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SOME RESULTS OF p -VALENT ANALYTIC FUNCTIONS DEFINED BY INTEGRAL OPERATOR

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In this paper we study different applications of the inclusion relationships, radius problems and some other interesting properties of p -valent functions which are defined by integral operator.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a, p]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of the functions $f \in \mathcal{H}(\mathbb{U})$ of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N}). \quad (1)$$

Let $f \in \mathcal{A}(p)$ be given by (1) and $g \in \mathcal{A}(p)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}.$$

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The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z) \quad (z \in \mathbb{U}).$$

Also, let $\mathcal{P}_k(\rho)$ be the class of functions $h(z) = 1 + \sum_{k=1}^{\infty} c_{k+p} z^{k+p}$ which are analytic in \mathbb{U} and satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re\{h(z)\} - \rho}{1 - \rho} \right| d\theta \leq k\pi, \quad (2)$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan and Parvatham [8].

We note that:

- (i) $\mathcal{P}_k(0) = \mathcal{P}_k$ ($k \geq 2$) (see Pinchuk [9]);
- (ii) $\mathcal{P}_2(\rho) = \mathcal{P}(\rho)$, the class of analytic functions with positive real part greater than ρ ;
- (iii) $\mathcal{P}_2(0) = \mathcal{P}$, the class of functions with positive real part. Let $h(z) \in \mathcal{P}_k(\rho)$, then we can write $h(z)$ of the form

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z) \quad (z \in \mathbb{U}; h_1, h_2 \in \mathcal{P}(\rho)).$$

Let us consider the integral operator (see Aouf et al. [1]):

$$\begin{aligned} & \mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z) \\ &= z^p + \frac{\Gamma(p + \alpha + \beta - \varepsilon + 1)}{\Gamma(p + \beta)} \sum_{k=1}^{\infty} \left[\frac{\Gamma(p + \beta + k)}{\Gamma(p + \alpha + \beta + k - \varepsilon + 1)} \right] a_{k+p} z^{k+p} \\ & \quad (\beta > -p; \alpha \geq \varepsilon - 1; \varepsilon > 0; p \in \mathbb{N}; z \in \mathbb{U}). \end{aligned} \quad (3)$$

From (3), it is easy to verify that

$$cz(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))' = (p + \alpha + \beta - \varepsilon) \mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z) - (\alpha + \beta - \varepsilon) \mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z). \quad (4)$$

Remark 1.1.

$$\begin{aligned} \text{(i) Putting } \varepsilon = 1, & \mathcal{Q}_{\beta,p}^{\alpha,1} f(z) = \mathcal{Q}_{\beta,p}^{\alpha} f(z) \\ &= z^p + \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} \sum_{k=1}^{\infty} \left[\frac{\Gamma(p + \beta + k)}{\Gamma(p + \alpha + \beta + k)} \right] a_{k+p} z^{k+p} \\ & \quad (\beta > -p; \alpha \geq 0; p \in \mathbb{N}; z \in \mathbb{U}), \end{aligned} \quad (5)$$

where the operator $\mathcal{Q}_{\beta,p}^\alpha$ is defined by Liu and Owa. [4];

(ii) Putting $\alpha = \varepsilon$ and $\beta = c$,

$$\begin{aligned} \mathcal{Q}_{c,p}^{\alpha,\alpha} f(z) &= J_{c,p} = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \\ &= z^p + \sum_{k=1}^\infty \left(\frac{c+p}{c+k+p} \right) a_{k+p} z^{k+p} \end{aligned} \tag{6}$$

$(c > -p; p \in \mathbb{N}; z \in \mathbb{U}),$

where $J_{c,p}$ is the familiar integral operator which was defined by Saitoh et al. [11];

(iii) Putting $p = \varepsilon = 1, \mathcal{Q}_{\beta,1}^{\alpha,1} f(z) = \mathcal{Q}_\beta^\alpha f(z)$

$$= z + \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + 1)} \sum_{k=2}^\infty \left[\frac{\Gamma(\beta + k)}{\Gamma(\alpha + \beta + k)} \right] a_k z^k$$

$(\beta > -1; \alpha > 0; z \in \mathbb{U}),$

where the operator \mathcal{Q}_β^α is defined by Jung et al. [3].

Using the operator $\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}$, we now define a subclass of $\mathcal{A}(p)$ as follows [12–14]:

Definition 1.2. Let $\beta > -1, \alpha \geq \varepsilon - 1, \varepsilon > 0, \mu > 0, \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, p \in \mathbb{N}$, we say that a function $f(z) \in \mathcal{A}(p)$ is in the class $\mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda, \mu, \rho)$ if it satisfies

$$\left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^{\mu-1} \right\} \tag{7}$$

$\in \mathcal{P}_k(\rho) \quad (z \in \mathbb{U}),$

where $k \geq 2, 0 \leq \rho < 1$ and $g \in \mathcal{A}(p)$ satisfies the condition

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right) \in \mathcal{P}(\eta) \quad (0 \leq \eta < 1; z \in \mathbb{U}). \tag{8}$$

We note that $\mathcal{F}_{\beta,p,k}^{\alpha,1}(\lambda, \mu, \rho) = \mathcal{F}_{\beta,p,k}^\alpha(\lambda, \mu, \rho)$ (see [6]).

In the present paper, we investigate some inclusion relationships, radius problem and some interesting properties of p -valent functions which are defined by integral operator $\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon}$.

2. Preliminaries

In this section, we recall some known results.

Lemma 2.1 ([5]). *Let $\theta(u, v)$ be a complex-valued function such that*

$$\theta : D \rightarrow \mathbb{C}, D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane})$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u, v)$ satisfies the following conditions :

- (i) $\theta(u, v)$ is continuous in D ;
- (ii) $(1, 0) \in D$ and $\Re\{\theta(1, 0)\} > 0$;
- (iii) for all $(iu_2, v_1) \in D$ such that

$$v_1 \leq -\frac{1}{2}(1 + u_2^2) \quad \Re\{\theta(iu_2, v_1)\} \leq 0.$$

Let

$$q(z) = 1 + q_1z + q_2z^2 + \dots$$

be analytic in \mathbb{U} such that $(q(z), zq'(z)) \in D (z \in \mathbb{U})$. If

$$\Re\{\theta(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 2.2 ([10]). *If $\varphi(z)$ is analytic in \mathbb{U} with $\varphi(0) = 1$ and λ_1 is a complex number satisfying $\Re(\lambda_1) \geq 0$ ($\lambda_1 \neq 0$), then*

$$\Re\{\varphi(z) + \lambda_1 z \varphi'(z)\} > \sigma \quad (0 \leq \sigma < 1).$$

Implies

$$\Re\varphi(z) > \sigma + (1 - \sigma)(2\gamma_1 - 1),$$

where γ_1 is given by

$$\gamma_1 = \gamma_1(\Re(\lambda_1)) = \int_0^1 \left(1 + t^{\Re(\lambda_1)}\right)^{-1} dt,$$

which is increasing function of $\Re(\lambda_1)$ and $\frac{1}{2} \leq \gamma_1 < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.3 ([2]). *let $q(z)$ be analytic in \mathbb{U} with $q(0) = 1$ and $\Re(q(z)) > 0$ ($z \in \mathbb{U}$). Then, for $|z| = r, z \in \mathbb{U}$,*

- (i) $\frac{1-r}{1+r} \leq \Re(q(z)) \leq |q(z)| \leq \frac{1+r}{1-r}$;
- (ii) $|q'(z)| \leq \frac{2\Re(q(z))}{1-r^2}$.

3. Main Results

Theorem 3.1. Let $f \in \mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda, \mu, \rho)$ and $\Re(\lambda) > 0$. Then $\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)}\right)^\mu \in \mathcal{P}_k(\gamma)$, where

$$\gamma = \frac{2\mu(p + \alpha + \beta - \varepsilon)\rho + \Re(\lambda)\delta}{2\mu(p + \alpha + \beta - \varepsilon) + \Re(\lambda)\delta}, \tag{9}$$

and $g \in \mathcal{A}(p)$ satisfies the condition (8) and

$$\delta = \frac{\Re h_0(z)}{|h_0(z)|^2}, h_0(z) = \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)}\right).$$

Proof. Set

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)}\right)^\mu = (1 - \gamma)h(z) + \gamma, \tag{10}$$

$h(0) = 1$, and $h(z)$ is analytic in \mathbb{U} and we can write

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z) \quad (h_1, h_2 \in \mathcal{P}(\gamma)). \tag{11}$$

Differentiating (10) with respect to z and using the identity (4), we have

$$\begin{aligned} & \left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)}\right)^\mu + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)}\right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)}\right)^{\mu-1} \right\} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ [(1 - \gamma)h_1(z) + \gamma] - \rho + \frac{\lambda(1 - \gamma)zh_1'(z)}{\mu(p + \alpha + \beta - \varepsilon)h_0(z)} \right\} \\ & - \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ [(1 - \gamma)h_2(z) + \gamma] - \rho + \frac{\lambda(1 - \gamma)zh_2'(z)}{\mu(p + \alpha + \beta - \varepsilon)h_0(z)} \right\}. \end{aligned}$$

Now, we form the functional $\psi(u, v)$ by choosing $u = h_i(z) = u_1 + iu_2$ and $v = zh_i'(z) = v_1 + iv_2$. Thus

$$\psi(u, v) = [(1 - \gamma)u + \gamma] - \rho + \frac{\lambda(1 - \gamma)v}{\mu(p + \alpha + \beta - \varepsilon)h_0(z)}.$$

The first and second conditions of Lemma 2.1 are satisfied by using $\psi(u, v)$. To prove the third condition:

$$\begin{aligned} \Re \{ \psi(iu_2, v_1) \} &= \gamma - \rho + \frac{\Re(\lambda)(1 - \gamma)v_1 \Re h_0(z)}{\mu(p + \alpha + \beta - \varepsilon)|h_0(z)|^2} \\ &= \gamma - \rho + \frac{\Re(\lambda)(1 - \gamma)v_1}{\mu(p + \alpha + \beta - \varepsilon)} \delta, \quad \delta = \frac{\Re h_0(z)}{|h_0(z)|^2} \end{aligned}$$

By using $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we have

$$\begin{aligned} \Re\{\psi(iu_2, v_1)\} &\leq \gamma - \rho - \frac{\Re(\lambda)(1 - \gamma)(1 + u_2^2)}{2\mu(p + \alpha + \beta - \varepsilon)} \delta \\ &= \frac{2\mu(p + \alpha + \beta - \varepsilon)(\gamma - \rho) - \Re(\lambda)\delta(1 - \gamma) - \Re(\lambda)\delta(1 - \gamma)u_2^2}{2\mu(p + \alpha + \beta - \varepsilon)} \\ &= \frac{C + D}{2E}, E > 0, \\ C &= 2\mu(p + \alpha + \beta - \varepsilon)(\gamma - \rho) - \Re(\lambda)\delta(1 - \gamma) \\ D &= -\Re(\lambda)\delta(1 - \gamma) \leq 0. \end{aligned}$$

Now, $\Re\{\psi(iu_2, v_1)\} \leq 0$, if $C \leq 0$ and this gives us γ as defined by (9). By applying Lemma 2.1 to conclude that $h_i \in \mathcal{P}$ for $z \in \mathbb{U}$ and thus $h \in \mathcal{P}_k$ which gives us the result. \square

Remark 3.2. We note that $\gamma = \rho$ when $\lambda = 0$.

Theorem 3.3. For $\lambda \geq 1$, let $f \in \mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda, \mu, \rho)$. Then

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \in \mathcal{P}_k(\rho) \quad (z \in \mathbb{U}).$$

Proof. We can write

$$\begin{aligned} \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) &= \left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right) + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \right\} + (\lambda - 1) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right). \end{aligned}$$

This implies that

$$\begin{aligned} \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) &= \frac{1}{\lambda} \left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right) + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \right\} + \left(1 - \frac{1}{\lambda}\right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right) \\ &= \frac{1}{\lambda} H_1(z) + \left(1 - \frac{1}{\lambda}\right) H_2(z) \quad (\lambda \geq 1). \end{aligned}$$

Since $H_1(z), H_2(z) \in \mathcal{P}_k(\rho)$, by Theorem 3.1, Definition 1.2 and since $\mathcal{P}_k(\rho)$ is a convex set (see [7]), the proof of Theorem 3.3 is completed. \square

Putting $g(z) = z^p$ in Theorem 3.3, we obtain the following theorem:

Theorem 3.4. Let $\lambda \in \mathbb{C}^*$ with $\Re(\lambda) > 0$. If $f \in \mathcal{A}(p)$ satisfies the condition

$$\left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{z^p} \right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^{\mu-1} \right\} \in \mathcal{P}_k(\rho)$$

$$(\mu > 0; z \in \mathbb{U}),$$

then

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu \in \mathcal{P}_k(\sigma),$$

where

$$\sigma = \rho + (1 - \rho) \left[2F_1\left(1, 1, \frac{\mu(p + \alpha + \beta - \varepsilon)}{\lambda_1} + 1; \frac{1}{2}\right) - 1 \right].$$

The value of σ is the best possible and cannot be improved.

Proof. Setting

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu = h(z) = \left(\frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) h_2(z),$$

where $h(0) = 1$ and h is analytic in U . By using (4) and some computations, we have

$$\left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{z^p} \right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^{\mu-1} \right\}$$

$$= \left\{ h(z) + \frac{\lambda z h'(z)}{\mu(p + \alpha + \beta - \varepsilon)} \right\} \in \mathcal{P}_k(\rho) \quad (z \in \mathbb{U}).$$

By using Lemma 2.2, we have $h_i(z) \in \mathcal{P}_k(\sigma)$, where

$$\sigma = \rho + (1 - \rho)(2\sigma_1 - 1), \tag{12}$$

with

$$\begin{aligned} \sigma_1 &= \int_0^1 (1 + t^{\frac{\Re(\lambda)}{\mu(p + \alpha + \beta - \varepsilon)}})^{-1} dt \\ &= \frac{\mu(p + \alpha + \beta - \varepsilon)}{\lambda_1} \int_0^1 u^{\frac{\mu(p + \alpha + \beta - \varepsilon)}{\lambda_1} - 1} (1 + u)^{-1} du, \quad (\lambda_1 = \Re(\lambda) > 0) \\ &= 2F_1\left(1, \frac{\mu(p + \alpha + \beta - \varepsilon)}{\lambda_1}, \frac{\mu(p + \alpha + \beta - \varepsilon)}{\lambda_1} + 1; -1\right) \\ &= 2F_1\left(1, 1, \frac{\mu(p + \alpha + \beta - \varepsilon)}{\lambda_1} + 1; \frac{1}{2}\right). \end{aligned}$$

Now, the proof of Theorem 3.4 is completed. □

Next, we consider the operator defined by

$$F_c = \left(\frac{p\mu + c}{z^c} \int_0^z t^{c-1} (f(t))^\mu dt \right)^{\frac{1}{\mu}} \quad (c > -p\mu; z \in \mathbb{U}). \quad (13)$$

It is clear that the function $F_c \in \mathcal{A}(p)$ and

$$z^c (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z))^\mu = (p\mu + c) \int_0^z t^{c-1} (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(t))^\mu dt \quad (z \in \mathbb{U}). \quad (14)$$

Theorem 3.5. *Let $\lambda > 0, \mu > 0$ and $c > -p\mu$. If $f \in \mathcal{A}(p)$ satisfies the condition*

$$\left\{ (1-\lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu + \lambda \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^{\mu-1} \right\} \in \mathcal{P}_k(\rho) \quad (15)$$

$$(\mu > 0; z \in \mathbb{U}),$$

then

$$\left\{ (1-\lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z)}{z^p} \right)^\mu + \lambda \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z)}{z^p} \right)^{\mu-1} \right\} \in \mathcal{P}_k(\alpha_1), \quad (16)$$

where $F_c(z)$ is given by (13) and

$$\alpha_1 = \rho + (1-\rho) \left[2F_1\left(1, 1, p\mu + c + 1; \frac{1}{2}\right) - 1 \right].$$

The value of α_1 is the best possible.

Proof. It is clear that $F_c \in \mathcal{A}(p)$ and differentiating both sides of (14), we have

$$(p\mu + c) z^{c-1} (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^\mu =$$

$$cz^{c-1} (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^\mu + \mu z^c (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^{\mu-1} (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))',$$

divide both sides by z^{c-1} and multiply by $z^{-p\mu}$, we get

$$(p\mu + c) (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^\mu = c (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^\mu + \mu z (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^{\mu-1} (\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z))',$$

$$(p\mu + c) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu = c \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu + p\mu z^p \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z))'}{pz^{p-1}} \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))^{\mu-1}}{z^{p\mu}},$$

hence

$$(p\mu + c) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu = c \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z)}{z^p} \right)^\mu + p\mu \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z)}{z^p} \right)^{\mu-1}. \quad (17)$$

Putting

$$G(z) = \left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z)}{z^p} \right)^\mu + \lambda \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} F_c(z)}{z^p} \right)^{\mu-1} \right\}, \tag{18}$$

where

$$G(z) = \left(\frac{k}{4} + \frac{1}{2} \right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) g_2(z).$$

Then $G(z)$ is analytic in \mathbb{U} with $G(0) = 1$. By differentiating (18) and using (17) in the resulting equation, we have

$$\begin{aligned} & \left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^\mu + \lambda \frac{(\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z))'}{pz^{p-1}} \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{z^p} \right)^{\mu-1} \right\} \\ &= \left\{ G(z) + \frac{zG'(z)}{p\mu + c} \right\} \in \mathcal{P}_k(\rho) \quad (z \in \mathbb{U}). \end{aligned}$$

Using Lemma 2.2, we note that $g_i(z) \in \mathcal{P}_k(\alpha_1)$, where

$$\alpha_1 = \rho + (1 - \rho)(2\sigma_2 - 1), \tag{19}$$

with

$$\sigma_2 = {}_2F_1\left(1, 1, p\mu + c + 1; \frac{1}{2}\right).$$

The proof of Theorem 3.5 is completed. □

Theorem 3.6. For $0 \leq \lambda_2 < \lambda_1$,

$$\mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda_1, \mu, \rho) \subset \mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda_2, \mu, \rho).$$

Proof. If $\lambda_2 = 0$, then the proof is immediate from Theorem 3.1. Let $\lambda_2 > 0$ and $f \in \mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda_1, \mu, \rho)$. Then there exist two functions $H_1, H_2 \in \mathcal{P}_k(\rho)$ such that

$$(1 - \lambda_1) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu + \lambda_1 \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^{\mu-1} = H_1(z),$$

and

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu = H_2(z).$$

Then

$$\begin{aligned} (1 - \lambda_2) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu + \lambda_2 \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^{\mu-1} \\ = \frac{\lambda_2}{\lambda_1} H_1(z) + \left(1 - \frac{\lambda_2}{\lambda_1}\right) H_2(z), \end{aligned} \quad (20)$$

and since $\mathcal{P}_k(\rho)$ is a convex set (see [8]), it follows that the right hand side of (20) belongs to $\mathcal{P}_k(\rho)$ and this completes the proof of Theorem 3.6. \square

Next, we can give the converse of Theorem 3.1 as follows:

Theorem 3.7. Let $\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu \in \mathcal{P}_k(\rho)$, with $\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right) \in \mathcal{P}(\eta)$, for $z \in \mathbb{U}$,

$0 \leq \eta < 1$.

Then $f \in \mathcal{F}_{\beta,p,k}^{\alpha,\varepsilon}(\lambda, \mu, \rho)$ for $|z| < r$, where r is given by

$$\begin{aligned} r = \mu(p + \alpha + \beta - \varepsilon) / \left(\{(1 - \eta)\mu(p + \alpha + \beta - \varepsilon) + |\lambda|\} \right. \\ \left. + \sqrt{\mu^2 \eta^2 (p + \alpha + \beta - \varepsilon)^2 + |\lambda|^2 + 2|\lambda|(1 - \eta)\mu(p + \alpha + \beta - \varepsilon)} \right). \end{aligned} \quad (21)$$

Proof. Let

$$\left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu = H, \quad \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right) = H_0,$$

then $H \in \mathcal{P}_k(\rho)$, $H_0 \in \mathcal{P}_k(\eta)$.

Proceeding as in Theorem 3.1, for $\beta > -1$, $\alpha \geq 0$, $\mu > 0$, $\lambda \in \mathbb{C}$, $k \geq 2$, $\rho \geq 0$, $\eta < 1$ and

$$H = (1 - \rho)h + \rho, \quad H_0 = (1 - \eta)h_0 + \eta, \quad \text{with } h \in \mathcal{P}_k, h_0 \in \mathcal{P}$$

we have

$$\begin{aligned} \frac{1}{1 - \rho} \left\{ (1 - \lambda) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^\mu + \lambda \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha-1,\varepsilon} g(z)} \right) \left(\frac{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} f(z)}{\mathcal{Q}_{\beta,p}^{\alpha,\varepsilon} g(z)} \right)^{\mu-1} - \rho \right\} \\ = \left\{ h(z) + \frac{\lambda}{\mu(p + \alpha + \beta - \varepsilon)} \frac{zh'(z)}{\{(1 - \eta)h_0(z) + \eta\}} \right\} \\ = \left(\frac{k}{4} + \frac{1}{2} \right) \left[h_1(z) + \frac{\lambda}{\mu(p + \alpha + \beta - \varepsilon)} \frac{zh'_1(z)}{\{(1 - \eta)h_0(z) + \eta\}} \right] \\ - \left(\frac{k}{4} - \frac{1}{2} \right) \left[h_2(z) + \frac{\lambda}{\mu(p + \alpha + \beta - \varepsilon)} \frac{zh'_2(z)}{\{(1 - \eta)h_0(z) + \eta\}} \right]. \end{aligned}$$

Using well known estimates, (see [3]), for $h_i \in \mathcal{P}$,

$$|zh_i(z)| \leq \frac{2r\Re h_i(z)}{1-r^2}, \quad \frac{1-r}{1+r} \leq |h_i(z)| \leq \frac{1+r}{1-r},$$

we have

$$\begin{aligned} & \Re \left\{ h_i(z) + \frac{\lambda}{\mu(p+\alpha+\beta-\varepsilon)} \frac{zh'_i(z)}{\{(1-\eta)h_0(z)+\eta\}} \right\} \\ & \geq \Re(h_i(z)) \left\{ 1 - \frac{2r|\lambda|}{\mu(p+\alpha+\beta-\varepsilon)} \frac{1}{1-r^2} \left(\frac{1+r}{(1-r(1-2\eta))} \right) \right\} \\ & \geq \Re(h_i(z)) \left\{ 1 - \frac{2r|\lambda|}{\mu(p+\alpha+\beta-\varepsilon)} \frac{1}{1-r} \left(\frac{1+r}{(1-r(1-2\eta))} \right) \right\} \\ & \geq \Re(h_i(z)) \left[\frac{\mu(p+\alpha+\beta-\varepsilon)[(1-r-(1-2\eta)r+r^2(1-2\eta)]-2r|\lambda|}{\mu(p+\alpha+\beta-\varepsilon)(1-r)\{1-(1-2\eta)r\}} \right] \\ & \geq \Re(h_i(z)) \left[\frac{\mu(p+\alpha+\beta-\varepsilon)r^2(1-2\eta)-2[(1-\eta)\mu(p+\alpha+\beta-\varepsilon)+|\lambda|]r}{\mu(p+\alpha+\beta-\varepsilon)(1-r)\{1-(1-2\eta)r\}} \right. \\ & \quad \left. + \frac{\mu(p+\alpha+\beta-\varepsilon)}{\mu(p+\alpha+\beta-\varepsilon)(1-r)\{1-(1-2\eta)r\}} \right] \end{aligned}$$

The right hand side of the last inequality is positive for $|z| < r$, where r is given by (21).

This completes the proof of Theorem 3.7. □

Remark 3.8. Putting $\varepsilon = 1$ in the above results, we obtain the results obtained by Muhammad [6].

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