f(z) = z + \frac{z^{m+1}}{C_{m+1}}. \tag{33}$$

Proof. We may write

$$\frac{1+w(z)}{1-w(z)} = C_{m+1}\left\{\frac{f(z)}{f_m(z)} - (1 - \frac{1}{C_{m+1}})\right\}$$

$$= \left\{\frac{1 + \sum_{n=2}^{m} a_n z^{n-1} + C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}}\right\}. \tag{34}$$

Then

$$w(z) = \frac{C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{m} a_n z^{n-1} + C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{C_{m+1} \sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{m} |a_n| - C_{m+1} \sum_{n=m+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leq 1$ if

$$2C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 2 - 2\sum_{n=2}^{m} |a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m} |a_n| + C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 1. \tag{35}$$

It is suffices to show that the left hand side of (35) is bounded above by $\sum_{n=2}^{\infty} C_n |a_n|$, which is equivalent to

$$\sum_{n=2}^{m} (C_n - 1) |a_n| + \sum_{n=m+1}^{\infty} (C_n - C_{m+1}) |a_n| \geq 0.$$

To see that the function f given by (33) gives the sharp result, we observe for $z = re^{i\pi/n}$ that

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{C_{m+1}}. \tag{36}$$

Letting $z \longrightarrow 1^-$, we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{1}{C_{m+1}}.$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. *If $f(z)$ is of the form (1) and satisfies the condition (12) and $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, then*

$$Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{C_{m+1}}{1 + C_{m+1}}.$$

The result is sharp for every m, with the extremely function $f(z)$ given by (33).

Proof. We may write

$$\frac{1 + w(z)}{1 - w(z)} = (1 + C_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \frac{C_{m+1}}{1 + C_{m+1}} \right\}$$

$$= \left\{ \frac{1 + \sum_{n=2}^{m} a_n z^{n-1} - C_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right\}, \tag{37}$$

where

$$w(z)=\frac{(1+C_{m+1})\sum\limits_{n=m+1}^{\infty}a_nz^{n-1}}{2+2\sum\limits_{n=2}^{m}a_nz^{n-1}+(C_{m+1}-1)\sum\limits_{n=m+1}^{\infty}a_nz^{n-1}},$$

and

$$|w(z)|\leq\frac{(1+C_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{m}|a_n|-(C_{m+1}-1)\sum\limits_{n=m+1}^{\infty}|a_n|}.$$

Now $|w(z)|\leq 1$ if and only if

$$2C_{m+1}\sum_{n=m+1}^{\infty}|a_n|\leq 2-2\sum_{n=2}^{m}|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m}|a_n|+C_{m+1}\sum_{n=m+1}^{\infty}|a_n|\leq 1. \tag{38}$$

It is suffices to show that the left hand side of (38) is bounded above by $\sum\limits_{n=2}^{\infty}C_n|a_n|$, which is equivalent to

$$\sum_{n=2}^{m}(C_n-1)|a_n|+\sum_{n=m+1}^{\infty}(C_n-C_{m+1})|a_n|\geq 0.$$

This completes the proof of Theorem 5.2. □

Theorem 5.3. *If $f(z)$ is of the form (1) and satisfies the condition (12) and $\frac{f(z)}{z}\neq 0$ $(0<|z|<1)$, then*

$$(a)\qquad Re\left\{\frac{f'(z)}{f_m'(z)}\right\}\geq 1-\frac{m+1}{C_{m+1}} \tag{39}$$

and

$$(b)\qquad Re\left\{\frac{f_m'(z)}{f'(z)}\right\}\geq\frac{C_{m+1}}{1+m+C_{m+1}}, \tag{40}$$

where

$$C_n\geq\begin{cases}1 & n=1,2,3,\ldots,m\\ n\dfrac{C_{m+1}}{m+1} & n=m+1,m+2,\ldots\end{cases}$$

and C_n is defined by(32). The estimates in (39) and (40) are sharp with the extremely function given by (33).

Proof. We prove only (a), which is similar in spirit of the proof of Theorem 5.1. The proof of (b) follows the pattern of that in Theorem 5.2. We write

$$\frac{1+w(z)}{1-w(z)} = C_{m+1}\left\{\frac{f'(z)}{f'_m(z)} - (1-\frac{1+m}{C_{m+1}})\right\}$$
$$= \left\{\frac{1+\sum\limits_{n=2}^{m} na_n z^{n-1} + \frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{m} na_n z^{n-1}}\right\},$$

where

$$w(z) = \frac{\frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{2+2\sum\limits_{n=2}^{m} na_n z^{n-1} + \frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{m} n|a_n| - \frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} n|a_n|}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n|a_n| + \frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n|a_n| \leq 1, \tag{41}$$

since the left hand side of (41) is bounded above by $\sum\limits_{n=2}^{\infty} C_n|a_n|$, this completes the proof of Theorem 5.3. □

6. Integral means

In [19] Silverman found that the function $f_2 = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured and settled in [20]:

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\delta} d\theta,$$

for all $f \in T$, $\delta > 0$ and $0 < r < 1$. In [19], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T, where $C(\alpha)$ *and* $T^*(\alpha)$ *are* the classes of convex and starlike functions of order α, $0 \leq \alpha < 1$, respectively.

In this section, we prove Silverman's conjecture for functions in the class $H_T(f,g;\alpha,k)$.

Lemma 6.1 ([12]). *If the functions fand g are analytic in U with $g \prec f$, then for $\delta > 0$ and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta.$$

Applying Theorems 2.1, 2.2 and Lemma 6.1 we prove the following theorem.

Theorem 6.2. *Suppose $f(z) \in H_T(f,g;\alpha,k), \delta > 0$, the sequence $\{b_n\}$ $(n \geq 2)$ is non-decreasing and $f_2(z)$ is defined by:*

$$f_2(z) = z - \frac{1-\alpha}{(2k+2-\alpha)b_2} z^2, \tag{42}$$

then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\delta} d\theta. \tag{43}$$

Proof. For $f(z)$ of the form (10) (43) is equivalent to prove that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\alpha)}{(2k+2-\alpha)b_2} z\right|^{\delta} d\theta.$$

By using Lemma 6.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{(1-\alpha)}{(2k+2-\alpha)b_2} z. \tag{44}$$

Setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{(1-\alpha)}{(2k+2-\alpha)b_2} w(z), \tag{45}$$

and using (14) and the hypothesis $\{b_n\}$ $(n \geq 2)$ is non-decreasing, we obtain

$$\begin{aligned}|w(z)| &= \left|\frac{(2k+2-\alpha)b_2}{(1-\alpha)}\sum_{n=2}^{\infty} a_n z^{n-1}\right| \\ &\leq |z|\sum_{n=2}^{\infty}\frac{(2k+2-\alpha)b_2}{(1-\alpha)}a_n \\ &\leq |z|\sum_{n=2}^{\infty}\frac{\left(kn^2+n-kn-\alpha\right)b_n}{(1-\alpha)}a_n \\ &\leq |z|\,.\end{aligned}$$

This completes the proof of Theorem 6.2. □

7. Neighborhood for the class $H_T(f,g;\alpha,k)$

In [8], Goodman and in [18], Ruscheweyh defined the δ- neighborhood of function T by

$$N_\delta(f) = \left\{h \in T : h(z) = z - \sum_{n=2}^{\infty} c_n z^n,\ \sum_{n=2}^{\infty} n\,|a_n - c_n| \leq \delta\right\}. \tag{46}$$

In particular, if

$$e(z) = z, \tag{47}$$

we immediately have

$$N_\delta(e) = \left\{h \in T : h(z) = z - \sum_{n=2}^{\infty} c_n z^n,\ \sum_{n=2}^{\infty} n\,|c_n| \leq \delta\right\}. \tag{48}$$

Theorem 7.1. *If* $b_n \geq b_2$ $(n \geq 2)$ *and*

$$\delta = \frac{2(1-\alpha)}{(2k+2-\alpha)b_2}, \tag{49}$$

then

$$H_T(f,g;\alpha,k) \subset N_\delta(e) \tag{50}$$

Proof. Let $f \in H_T(f,g;\alpha,k)$. Then, in view of the assertion (14) of Theorem 2.2 and the given condition that $b_n \geq b_2$ $(n \geq 2)$, we have

$$(2k+2-\alpha)b_2\sum_{n=2}^{\infty}a_n \leq \sum_{n=2}^{\infty}\left(kn^2+n-kn-\alpha\right)b_na_n$$
$$\leq (1-\alpha),$$

so that

$$\sum_{n=2}^{\infty}a_n \leq \frac{(1-\alpha)}{(2k+2-\alpha)b_2}. \tag{51}$$

Making use of (14) again, in conjunction with (51), we get

$$\begin{aligned} b_2\sum_{n=2}^{\infty}na_n &\leq (1-\alpha)+(\alpha-2k)\,b_2\sum_{n=2}^{\infty}a_n \\ &\leq (1-\alpha)+(\alpha-2k)\,b_2\frac{(1-\alpha)}{(2k+2-\alpha)b_2} \\ &\leq \frac{2\,(1-\alpha)}{(2k+2-\alpha)}. \end{aligned}$$

Hence

$$\sum_{n=2}^{\infty}na_n \leq \frac{2\,(1-\alpha)}{(2k+2-\alpha)b_2}=\delta, \tag{52}$$

which, by means of the definition (48). This completes the proof of Theorem 7.1. □

Now we determine the neighborhood for the class $H_T^{(\gamma)}(f,g;\alpha,k)$, which we define as follows. A function $f(z)\in T$ is said to the class $H_T^{(\gamma)}(f,g;\alpha,k)$ if there exists a function $\zeta(z)\in H_T(f,g;\alpha,k)$ such that

$$\left|\frac{f(z)}{\zeta(z)}-1\right| < 1-\gamma\ (0\leq\gamma<1) \tag{53}$$

Theorem 7.2. *If* $\zeta(z)\in H_T(f,g;\alpha,k)$ *and*

$$\gamma = 1-\frac{\delta(2k+2-\alpha)b_2}{2\left[(2k+2-\alpha)b_2-(1-\alpha)\right]} \tag{54}$$

then

$$N_\delta(\zeta)\subset H_T^{(\gamma)}(f,g;\alpha,k) \tag{55}$$

where

$$\delta \leq 2-2\,(1-\alpha)\left[(2k+2-\alpha)b_2\right]^{-1}. \tag{56}$$

Proof. Suppose that $\zeta(z) \in N_\delta(\zeta)$. We find from (46) that

$$\sum_{n=2}^{\infty} n\,|a_n - c_n| \leq \delta, \tag{57}$$

which readily implies that

$$\sum_{n=2}^{\infty} |a_n - c_n| \leq \frac{\delta}{2}. \tag{58}$$

Next, since $\zeta(z) \in H_T(f,g;\alpha,k)$, we have [cf. equation (51)] that

$$\sum_{n=2}^{\infty} c_n \leq \frac{(1-\alpha)}{(2k+2-\alpha)b_2}, \tag{59}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{\zeta(z)} - 1\right| &\leq \frac{\sum\limits_{n=2}^{\infty} |a_n - c_n|}{1 - \sum\limits_{n=2}^{\infty} c_n} \\ &\leq \frac{\delta}{2} \frac{(2+2k-\alpha)\,b_2}{[(2k+2-\alpha)b_2 - (1-\alpha)]} \\ &= 1-\gamma,\end{aligned}$$

thus, by the above definition, $f(z) \in H_T^{(\gamma)}(f,g;\alpha,k)$ for γ given by (54). This completes the proof of Theorem 7.2. $\square$

Remark 7.3. (i) Taking $g(z) = z + \sum\limits_{n=2}^{\infty} \sigma_n z^n \ (q,s \in \mathbb{N}, k \geq 0, 0 \leq \alpha < 1)$, where σ_n given by (5), in the above results we obtain the corresponding results for the class $TW_s^q(\alpha,k)$, we obtain the results obtained by Dziok and Murugusundaramoorthy (see [5]);
(ii) Taking $g(z) = \frac{z}{1-z}$ and $g(z) = z + \sum\limits_{n=2}^{\infty} \left(\frac{l+1+\mu(n-1)}{l+1}\right)^m z^n$, respectively, in the above results we obtain the corresponding results for the classes $S_T(k,\alpha)$ and $T\mathfrak{L}_m(\mu,l,\alpha,k)$, respectively.

Acknowledgements

The authors would like to thank the referees of the paper for their helpful suggestions.

REFERENCES

[1] B. C. Carlson - D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, J. Math. Anal. Appl. 15 (1984), 737–745.

[2] A. Cătaş - G. I. Oros - G. Oros, *Differential subordinations associated with multiplier transformations*, Abstract Appl. Anal. 2008 (2008), ID 845724, 1–11.

[3] N. E. Cho - O. S. Kwon - H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl. 292 (2004), 470–483.

[4] J. H. Choi - M. Saigo - H. M. Srivastava, *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl. 276 (2002), 432–445.

[5] J. Dziok - G. Murugusundaramoorthy, *A generalized class of starlike functions associated with the wright hypergeometric function*, Mat. Vesnik 62 (4) (2010), 271–283.

[6] J. Dziok - R. K. Raina, *Families of analytic functions associated with the Wright's generalized hypergeometric function*, Demonstratio Math. 37 (2004), 533–542.

[7] J. Dziok - H. M. Srivastava, *Classes of analytic functions with the generalized hypergeometric function*, Appl. Math. Comput. 103 (1999), 1–13.

[8] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. 8 (3) (1957), 598–601.

[9] Z. Lewandowski - S. S. Miller - E. Zlotkiewicz, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. 65 (1976), 111–117.

[10] J. L. Li - S. Owa, *Sufficient conditions for starlikness*, Indian. J. Pure Appl. Math. 33 (2002), 313–318.

[11] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. 16 (1969), 755–758.

[12] J. L. Littlewood, *On inequalities in the theory functions*, Proc. London Math. Soc. 23 (1925), 481–519.

[13] M. S. Liu - Y. C. Zhu - H. M. Srivastava, *Properties and characteristics of certain subclasses of starlike functions of order* β, Math. Comput. Modelling 48 (2008), 402–419.

[14] K. I. Noor, *On new classes of integral operators*, J. Natur. Geom. 16 (1999), 71–80.

[15] S. Owa - H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. 39 (1987), 1057–1077.

[16] C. Ramesha - S. Kumar - K. S. Padmanabhan, *A sufficient condition for starlikeness*, Chinese J. Math. 33 (1995), 167–171.

[17] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49 (1975), 109–115.

[18] St. Ruscheweyh, *Neighbohoods of univalent functions*, Proc. Amer. Math. Soc. 81 (1981), 521–527.

[19] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51 (1975), 109–116.

[20] H. Silverman, *Integral means for univalent functions with negative coefficients*, Houston J. Math. 23 (1997), 169–174.

[21] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. 209 (1997), 221–227.

[22] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London. Math. Soc. 46 (1946), 389–408.

MOHAMED KAMAL AOUF
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

ADELA O. MOSTAFA
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
e-mail: adelaeg254@yahoo.com

EMAN AHMED ADWAN
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
e-mail: eman.a2009@yahoo.com