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SOME STRUCTURE THEOREMS ON LOCALLY CONVEX CONES OF LINEAR OPERATORS

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In this paper we investigate the structure of $\mathcal{C}(\mathcal{P},\mathcal{Q})$ (the cone of all continuous linear operators from locally convex cone $(\mathcal{P},\mathfrak{U})$ into locally convex cone $(\mathcal{Q},\mathcal{W})$), when $(\mathcal{P},\mathfrak{U})$ or $(\mathcal{Q},\mathcal{W})$ are inductive or projective limit locally convex cones. We consider some special convex quasiuniform structures on $\mathcal{C}(\mathcal{P},\mathcal{Q})$, and prove some structure theorems.

1. Introduction

The theory of locally convex cones as developed in [5] and [13] uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the later. For recent researches see [1–3, 9, 12].

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a+\beta a$, $\alpha(a+b) = \alpha a+\alpha b$, $1a = a$ and $0a = 0$ for all $a,b \in \mathcal{P}$ and $\alpha,\beta \geq 0$.

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Let $\mathcal{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:
(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a,a) : a \in \mathcal{P}\}$);
(U_2) for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu) U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a,b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a,c) \in U$ and $(c,b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone if

(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0,a) \in \rho U$.

We say that the convex subset E of $\mathcal{P}^2$ is uniformly convex whenever E has properties (U1) and (U3). The uniformly convex subsets play an important role in the construction of a convex quasiuniform structure. With every collection of uniformly convex subsets we can obtain a convex quasiuniform structure (see [1], Proposition 2.2). With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b,a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in P : (a,b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ if for every $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

In locally convex cone $(\mathcal{P}, \mathfrak{U})$ the *closure* of $a \in \mathcal{P}$ is defined to be the set

$$\overline{a} = \bigcap_{U \in \mathfrak{U}} U(a)$$

(see [5], chapter I). The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called *separated* if $\overline{a} = \overline{b}$ implies $a = b$ for $a, b \in \mathcal{P}$. It is proved in [5] that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ is separated if and only if its symmetric topology is Hausdorff.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. We set $\tilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a,b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}.$$

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a+\varepsilon]$ are the upper and the intervals $[a-\varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a)+T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$, where $(T \times T)(U) = \{(T(a), T(b)) \in \mathcal{Q}^2 : (a, b) \in U\}$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. We shall say that the subset F of $\mathcal{P}^2$ is u-bounded if it is absorbed by each $U \in \mathfrak{U}$. The subset B of $\mathcal{P}$ is called bounded below (or above) whenever $\{0\} \times B$ (or $B \times \{0\}$) is u-bounded. The subset B is called bounded if it is bounded below and above. An element $a \in \mathcal{P}$ is called bounded below (or above) whenever $\{a\}$ is so (recall that every $a \in \mathcal{P}$ is required to be bounded below by (U_5)).

The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called a *uc*-cone whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$. It is proved in [1] that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ is a *uc*-cone if and only if $\mathfrak{U}$ has a u-bounded element.

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. The linear operator $T : \mathcal{P} \to \mathcal{Q}$ is called u-bounded whenever for every u-bounded subset B of $\mathcal{P}^2$, $(T \times T)(B)$ is u-bounded. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called bornological if every u-bounded linear operator from $(\mathcal{P}, \mathfrak{U})$ into any locally convex cone is continuous.

The projective and inductive limits of locally convex cones have been investigated in [10]. Also, the strict inductive limit of locally convex cones has been defined in [9]. The products and direct sums as a special case of projective and inductive limits have been investigated in [8]. The dual of projective and inductive limits of locally convex cones have been investigated in [7]. In this paper we want to study the structure of $\mathcal{C}(\mathcal{P}, \mathcal{Q})$ (the cone of continuous linear operators), when $(\mathcal{P}, \mathfrak{U})$ or $(\mathcal{Q}, \mathcal{W})$ are the inductive or projective limit locally convex cones. The structure of $\mathcal{C}(\mathcal{P}, \mathcal{Q})$, when $\mathcal{P}$ or $\mathcal{Q}$ are products or direct sums of some locally convex cones is an interesting special case that investigated in this paper. We review some results from [10]. For every $\gamma \in \Gamma$ let $(\mathcal{P}_\gamma, \mathfrak{U}_\gamma)$ be a locally convex cone. If $\mathcal{P}$ is a cone and for every $\gamma \in \Gamma$, u_γ is a linear mapping of $\mathcal{P}$ into $\mathcal{P}_\gamma$, then there is a coarsest convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ that makes all u_γ continuous. $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone and it is called the projective limit of the locally convex cones $(\mathcal{P}_\gamma, \mathfrak{U}_\gamma)$, $\gamma \in \Gamma$. If $\mathcal{P} = \prod_{\gamma\in\Gamma} \mathcal{P}_\gamma$, then $\mathcal{P}$ can be made into a locally convex cone by regarding it as the projective limit of the locally convex cones $(\mathcal{P}_\gamma, \mathfrak{U}_\gamma)$ by the projections mapping $\pi_\gamma : \mathcal{P} \to \mathcal{P}_\gamma, \pi_\gamma((x_\gamma)_{\gamma\in\Gamma}) = x_\gamma$.

For each $\gamma \in \Gamma$, let $(\mathcal{P}_\gamma, \mathfrak{U}_\gamma)$ be a locally convex cone. Suppose $\mathcal{P}$ is a cone and for every $\gamma \in \Gamma$, $v_\gamma : \mathcal{P}_\gamma \to \mathcal{P}$ is a linear mapping such that $\mathcal{P} = span(\bigcup_{\gamma\in\Gamma} v_\gamma(\mathcal{P}_\gamma))$. Then there is the finest convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ that makes all v_γ continuous. $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone and it is called the inductive limit of locally convex cones $(\mathcal{P}_\gamma, \mathfrak{U}_\gamma)$, $\gamma \in \Gamma$. The subcone of $\mathcal{P} = \prod_{\gamma\in\Gamma} \mathcal{P}_\gamma$ spanned by $\bigcup_{\gamma\in\Gamma} j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma : \mathcal{P}_\gamma \to \prod_{\gamma\in\Gamma} \mathcal{P}_\gamma$ is the injection mapping, is called the direct sum of cones $\mathcal{P}_\gamma$, $\gamma \in \Gamma$ and denoted by $\bigoplus_{\gamma\in\Gamma} \mathcal{P}_\gamma$. If we consider the product convex quasiunifom structure on $\bigoplus_{\gamma\in\Gamma} \mathcal{P}_\gamma$, then it induces the original convex quasiunifom structure on each $\mathcal{P}_\gamma$. The finest such convex quasiunifom structure on $\bigoplus_{\gamma\in\Gamma} \mathcal{P}_\gamma$ is obtained by regarding $\bigoplus_{\gamma\in\Gamma} \mathcal{P}_\gamma$ as the inductive limit of locally convex cones $(\mathcal{P}_\gamma, \mathfrak{U}_\gamma)$, $\gamma \in \Gamma$ (see [8]).

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone and $\mathcal{P}^*$ be its dual. In the following we denote by $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*)$ the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $\mu \in \mathcal{P}^*$ continuous. Similarly, $\mathfrak{U}_\sigma(\mathcal{P}^*, \mathcal{P})$ is the the coarsest convex quasiuniform structure that makes all $a \in \mathcal{P}$ continious, as linear functionals on $\mathcal{P}^*$. In fact, $(\mathcal{P}, \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*))$ is the projective limit of $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ by the functionals $\mu \in \mathcal{P}^*$.

2. Some structure theorems

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. We denote the cone of all continuous linear operators from $\mathcal{P}$ into $\mathcal{Q}$ by $\mathcal{C}(\mathcal{P}, \mathcal{Q})$. If $(\mathcal{Q}, \mathcal{W}) = (\overline{\mathbb{R}}, \mathcal{V})$, then $\mathcal{C}(\mathcal{P}, \mathcal{Q}) = \mathcal{P}^*$. We define a convex quasiuniform structure on $\mathcal{C}(\mathcal{P}, \mathcal{Q})$. Let $\mathcal{B}$ be a collection of bounded below subsets of $(\mathcal{P}, \mathfrak{U})$ such that

$$\text{for every } A, B \in \mathcal{B} \text{ there is } C \in \mathcal{B} \text{ such that } A \cup B \subseteq C. \qquad (UW)$$

For $B \in \mathcal{B}$ and $W \in \mathcal{W}$ we set

$$V_{B,W} = \{(S,T) \in \mathcal{C}(\mathcal{P}, \mathcal{Q}) \times \mathcal{C}(\mathcal{P}, \mathcal{Q}) : (S(b), T(b)) \in W\}.$$

Then $\mathcal{V}_{\mathcal{B},\mathcal{W}} = \{V_{B,W} : B \in \mathcal{B}, W \in \mathcal{W}\}$ is a convex quasiuniform structure on $\mathcal{C}(\mathcal{P}, \mathcal{Q})$. We prove that the elements of $\mathcal{C}(\mathcal{P}, \mathcal{Q})$ are bounded below with respect to the convex quasiuniform structure $\mathcal{V}_{\mathcal{B},\mathcal{W}}$. Let $V_{B,W} \in \mathcal{V}_{\mathcal{B},\mathcal{W}}$ and $T \in \mathcal{C}(\mathcal{P}, \mathcal{Q})$. Since B is bounded below and T is continuous, we realize that $T(B)$ is bounded below in $(\mathcal{Q}, \mathcal{W})$. Then there is $\lambda > 0$ such that $(0, T(b)) \in \lambda W$ for all $b \in B$. This shows that $(0, T) \in \lambda V_{B,W}$. Therefore $(\mathcal{C}(\mathcal{P}, \mathcal{Q}), \mathcal{V}_{\mathcal{B},\mathcal{W}})$ is a locally convex cone.

Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones. If $\mathcal{B}$ is the collection of all bounded below or bounded subsets of $(\mathcal{P},\mathfrak{U})$, then we denote the corresponding convex quasiuniform structure on $\mathcal{C}(\mathcal{P},\mathcal{Q})$ by $\mathcal{V}_{b\beta}$ or $\mathcal{V}_{\beta}$. Obviously, $\mathcal{V}_{b\beta}$ is finer than $\mathcal{V}_{\beta}$, since every bounded subset of $\mathcal{P}$ is bounded below.

Proposition 2.1. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be uc-cones. Then $\mathcal{C}(\mathcal{P},\mathcal{Q})$ is a uc-cone if endowed with the convex quasiuniform structures $\mathcal{V}_{b\beta}$ and $\mathcal{V}_{\beta}$.*

Proof. Let $\mathfrak{U}=\{\alpha U:\alpha>0\}$ and $\mathcal{W}=\{\alpha W:\alpha>0\}$. We set $B=(0)U$. We shall prove that $\mathcal{V}_{b\beta}$ is equivalent to the convex quasiuniform structure $\{\varepsilon V_{B,W}:\varepsilon>0\}$. It is enough to show that $\{\varepsilon V_{B,W}:\varepsilon>0\}$ is finer than $\mathcal{V}_{b\beta}$. Let $V_{A,\alpha W}\in\mathcal{V}_{b\beta}$. Then we have $A\subseteq\lambda B$ for some $\lambda>0$. We claim that $\frac{\alpha}{\lambda}V_{B,W}\subseteq V_{A,\alpha W}$. Let $(S,T)\in\frac{\alpha}{\lambda}V_{B,W}$. Then $(S(\lambda a),T(\lambda a))\in\alpha W$ for all $a\in B$. If we set $b=\lambda a$, then we have $(S(b),T(b))\in\alpha W$ for all $b\in\lambda B$. Since $A\subseteq\lambda B$, this shows that $(S(b),T(b))\in\alpha W$ for all $b\in A$. Therefore $(S,T)\in V_{A,\alpha W}$.

In a similar way one can prove that $\mathcal{V}_{\beta}$ is equivalent to the convex quasiuniform structure $\{\alpha V_{B',W}:\alpha>0\}$, where $B'=U(0)U$. □

Example 2.2. Suppose that $(\mathcal{P},\mathfrak{U})=(\mathcal{Q},\mathcal{W})=(\overline{\mathbb{R}},\tilde{\mathcal{V}})$. Then $\mathcal{C}(\overline{\mathbb{R}},\overline{\mathbb{R}})=\overline{\mathbb{R}}^{*}=[0,+\infty)\cup\{\overline{0}\}$, where $\overline{0}$ is a functional on $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$ acting as follows:

$$\overline{0}(a)=\begin{cases}0 & a\in\mathbb{R}\\ +\infty & else.\end{cases}$$

In this example we have $\mathcal{V}_{b\beta}=\{\alpha V_{[-1,+\infty],\tilde{1}}:\alpha>0\}$ and $\mathcal{V}_{\beta}=\{\alpha V_{[-1,+1],\tilde{1}}:\alpha>0\}$ by Proposition 2.1. The upper, lower and symmetric neighborhoods of $\overline{0}$ in $(\mathcal{C}(\overline{\mathbb{R}},\overline{\mathbb{R}}),\mathcal{V}_{b\beta})$ are as follows:

$$V_{[-1,+\infty],\tilde{1}}(\overline{0})=\{0,\overline{0}\},\ (\overline{0})V_{[-1,+\infty],\tilde{1}}=\{\overline{0}\}\text{ and }V_{[-1,+\infty],\tilde{1}}(\overline{0})V_{[-1,+\infty],\tilde{1}}=\{\overline{0}\}.$$

Then the functional $\overline{0}$ is an isolated point in the lower and symmetric topologies of $(\mathcal{C}(\overline{\mathbb{R}},\overline{\mathbb{R}}),\mathcal{V}_{b\beta})$. Similarly in $(\mathcal{C}(\overline{\mathbb{R}},\overline{\mathbb{R}}),\mathcal{V}_{\beta})$ we have

$$V_{[-1,+1],\tilde{1}}(\overline{0})=\{0,\overline{0}\},\ (\overline{0})V_{[-1,+1],\tilde{1}}=\{0,\overline{0}\}\text{ and }V_{[-1,+1],\tilde{1}}(\overline{0})V_{[-1,+1],\tilde{1}}=\{0,\overline{0}\}.$$

We shall say that a subset H of $\mathcal{C}(\mathcal{P},\mathcal{Q})$ is equicontinuous whenever for each $W\in\mathcal{W}$ there is $U\in\mathfrak{U}$ such that $(S\times S)(U)\subseteq W$ for all $S\in H$. Every equicontinuous subset H of $\mathcal{C}(\mathcal{P},\mathcal{Q})$ is bounded below with respect to the convex quasiuniform structure $\mathcal{V}_{\mathcal{B},\mathcal{W}}$. Indeed, let $V_{B,W}\in\mathcal{V}_{\mathcal{B},\mathcal{W}}$. Then there is $U\in\mathfrak{U}$ such that $(S\times S)(U)\subseteq W$ for all $S\in H$. Also, there is $\lambda>0$ such that $(\{0\}\times B)\subseteq\lambda U$, since B is bounded below in $(\mathcal{P},\mathfrak{U})$. We claim that $\{0\}\times H\subseteq\lambda V_{B,W}$. Let $S\in H$. Then $(S\times S)(U)\subseteq W$. This shows that $(S\times S)(\frac{1}{\lambda}(\{0\}\times B))\subseteq W$, since $\frac{1}{\lambda}(\{0\}\times B)\subseteq U$. Therefore $(0,\frac{1}{\lambda}S(b))\in W$ for all $b\in B$, yields $(0,S)\in\lambda V_{B,W}$.

Proposition 2.3. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones and let $\mathcal{B}$ be a collection of bounded below subsets of $\mathcal{P}$ which has property (UW). If $\mathcal{P} = \bigcup_{B\in\mathcal{B}} B$, then for every $a \in \mathcal{P}$ the linear operator $\delta_a : \mathcal{C}(\mathcal{P},\mathcal{Q}) \to \mathcal{Q}, \delta_a(T) = T(a)$ is continuous.*

Proof. Let $W \in \mathcal{W}$ and $a \in \mathcal{P}$. There is $B \in \mathcal{B}$ such that $a \in B$. We prove that $(\delta_a \times \delta_a)(V_{B,W}) \subseteq W$. Let $(S,T) \in V_{B,W}$. Then $(S(b),T(b)) \in W$ for all $b \in B$. This shows that $(S(a),T(a)) \in W$, since $a \in B$. Then $(\delta_a(S),\delta_a(T)) \in W$. This yields that $(\delta_a \times \delta_a)(S,T) \in W$. □

Theorem 2.4. *Let $(\mathcal{P},\mathfrak{U})$ be the inductive limit of the locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ by the linear mappings u_γ, $\gamma \in \Gamma$ and let $(\mathcal{Q},\mathcal{W})$ be a locally convex cone. Let $\mathcal{B}_\gamma$ be a class of bounded below subsets of $\mathcal{P}_\gamma$ for every $\gamma \in \Gamma$, which has (UW), and let $\mathcal{B}$ be the class of all finite unions of the sets contained in $\bigcup_{\gamma\in\Gamma} u_\gamma(\mathcal{B}_\gamma)$. Then $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is the projective limit of the locally convex cones $(\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q}),\mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})$ by the linear mappings $T_\gamma : \mathcal{C}(\mathcal{P},\mathcal{Q}) \to \mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q})$, $T_\gamma(A) = A \circ u_\gamma$, for $A \in \mathcal{C}(\mathcal{P},\mathcal{Q})$.*

Proof. Obviously, $\mathcal{B}$ has property (UW). Then $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is a locally convex cone. Now, we prove that $T_\gamma : (\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}}) \to (\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q}),\mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})$ is continuous for each $\gamma \in \Gamma$. Let $V_{B_\gamma,W} \in \mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}}$. We set $B = u_\gamma(B_\gamma)$. Obviously, we have $B \in \mathcal{B}$. We prove that $(T_\gamma \times T_\gamma)(V_{B,W}) \subseteq V_{B_\gamma,W}$. Let $(S,A) \in V_{B,W}$. Then $(S(b),A(b)) \in W$ for all $b \in B$. For every $b \in B$ there is $b_\gamma \in B_\gamma$ such that $b = u_\gamma(b_\gamma)$. This shows that $(S \circ u_\gamma(b_\gamma), A \circ u_\gamma(b_\gamma)) \in W$ and then $(T_\gamma(S),T_\gamma(A)) \in V_{B_\gamma,W}$. Now, let $\mathcal{H}$ be a convex quasiuniform structure on $\mathcal{C}(\mathcal{P},\mathcal{Q})$, that makes all T_γ continuous. We shall prove that $\mathcal{H}$ is finer than $\mathcal{V}_{\mathcal{B},\mathcal{W}}$. Let $B = u_\gamma(B_\gamma)$. There is $H \in \mathcal{H}$ such that $(T_\gamma \times T_\gamma)(H) \subseteq V_{B_\gamma,W}$. We show that $H \subseteq V_{B,W}$. If $(S,A) \in H$, then $(T_\gamma(S),T_\gamma(A)) \in V_{B_\gamma,W}$. Then $(S(u_\gamma(b_\gamma)),A(u_\gamma(b_\gamma))) \in W$ for all $b_\gamma \in B_\gamma$. This yields that $(S(b),A(b)) \in W$ for all $b \in B$. Therefore $(S,A) \in V_{B,W}$. □

Corollary 2.5. *Let $(\mathcal{P},\mathfrak{U}) = \bigoplus_{\gamma\in\Gamma}(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ and let $(\mathcal{Q},\mathcal{W})$ be a locally convex cone. Suppose $\mathcal{B}_\gamma$ is a class of bounded below subsets of $\mathcal{P}_\gamma$ for every $\gamma \in \Gamma$, which has property (UW) and $\mathcal{B}$ is the class of all finite unions of the sets contained in $\bigcup_{\gamma\in\Gamma} j_\gamma(\mathcal{B}_\gamma)$. Then $(\mathcal{L}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}}) = \prod_{\gamma\in\Gamma}(\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q}),\mathcal{V}_{\mathcal{B}_\gamma,\mathcal{W}})$.*

Corollary 2.6. *Let $(\mathcal{P},\mathfrak{U}) = \bigoplus_{\gamma\in\Gamma}(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$. Suppose $\mathcal{B}_\gamma$ is a class of bounded below subsets of $\mathcal{P}_\gamma$ for every $\gamma \in \Gamma$, which has property (UW) and $\mathcal{B}$ is the class of all finite unions of the sets contained in $\bigcup_{\gamma\in\Gamma} j_\gamma(\mathcal{B}_\gamma)$. Then $(\mathcal{P}^*,\mathcal{V}_{\mathcal{B},\tilde{\mathcal{V}}}) = \prod_{\gamma\in\Gamma}(\mathcal{P}^*_\gamma,\mathcal{V}_{\mathcal{B}_\gamma,\tilde{\mathcal{V}}})$.*

Example 2.7. Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones and $\sim$ be an equivalence relation on $\mathcal{P}$ which is compatible with the algebraic operations of $\mathcal{P}$ (see

[12]). We denote the equivalence class of an element $a \in \mathcal{P}$ by $[a]$ and set

$$[\mathcal{P}] = \{[a] \mid a \in \mathcal{P}\}.$$

The operations $[a]+[b] = [a+b]$ and $\alpha[a] = [\alpha a]$ are well-defined for $a,b \in \mathcal{P}$ and $\alpha \geq 0$ and $[\mathcal{P}]$ becomes a cone with these operations, which had been called the quotient cone. On $[\mathcal{P}]$ we consider the finest convex quasiuniform structure $[\mathfrak{U}]$, that makes the projection mapping $\pi : \mathcal{P} \to [\mathcal{P}], \pi(a) = [a]$ continuous. In fact, $([\mathcal{P}],[\mathfrak{U}])$ is the inductive limit of $(\mathcal{P},\mathfrak{U})$ under the projection mapping. Suppose that $\mathcal{B}$ is a collection of bounded below subsets of $\mathcal{P}$, which has property (UW) and suppose $[\mathcal{B}]$ is the collection of all finite unions of the sets contained in $\pi(\mathcal{B})$. Then $(\mathcal{C}([\mathcal{P}],\mathcal{Q}),\mathcal{V}_{[\mathcal{B}],\mathcal{W}})$ is the projective limit of the locally convex cone $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ by the linear mapping $T : \mathcal{C}([\mathcal{P}],\mathcal{Q}) \to \mathcal{C}(\mathcal{P},\mathcal{Q})$, $T(A) = A \circ \pi$ by Theorem 2.4.

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. For a uniformly convex u-bounded subset H of $\mathcal{P}^2$, we set

$$\mathcal{P}_H = \{a \in \mathcal{P} : \exists \lambda > 0, (0,a) \in \lambda H\} \text{ and } \mathfrak{U}_H = \{\alpha(H \cap \mathcal{P}_H^2) : \alpha > 0\}.$$

Then $(\mathcal{P}_H,\mathfrak{U}_H)$ is a uc-cone.

Remark 2.8. Suppose $(\mathcal{P},\mathfrak{U})$ is a bornological cone and $\mathcal{H}$ is the collection of all uniformly convex u-bounded subsets of $\mathcal{P}^2$, then it is proved in [1] that $(\mathcal{P},\mathfrak{U})$ is the inductive limit of uc-subcones $(\mathcal{P}_H,\mathfrak{U}_H)_{H\in\mathcal{H}}$, with the inclusion mappings $I_H : \mathcal{P}_H \to \mathcal{P}$. Now for every $H \in \mathcal{H}$, suppose $\mathcal{B}_H$ is a collection of bounded below subsets of $(\mathcal{P}_H,\mathfrak{U}_H)$, which has property (UW) and suppose $\mathcal{B}$ is the class of all finite unions of the sets contained in $\bigcup_{H\in\mathcal{H}} \mathcal{B}_H$. Then $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is the projective limit of the locally convex cones $(\mathcal{C}(\mathcal{P}_H,\mathcal{Q}),\mathcal{V}_{\mathcal{B}_H,\mathcal{W}})$ with the linear mappings $T_H : \mathcal{C}(\mathcal{P},\mathcal{Q}) \to \mathcal{C}(\mathcal{P}_H,\mathcal{Q})$, $T_H(A) = AoI_H$, by Theorem 2.4. If $(\mathcal{Q},\mathcal{W})$ is a uc-cone, then for every $H \in \mathcal{H}$, $(\mathcal{C}(\mathcal{P}_H,\mathcal{Q}),\mathcal{V}_{\mathcal{B}_H,W})$ is a uc- cone by Proposition 2.1. Therefore $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is the projective limit of uc-cones in this case.

Definition 2.9. Let $\mathcal{P}$ be a cone. We shall say that the subset B of $\mathcal{P}\setminus\{0\}$ is a base for $\mathcal{P}$ whenever

(1) for every $a \in \mathcal{P}$ there are $n \in \mathbb{N}$, $b_1,...,b_n \in B$ and $\alpha_1,...,\alpha_n \geq 0$ such that $a = \sum_{i=1}^n \alpha_i b_i$, in the other words $\mathcal{P} = span(B)$,

(2) for every $B' \subsetneq B$, $\mathcal{P} \neq span(B')$.

Let B be a base for the cone $\mathcal{P}$. For $b \in B$ we set $\mathcal{P}_b = \{\alpha b : \alpha \geq 0\}$. Then we have $\mathcal{P} = \bigoplus_{b\in B} \mathcal{P}_b$. Indeed, (1) shows that $\mathcal{P} \subseteq \bigoplus_{b\in B} \mathcal{P}_b$. We prove that for $b_1,b_2 \in B$, $\mathcal{P}_{b_1} \cap \mathcal{P}_{b_2} = \{0\}$. If $a \in \mathcal{P}_{b_1} \cap \mathcal{P}_{b_2}$ and $a \neq 0$, then $a = \alpha_1 b_1 = \alpha_2 b_2$ for some $\alpha_1,\alpha_2 > 0$. Then $b_2 = \frac{\alpha_1}{\alpha_2} b_1$. This shows that $\mathcal{P} = span(B\setminus\{b_1\})$.

This is a contradiction by (2). Now, we suppose that $(\mathcal{P},\mathfrak{U})$ is a locally convex cone and for $b \in B$, $\mathfrak{U}_b$ is the convex quasiuniform structure on $\mathcal{P}_b$ induced by $\mathfrak{U}$. Then it is easy to see that $(\mathcal{P},\mathfrak{U}) = \bigoplus_{b\in B}(\mathcal{P}_b,\mathfrak{U}_b)$.

Example 2.10. Let S be the cone of all sequences in $\overline{\mathbb{R}}$. For $i \in \mathbb{N}$, we define the sequences $(a^i_n)_{n\in\mathbb{N}}$, $(b^i_n)_{n\in\mathbb{N}}$ and $(c^i_n)_{n\in\mathbb{N}}$ as following:

$$a^i_n = \begin{cases} 1 & n=i \\ 0 & else \end{cases}, \; b^i_n = \begin{cases} -1 & n=i \\ 0 & else \end{cases} \text{ and } c^i_n = \begin{cases} +\infty & n=i \\ 0 & else. \end{cases}$$

Then $B = \{(a^i_n)_{n\in\mathbb{N}}, (b^i_n)_{n\in\mathbb{N}}, (c^i_n)_{n\in\mathbb{N}} : i \in \mathbb{N}\}$ is a base for S. For $\delta > 0$, we set

$$\tilde{\delta} = \{((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) \in S^2 : a_n \leq b_n + \delta, \forall n \in \mathbb{N}\}.$$

Then $\mathfrak{U} = \{\tilde{\delta} : \delta > 0\}$ is a convex quasiuniform structure on S. If $\mathcal{P}$ is the subcone of all bounded below elements of S with respect to $\mathfrak{U}$, then $(\mathcal{P},\mathfrak{U})$ is a locally convex cone. The above discussion yields that $(\mathcal{P},\mathfrak{U}) = \bigoplus_{b\in B}(\mathcal{P}_b,\mathfrak{U}_b)$. Now, let $(\mathcal{Q},\mathcal{W})$ be a locally convex cone and $\mathcal{B}_b$ be a collection of bounded below subsets of $\mathcal{P}_b$ which have (UW). If we assume that $\mathcal{B}$ is the collection of all the sets contained in $\bigcup_{b\in B}\mathcal{B}_b$, then we have

$$(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}}) = \prod_{b\in B}(\mathcal{C}(\mathcal{P}_b,\mathcal{Q}),\mathcal{V}_{\mathcal{B}_b,\mathcal{W}}), \qquad (1)$$

by Corollary 2.5. For $i \in \mathbb{N}$ and $b = (a^i_n)_{n\in\mathbb{N}}$ or $b = (b^i_n)_{n\in\mathbb{N}}$ we have $(\mathcal{P}_b,\mathfrak{U}_b)^* = [0,+\infty)$. Also for $b = (c^i_n)_{n\in\mathbb{N}}$ we have $(\mathcal{P}_b,\mathfrak{U}_b)^* = \{0,+\infty\}$. Now, formula (1) with $(\mathcal{Q},\mathcal{W}) = (\overline{\mathbb{R}},\tilde{\mathcal{V}})$ implies that

$$\begin{aligned}(\mathcal{P}^*,\mathcal{V}_{\mathcal{B},\tilde{\mathcal{V}}}) = \Big(\prod_{i=1}^{\infty}([0,+\infty),\mathcal{V}_{\mathcal{B}_b,\tilde{\mathcal{V}}})\Big) \times \Big(\prod_{i=1}^{\infty}([0,+\infty),\mathcal{V}_{\mathcal{B}_b,\tilde{\mathcal{V}}})\Big) \\ \times \Big(\prod_{i=1}^{\infty}(\{0,+\infty\},\mathcal{V}_{\mathcal{B}_b,\tilde{\mathcal{V}}})\Big).\end{aligned}$$

Lemma 2.11. In a separated locally convex cone the only bounded subcone is $\{0\}$.

Proof. Let $(\mathcal{P},\mathfrak{U})$ be a separated locally convex cone and $\mathcal{Q}$ be a bounded subcone of $\mathcal{P}$. Then for every $U \in \mathfrak{U}$ there is $\lambda > 0$ such that $(0,q) \in \lambda U$ and $(q,0) \in \lambda U$ for all $q \in \mathcal{Q}$. Let $q \in \mathcal{Q}$ be a fixed element. We have $(0,nq) \in \lambda U$ and $(nq,0) \in \lambda U$ for all $n \in \mathbb{N}$, since $\mathcal{Q}$ is a subcone. This yields that

$$q \in \bigcap_{n\in\mathbb{N}}(\frac{\lambda}{n}U)(0)(\frac{\lambda}{n}U).$$

Therefore $q = 0$, since the symmetric topology of $(\mathcal{P},\mathfrak{U})$ is Hausdorff. □

The situation is more telling if we assume $(\mathcal{P},\mathfrak{U})$ to be a projective limit locally convex cone. We suppose first that $(\mathcal{P},\mathfrak{U})=\prod_{\gamma\in\Gamma}(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ and $(\mathcal{Q},\mathcal{W})$ is a locally convex cone. Let $S\in\mathcal{C}(\mathcal{P},\mathcal{Q})$. If S_γ is the restriction of S to $\mathcal{P}_\gamma$ and p_γ is the projection mapping, then for $(a_\gamma)_{\gamma\in\Gamma}\in\mathcal{P}$ we have $S_\gamma(a_\gamma)=S\circ p_\gamma((a_\gamma)_{\gamma\in\Gamma})$ and $S_\gamma\circ p_\gamma=S\circ p_\gamma\in\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q})$. If only finitely many S_γ are non zero, then $\sum_{i=1}^n S_{\gamma_i}\in\bigoplus_{\gamma\in\Gamma}\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q})$ and $S=\sum_{i=1}^n S_{\gamma_i}\circ p_{\gamma_i}\in\mathcal{C}(\mathcal{P},\mathcal{Q})$. This shows that

$$\bigoplus_{\gamma\in\Gamma}\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q})\subset\mathcal{C}(\prod_{\gamma\in\Gamma}\mathcal{P}_\gamma,\mathcal{Q}).$$

Generally $\bigoplus_{\gamma\in\Gamma}\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q})$ is a proper subset of $\mathcal{C}(\prod_{\gamma\in\Gamma}\mathcal{P}_\gamma,\mathcal{Q})$. For example consider the cone $\mathcal{P}=\prod_{i=1}^\infty\mathcal{P}_i$, where $\mathcal{P}_i=\overline{\mathbb{R}}$ for all $i\in\mathbb{N}$. Then the range of every linear operator $T\in\bigoplus_{i=1}^n\mathcal{C}(\mathcal{P}_i,\mathcal{P})$ has a base with finite elements, but it is not true for the identity mapping $I\in\mathcal{C}(\mathcal{P},\mathcal{P})$.

Under an additional condition we have the equality in the above.

Proposition 2.12. *Let $(\mathcal{P},\mathfrak{U})=\prod_{\gamma\in\Gamma}(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$, where all elements of $\mathcal{P}_\gamma$ are bounded above for all $\gamma\in\Gamma$. Also, let $(\mathcal{Q},\mathcal{W})$ be a separated locally convex cone with a sequence $C_1\subset C_2\subset\ldots$ of bounded subsets such that every bounded subset of $\mathcal{Q}$ contained in some C_i, $i\in\mathbb{N}$. Then*
(a) Algebricaly, we have

$$\mathcal{C}(\mathcal{P},\mathcal{Q})=\bigoplus_{\gamma\in\Gamma}\mathcal{C}(\mathcal{P}_\gamma,\mathcal{Q}).$$

(b) If for every $\gamma\in\Gamma$, $\mathcal{B}_\gamma$ is a collection of bounded below subsets of $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ and $\mathcal{B}$ is the collection af all sets $\prod_{\gamma\in\Gamma}B_\gamma$, where $B_\gamma\in\mathcal{B}_\gamma$, then the iductive limit convex quasiuniform structure on $\mathcal{C}(\mathcal{P},\mathcal{Q})$ is finer than $\mathcal{V}_{\mathcal{B},\mathcal{W}}$.

Proof. For (a) assume that there exists $S\in\mathcal{C}(\mathcal{P},\mathcal{Q})$ such that

$$S\notin\bigoplus_{\gamma\in\Gamma}\mathcal{C}(\mathcal{P}_\gamma,\mathfrak{U}_\gamma).$$

Then there are infinitely many restrictions S_{γ_n}, $n=1,2,\ldots$ such that $S_{\gamma_n}\neq 0$. Then there is $a_{\gamma_n}\in\mathcal{P}_{\gamma_n}$ such that $b_{\gamma_n}=S_{\gamma_n}(a_{\gamma_n})\notin C_n$ for all $n\in\mathbb{N}$, by Lemma 2.11. The net $(a_{\gamma_n})_{n\in\mathbb{N}}$ is bounded in $(\mathcal{P},\mathfrak{U})$, since all of its component are bounded by the assumption, but $S((a_{\gamma_n})_{n\in\mathbb{N}})=\sum_{n=1}^\infty S_{\gamma_n}(a_{\gamma_n})=\sum_{n=1}^\infty b_n$ is unbounded in $(\mathcal{Q},\mathcal{W})$. This is a contradiction, because S is continuous. Then

$$\mathcal{C}(\mathcal{P},\mathcal{Q})\subseteq\bigoplus_{\gamma\in\Gamma}\mathcal{C}(\mathcal{P}_\gamma,\mathfrak{U}_\gamma).$$

For (b), let $V_{B,W}\in\mathcal{V}_{\mathcal{B},\mathcal{W}}$, where $B=\prod_{\gamma\in\Gamma}B_\gamma$, $B_\gamma\in\mathcal{B}_\gamma$ and $W\in\mathcal{W}$. It is enough to show that $\bigcup_{\gamma\in\Gamma}(j_\gamma\times j_\gamma)(V_{B_\gamma,W})\subseteq V_{B,W}$. For $\gamma'\in\Gamma$, let $(S_{\gamma'},T_{\gamma'})\in$

$V_{B_{\gamma'},W}$. Then for each $b_{\gamma'} \in B_{\gamma'}$, we have $(S_{\gamma'}(b_{\gamma'}), T_{\gamma'}(b_{\gamma'})) \in W$. Now, since for $(b_\gamma)_{\gamma\in\Gamma} \in B$ we have $(j_{\gamma'}(S_{\gamma'}))((b_\gamma)_{\gamma\in\Gamma}) = S_{\gamma'}(b_{\gamma'})$ and $(j_{\gamma'}(T_{\gamma'}))((b_\gamma)_{\gamma\in\Gamma}) = T_{\gamma'}(b_{\gamma'})$ by (a), we conclude that

$$(j_{\gamma'} \times j_{\gamma'})(S_{\gamma'}, T_{\gamma'}) \in V_{B,W}.$$

□

Theorem 2.13. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and let $(\mathcal{Q},\mathcal{W})$ be the projective limit of the locally convex cones $(\mathcal{Q}_\gamma,\mathcal{W}_\gamma)$ by the linear mappings v_γ, $\gamma\in\Gamma$. If $\mathcal{B}$ is a collection of bounded below subsets of $(\mathcal{P},\mathfrak{U})$ which has property (UW), then the locally convex cone $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is the projective limit of the locally convex cones $(\mathcal{C}(\mathcal{P},\mathcal{Q}_\gamma),\mathcal{V}_{\mathcal{B},\mathcal{W}_\gamma})$, $\gamma\in\Gamma$, by the linear mappings $T_\gamma : \mathcal{C}(\mathcal{P},\mathcal{Q}) \to \mathcal{C}(\mathcal{P},\mathcal{Q}_\gamma)$, $T_\gamma(A) = v_\gamma \circ A$.*

Proof. Firstly, we prove that for every γ, T_γ is continuous. Let $V_{B,W_\gamma} \in \mathcal{V}_{\mathcal{B},\mathcal{W}_\gamma}$. Since v_γ is continuous, there is $W \in \mathcal{W}$ such that $(v_\gamma \times v_\gamma)(W) \subseteq W_\gamma$. We show $(T_\gamma \times T_\gamma)(V_{B,W}) \subseteq V_{B,W_\gamma}$. If $(S,A) \in V_{B,W}$, then $(S(b),A(b)) \in W$ for all $b \in B$. Therefore $(v_\gamma\circ S(b), v_\gamma\circ A(b)) \in W_\gamma$ and then $(T_\gamma \times T\gamma)(S,A) = (v_\gamma\circ S, v_\gamma\circ A) \in V_{B,W_\gamma}$. Now, we prove that $\mathcal{V}_{\mathcal{B},\mathcal{W}}$ is the coarsest convex quasiuniform structure on $\mathcal{L}(\mathcal{P},\mathcal{Q}_\gamma)$ that makes all T_γ, $\gamma\in\Gamma$ continuous. For this aim let $\mathcal{H}$ be another convex quasiuniform structure on $\mathcal{C}(\mathcal{P},\mathcal{Q}_\gamma)$ that makes all T_γ, $\gamma\in\Gamma$ continuous. We shall prove that $\mathcal{H}$ is finer than $\mathcal{V}_{\mathcal{B},\mathcal{W}}$. There are $n \in \mathbb{N}$ and $\gamma_1,...,\gamma_n \in \Gamma$ such that $\bigcap_{i=1}^n (v_{\gamma_i} \times v_{\gamma_i})^{-1}(W_{\gamma_i}) \subseteq W$, since $(\mathcal{Q},\mathcal{W})$ is the projective limit of $(\mathcal{Q}_\gamma,\mathcal{W}_\gamma)$, $\gamma\in\Gamma$. For every $i = 1,\cdots,n$ there is $H_i \in \mathcal{H}$ such that $(T_\gamma \times T_\gamma)(H_i) \subseteq V_{B,W_{\gamma_i}}$. Since $\mathcal{H}$ is a convex quasiunifom structure, there is $H \in \mathcal{H}$ such that $H \subseteq \bigcap_{i=1}^n H_i$. We claim that $H \subseteq V_{B,W}$. Let $(S,A) \in H$. Then for every $i = 1,\cdots,n$, we have $(S,A) \in H_i$. This shows that

$$(T_\gamma(S), T_\gamma(A)) = (v_\gamma\circ S, v_\gamma\circ A) \in V_{B,W_{\gamma_i}}.$$

Then for every $i = 1,...,n$, $(v_\gamma\circ S(b), v_\gamma\circ A(b)) \in W_{\gamma_i}$ for all $b \in B$. Therefore

$$(S(b),A(b)) \in \bigcap_{i=1}^{n} V_\gamma^{-1}(W_{\gamma_i}) \subseteq W,$$

for all $b \in B$. This yields that $(S,A) \in V_{B,W}$. □

Corollary 2.14. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and let*

$$(\mathcal{Q},\mathcal{W}) = \prod_{\gamma\in\Gamma}(\mathcal{Q}_\gamma,\mathcal{W}_\gamma).$$

If $\mathcal{B}$ is a collection of bounded below subsets of $\mathcal{P}$ which has property (UW), then $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}}) = \prod_{\gamma\in\Gamma}(\mathcal{C}(\mathcal{P},\mathcal{Q}_\gamma),\mathcal{V}_{\mathcal{B},\mathcal{W}_\gamma})$.

Example 2.15. Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones. We consider the locally convex cone $(\mathcal{Q},\mathcal{W}_\sigma(\mathcal{Q},\mathcal{Q}^*))$. We note that $(\mathcal{Q},\mathcal{W}_\sigma(\mathcal{Q},\mathcal{Q}^*))$ is the projective limit of $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$ under the functionals $\mu \in \mathcal{Q}^*$. If $\mathcal{B}$ is a collection of bounded below subsets of $(\mathcal{P},\mathfrak{U})$ which has property (UW), then the locally convex cone $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}_\sigma(\mathcal{Q},\mathcal{Q}^*)})$ is the projective limit of the locally convex cone $(\mathcal{P}^*,\mathcal{V}_{\mathcal{B},\tilde{\mathcal{V}}})$ by the linear mappings $T_\mu : \mathcal{C}(\mathcal{P},\mathcal{Q}) \to \mathcal{P}^*, T_\mu(A) = \mu o A$, $\mu \in \mathcal{Q}^*$, by Theorem 2.13.

In the following proposition we present some conditions under which the locally convex cone $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is separated.

Proposition 2.16. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones and $\mathcal{B}$ be a collection of bounded below subsets of $\mathcal{P}$, which have (UW). If $(\mathcal{Q},\mathcal{W})$ is separated and $\mathcal{P} = \bigcup_{B\in\mathcal{B}} B$, then $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is separated.*

Proof. It is sufficient to show that the symmetric topology of $(\mathcal{C}(\mathcal{P},\mathcal{Q}),\mathcal{V}_{\mathcal{B},\mathcal{W}})$ is Hausdorff. Let $S,T \in \mathcal{C}(\mathcal{P},\mathcal{Q})$ and $S \neq T$. There is $a \in \mathcal{P}$ such that $S(a) \neq T(a)$. Since $(\mathcal{Q},\mathcal{W})$ is separated, there are $W,W' \in \mathcal{W}$ such that

$$W(S(a))W \cap W'(T(a))W' = \emptyset.$$

We have $a \in B$ for some $B \in \mathcal{B}$, since $\mathcal{Q} = \bigcup_{B\in\mathcal{B}} B$. Now, we claim that

$$\mathcal{V}_{B,W}(S)\mathcal{V}_{B,W} \cap \mathcal{V}_{B,W'}(T)\mathcal{V}_{B,W'} = \emptyset.$$

If $K \in \mathcal{V}_{B,W}(S)\mathcal{V}_{B,W} \cap \mathcal{V}_{B,W'}(T)\mathcal{V}_{B,W'}$, then

$$K(a) \in W(S(a))W \cap W'(T(a))W',$$

and this is a contradiction. □

Example 2.17. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and $\mathcal{B}$ be the collection of all finite subsets of $\mathcal{P}$. If we set $(\mathcal{Q},\mathcal{W}) = (\overline{\mathbb{R}},\tilde{\mathcal{V}})$, then $\mathcal{P}^* = \mathcal{C}(\mathcal{P},\mathcal{Q})$, endowed with the convex quasiuniform structure $\mathcal{V}_{\mathcal{B},\tilde{\mathcal{V}}}$ is a separated locally convex cone by Proposition 2.16. We note that the convex quasiuniform structure $\mathcal{V}_{\mathcal{B},\tilde{\mathcal{V}}}$ is equivalent with $\mathfrak{U}_\sigma(\mathcal{P}^*,\mathcal{P})$ on $\mathcal{P}^*$. Then the locally convex cone $(\mathcal{P}^*,\mathfrak{U}_\sigma(\mathcal{P}^*,\mathcal{P}))$ is separated.

REFERENCES

[1] D. Ayaseh and A. Ranjbari, Bornological Convergence in Locally Convex Cones, Mediterr. J. Math., 13 (4), 1921-1931(2016).

[2] D. Ayaseh and A. Ranjbari, Bornological Locally Convex Cones, Le Matematiche, 69(2), 267-284(2014).

[3] D. Ayaseh and A. Ranjbari, Locally Convex Quotient Lattice Cones, Math. Nachr., 287(10), 1083-1092(2014).

[4] D. Ayaseh and A. Ranjbari, Some notes on bornological and nonbornological locally convex cones, Le Matematiche, 70(2), 235-241(2015).

[5] K. Keimel and W. Roth, Ordered cones and approximation, Lecture Notes in Mathematics, vol. 1517, 1992, Springer Verlag, Heidelberg-Berlin-New York.

[6] G. Kothe, Topological Vector Spaces II, Springer-Verlag New York Heidelberg Berlin, 1979.

[7] M. R. Motallebi and H. Sai u, Duality on Locally Convex Cones, J. Math. Anal. Appl., 337 (2008) 888-905.

[8] M. R. Motallebi and H. Saiflu, Products and Direct Sums in Locally Convex Cones, Canad. Math. Bull. 55, 783-798(2012).

[9] A. Ranjbari, Strict inductive limits in locally convex cones, Positivity 15(3), 465-471(2011).

[10] A. Ranjbari and H. Saiflu, Projective and inductive limits in locally convex cones, J. Math. Anal. Appl., 332, 1097-1108(2007).

[11] A. P. Robertson and W. Robertson, Topological vector spaces, Cambridge Univesity Press 1964, 1973.

[12] W. Roth, Locally convex quotient cones, J. Convex Anal. 18(4),903- 913(2011).

[13] W. Roth, Operator-valued measures and integrals for cone-valued functions, Lecture Notes in Mathematics, vol. 1964, 2009, Springer Verlag, Heidelberg-Berlin-New York.

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