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EXPANSION FORMULAS FOR APOSTOL TYPE Q -APPELL POLYNOMIALS, AND THEIR SPECIAL CASES

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We present identities of various kinds for generalized q -Apostol-Bernoulli and Apostol-Euler polynomials and power sums, which resemble q -analogues of formulas from the 2009 paper by Liu and Wang. These formulas are divided into two types: formulas with only q -Apostol-Bernoulli, and only q -Apostol-Euler polynomials, or so-called mixed formulas, which contain polynomials of both kinds. This can be seen as a logical consequence of the fact that the q -Appell polynomials form a commutative ring. The functional equations for Ward numbers operating on the q -exponential function, as well as symmetry arguments, are essential for many of the proofs. We conclude by finding multiplication formulas for two q -Appell polynomials of general form. This brings us to the q -H polynomials, which were discussed in a previous paper.

1. Introduction

In the second article on q -analogues of two Appell polynomials [4], the Apostol-Bernoulli and Apostol-Euler polynomials, focus was on multiplication formulas and on formulas including (multiple) λ power sums. In this article we will find a corresponding multiplication formula for a more general q -Appell polynomial, which is a generalization of both q -Apostol-Euler and q -Apostol-H polynomials.

There are many new formulas on this subject, both Apostol-Appell and similar Appell, which have recently been published; in all cases the limit $\lambda \rightarrow 1$ is straightforward. Sometimes we write q -analogue of etc., not bothering about the above dichotomy.

This paper is organized as follows: In section 1 we give a general introduction and the definitions. In section 2 we present formulas with only q -Apostol-Bernoulli, and only q -Apostol-Euler polynomials. In section 3 we present mixed formulas for these polynomials. In section 4, two general polynomials are defined, which generalize the q -Apostol-Bernoulli and q -Apostol-Euler polynomials. Then multiplication formulas for these polynomials are proved, which specialize to the q -Apostol-H polynomials.

We now start with the definitions. Some of the notation is well-known and can be found in the book [1]. The variables i, j, k, l, m, n, v will denote positive integers, and λ, μ will denote complex numbers when nothing else is stated.

Definition 1.1. The Gauss q -binomial coefficient are defined by

$$\binom{n}{k}_q \equiv \frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!}, k = 0, 1, \dots, n. \quad (1)$$

Let a and b be any elements with commutative multiplication. Then the NWA q -addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}, n = 0, 1, 2, \dots \quad (2)$$

If $0 < |q| < 1$ and $|z| < |1 - q|^{-1}$, the q -exponential function is defined by

$$E_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \quad (3)$$

The following theorem shows how Ward numbers usually appear in applications.

Theorem 1.1. Assume that $n, k \in \mathbb{N}$. Then

$$(\bar{n}_q)^k = \sum_{m_1 + \dots + m_n = k} \binom{k}{m_1, \dots, m_n}_q, \quad (4)$$

where each partition of k is multiplied with its number of permutations.

Theorem 1.2. Functional equations for Ward numbers operating on the q -exponential function. First assume that the letters \bar{m}_q and \bar{n}_q are independent,

i.e. come from two different functions, when operating with the functional. Furthermore, $mnt < \frac{1}{1-q}$. Then we have

$$E_q(\overline{m}_q \overline{n}_q t) = E_q(\overline{mn}_q t). \quad (5)$$

Furthermore,

$$E_q(\overline{j}_q m_q) = E_q(\overline{j}_q)^m = E_q(\overline{m}_q)^j. \quad (6)$$

Compare with the semiring of Ward numbers [1, p. 167].

Proof. Formula (5) is proved as follows:

$$E_q(\overline{m}_q \overline{n}_q t) = E_q((1 \oplus_q 1 \oplus_q \cdots \oplus_q 1) \overline{n}_q t), \quad (7)$$

where the number of 1s to the left is m . But this means exactly $E_q(\overline{n}_q t)^m$, and the result follows. \square

Definition 1.2. The generalized NWA q -Apostol-Bernoulli polynomials $\mathcal{B}_{\text{NWA}, \lambda, \nu, q}^{(n)}(x)$ are defined by

$$\frac{t^n}{(\lambda E_q(t) - 1)^n} E_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{B}_{\text{NWA}, \lambda, \nu, q}^{(n)}(x)}{\{\nu\}_q!}, \quad |t + \log \lambda| < 2\pi. \quad (8)$$

Definition 1.3. The generalized NWA q -Apostol-Euler polynomials $\mathcal{F}_{\text{NWA}, \lambda, \nu, q}^{(n)}(x)$ are defined by

$$\frac{2^n}{(\lambda E_q(t) + 1)^n} E_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{F}_{\text{NWA}, \lambda, \nu, q}^{(n)}(x)}{\{\nu\}_q!}, \quad |t + \log \lambda| < \pi. \quad (9)$$

Definition 1.4. The generalized NWA q - \mathcal{H} polynomials are defined by

$$\frac{(2t)^n}{(\lambda E_q(t) + 1)^n} E_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{H}_{\text{NWA}, \lambda, \nu, q}^{(n)}(x)}{\{\nu\}_q!}, \quad |t + \log \lambda| < \pi. \quad (10)$$

Definition 1.5. The generalized JHC q - \mathcal{H} polynomials are defined by

$$\frac{(2t)^n}{(\lambda E_{\frac{1}{q}}(t) + 1)^n} E_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{H}_{\text{JHC}, \lambda, \nu, q}^{(n)}(x)}{\{\nu\}_q!}, \quad |t + \log \lambda| < \pi. \quad (11)$$

Definition 1.6. The generating function for $H_{\text{NWA},v,q}^{(n)}(x)$ is given by

$$\frac{(2t)^n}{(E_q(t) + 1)^n} E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v H_{\text{NWA},v,q}^{(n)}(x)}{\{v\}_q!}, \quad |t| < 2\pi. \quad (12)$$

Definition 1.7. The generating function for $H_{\text{JHC},v,q}^{(n)}(x)$ is given by

$$\frac{(2t)^n}{(E_{\frac{1}{q}}(t) + 1)^n} E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v H_{\text{JHC},v,q}^{(n)}(x)}{\{v\}_q!}, \quad |t| < 2\pi. \quad (13)$$

The polynomials in (12) and (13) are q -analogues of the generalized H polynomials.

Definition 1.8. The polynomials $b_{\lambda,v,q}^{(n)}(x)$ are defined by

$$\frac{t^n g(t)}{(\lambda E_q(t) - 1)^n} E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v b_{\lambda,v,q}^{(n)}(x)}{\{v\}_q!}. \quad (14)$$

Definition 1.9. The e polynomials are defined by

$$\frac{2^n g(t)}{(\lambda E_q(t) + 1)^n} E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v e_{\lambda,v,q}^{(n)}(x)}{\{v\}_q!}. \quad (15)$$

The f polynomials are more general forms of the JHC q - \mathcal{H} polynomials.

Definition 1.10. The f polynomials $f_{\lambda,v,q}^{(n)}(x)$ are defined by

$$\frac{2^n g(t)}{(\lambda E_{\frac{1}{q}}(t) + 1)^n} E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v f_{\lambda,v,q}^{(n)}(x)}{\{v\}_q!}. \quad (16)$$

Definition 1.11. A q -analogue of [7, (20) p. 381], the multiple q -power sum is defined by

$$s_{\text{NWA},\lambda,m,q}^{(l)}(n) \equiv \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} \lambda^k (\overline{k}_q)^m, \quad (17)$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$, $\forall j_i \geq 0$.

Definition 1.12. A q -analogue of [7, (46) p. 386], the multiple alternating q -power sum is defined by

$$\sigma_{\text{NWA},\lambda,m,q}^{(l)}(n) \equiv (-1)^l \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} (-\lambda)^k (\overline{k}_q)^m, \quad (18)$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$, $\forall j_i \geq 0$.

Theorem 1.3. A symmetry relation for the generalized q - \mathcal{H} numbers.

$$(-1)^v \mathcal{H}_{\text{JHC}, \lambda^{-1}, v, q} = \mathcal{H}_{\text{NWA}, \lambda, v, q}. \quad (19)$$

Proof. A simple computation with generating functions shows the way:

$$\begin{aligned} \sum_{v=0}^{\infty} \frac{(-t)^v \mathcal{H}_{\text{JHC}, \lambda^{-1}, v, q}}{\{v\}_q!} &= \frac{-2t}{\lambda^{-1} \mathbb{E}_{\frac{1}{q}}(-t) + 1} = \frac{-2t \lambda \mathbb{E}_q(t)}{\lambda \mathbb{E}_q(t) + 1} \\ &= -\lambda \sum_{v=0}^{\infty} \frac{t^v \mathcal{H}_{\text{NWA}, \lambda, v, q}(1)}{\{v\}_q!}. \end{aligned} \quad (20)$$

Equating the coefficients of t^v gives (19). \square

Theorem 1.4. Assume that $g(t)$ in (15) and (16) are equal and even functions. Then

$$f_{\lambda^{-1}, v, q}^{(n)}(x) = (-1)^v \lambda^n e_{\lambda, v, q}^{(n)}(\bar{n}_q \ominus_q x). \quad (21)$$

This implies a complementary argument theorem for the generalized q - \mathcal{H} polynomials.

Theorem 1.5.

$$\mathcal{H}_{\text{JHC}, \lambda^{-1}, v, q}^{(n)}(x) = (-1)^v \lambda^n \mathcal{H}_{\text{NWA}, \lambda, v, q}^{(n)}(\bar{n}_q \ominus_q x), n \text{ even.} \quad (22)$$

$$\mathcal{H}_{\text{JHC}, \lambda^{-1}, v, q}^{(n)}(x) = (-1)^{v+1} \lambda^n \mathcal{H}_{\text{NWA}, \lambda, v, q}^{(n)}(\bar{n}_q \ominus_q x), n \text{ odd.} \quad (23)$$

Definition 1.13. The following functions named the q -power sum, and the alternate q -power sum (with respect to λ), were introduced in [4].

$$s_{\text{NWA}, \lambda, m, q}(n) \equiv \sum_{k=0}^{n-1} \lambda^k (\bar{k}_q)^m \text{ and } \sigma_{\text{NWA}, \lambda, m, q}(n) \equiv \sum_{k=0}^{n-1} (-1)^k \lambda^k (\bar{k}_q)^m. \quad (24)$$

Their respective generating functions are

$$\sum_{m=0}^{\infty} s_{\text{NWA}, \lambda, m, q}(n) \frac{t^m}{\{m\}_q!} = \frac{\lambda^n \mathbb{E}_q(\bar{n}_q t) - 1}{\lambda \mathbb{E}_q(t) - 1} \quad (25)$$

and

$$\sum_{m=0}^{\infty} \sigma_{\text{NWA}, \lambda, m, q}(n) \frac{t^m}{\{m\}_q!} = \frac{(-1)^{n+1} \lambda^n \mathbb{E}_q(\bar{n}_q t) + 1}{\lambda \mathbb{E}_q(t) + 1}. \quad (26)$$

2. The first expansion formulas

Theorem 2.1. *A triple sum of NWA q -Apostol-Euler polynomials is equal to another triple sum of NWA q -Apostol-Euler polynomials.*

$$\begin{aligned}
& \sum_{|\mathbf{v}|=n} \binom{n}{\bar{\mathbf{v}}}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} \mathcal{F}_{\text{NWA}, \lambda^i, v_1, q}^{(k)}(\bar{j}_q x) \mathcal{F}_{\text{NWA}, \lambda^j, v_2, q}^{(k-1)}(\bar{i}_q y) \sigma_{\text{NWA}, \lambda^i, v_3, q}(i) (\bar{j}_q)^{v_3} \\
&= \sum_{v=0}^n \binom{n}{\mathbf{v}}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{F}_{\text{NWA}, \lambda^i, n-v, q}^{(k-1)}(\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \\
& \mathcal{F}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m}{\bar{i}_q} \right).
\end{aligned} \tag{27}$$

Proof. Define the following function, note that $f_q(t)$ is symmetric when i, j have the same parity.

$$\begin{aligned}
f_q(t) &\equiv \frac{E_q(\bar{i}\bar{j}_q(x \oplus y)t) ((-1)^{i+1} \lambda^{ij} E_q(\bar{i}\bar{j}_q t) + 1)}{(\lambda^i E_q(\bar{i}_q t) + 1)^k (\lambda^j E_q(\bar{j}_q t) + 1)^k} = 2^{1-2k} E_q(\bar{i}\bar{j}_q(x \oplus y)t) \\
& \left(\frac{2}{\lambda^i E_q(\bar{i}_q t) + 1} \right)^k \left(\frac{2}{\lambda^j E_q(\bar{j}_q t) + 1} \right)^{k-1} \left(\frac{(-1)^{i+1} \lambda^{ij} E_q(\bar{i}\bar{j}_q t) + 1}{\lambda^j E_q(\bar{j}_q t) + 1} \right).
\end{aligned} \tag{28}$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$\begin{aligned}
f_q(t) &\stackrel{\text{by (26,9)}}{=} 2^{1-2k} \left(\sum_{v=0}^{\infty} \mathcal{F}_{\text{NWA}, \lambda^i, v, q}^{(k)}(\bar{j}_q x) \frac{(\bar{i}_q t)^v}{\{\mathbf{v}\}_q!} \right) \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA}, \lambda^j, m, q}(i) \frac{(\bar{j}_q t)^m}{\{\mathbf{m}\}_q!} \right) \\
& \left(\sum_{l=0}^{\infty} \mathcal{F}_{\text{NWA}, \lambda^j, l, q}^{(k-1)}(\bar{i}_q y) \frac{(\bar{j}_q t)^l}{\{\mathbf{l}\}_q!} \right) = 2^{1-2k} \frac{2^k}{(\lambda^i E_q(\bar{i}_q t) + 1)^k} \\
& \frac{2^{k-1}}{(\lambda^j E_q(\bar{j}_q t) + 1)^{k-1}} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} E_q \left(\left(\bar{j}_q x \oplus_q \bar{j}_q y \oplus_q \frac{\bar{j}m}{\bar{i}_q} \right) \bar{i}_q t \right) \\
&= 2^{1-2k} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{\mathbf{l}\}_q!} \mathcal{F}_{\text{NWA}, \lambda^i, l, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m}{\bar{i}_q} \right) \\
& \sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{\mathbf{n}\}_q!} \mathcal{F}_{\text{NWA}, \lambda^j, n, q}^{(k-1)}(\bar{i}_q y).
\end{aligned} \tag{29}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{\mathbf{n}\}_q!}$. \square

Theorem 2.2. *Almost a q -analogue of [5, p. 3351]. Assume that i and j are either both odd, or both even. Then we have*

$$\begin{aligned}
& \sum_{v=0}^n \binom{n}{v}_q (\bar{j}_q)^v (\bar{i}_q)^{n-v} \mathcal{F}_{\text{NWA}, \lambda^i, n-v, q}^{(k-1)}(\bar{j}_q y) \sum_{m=0}^{j-1} \lambda^{im} (-1)^m \mathcal{F}_{\text{NWA}, \lambda^j, v, q}^{(k)} \left(\bar{i}_q x \oplus_q \frac{\bar{i}m_q}{\bar{j}_q} \right) \\
&= \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{F}_{\text{NWA}, \lambda^j, n-v, q}^{(k-1)}(\bar{i}_q y) \\
& \quad \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \mathcal{F}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right)
\end{aligned} \tag{30}$$

Proof. This follows from the previous proof, and then using the symmetry for i and j . \square

Theorem 2.3. *A triple sum of NWA q -Apostol-Bernoulli polynomials is equal to a double sum of NWA q -Apostol-Bernoulli polynomials.*

$$\begin{aligned}
& \sum_{|v|=n} \binom{n}{v}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} (\bar{j}_q)^{v_3} \mathcal{B}_{\text{NWA}, \lambda^i, v_1, q}^{(k)}(\bar{j}_q x) \mathcal{B}_{\text{NWA}, \lambda^j, v_2, q}^{(k-1)}(\bar{i}_q y) s_{\text{NWA}, \lambda^j, v_3, q}(i) \\
&= \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{B}_{\text{NWA}, \lambda^j, n-v, q}^{(k-1)}(\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{B}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right)
\end{aligned} \tag{31}$$

Proof. Define the following symmetric function

$$\begin{aligned}
\phi_q(t) &\equiv \frac{\mathbf{E}_q(\bar{i}\bar{j}_q(x \oplus y)t)(\lambda^{ij}\mathbf{E}_q(\bar{i}\bar{j}_qt) - 1)}{(\lambda^i\mathbf{E}_q(\bar{i}qt) - 1)^k(\lambda^j\mathbf{E}_q(\bar{j}qt) - 1)^k} t^k = \mathbf{E}_q(\bar{i}\bar{j}_q(x \oplus y)t) \\
& \left(\frac{\bar{i}qt}{\lambda^i\mathbf{E}_q(\bar{i}qt) - 1} \right)^k \left(\frac{\bar{j}qt}{\lambda^j\mathbf{E}_q(\bar{j}qt) - 1} \right)^{k-1} \left(\frac{\lambda^{ij}\mathbf{E}_q(\bar{i}\bar{j}_qt) - 1}{\lambda^j\mathbf{E}_q(\bar{j}qt) - 1} \right) \frac{t^{1-2k}}{(\bar{i}_q)^k(\bar{j}_q)^{k-1}}.
\end{aligned} \tag{32}$$

By using the formula for a geometric sequence, we can expand $\phi_q(t)$ in two

ways:

$$\begin{aligned}
\phi_q(t) &\stackrel{\text{by (25)}}{=} \left(\sum_{v=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,v,q}^{(k)}(\bar{j}_q x) \frac{(\bar{i}_q t)^v}{\{v\}_q!} \right) \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^j,m,q}(i) \frac{(\bar{j}_q t)^m}{\{m\}_q!} \right) \\
&\left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^j,l,q}^{(k-1)}(\bar{i}_q y) \frac{(\bar{j}_q t)^l}{\{l\}_q!} \right) \frac{t^{1-2k}}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} = \frac{(\bar{i}_q t)^k}{(\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1)^k} \\
&\frac{(\bar{j}_q t)^{k-1}}{(\lambda^j \mathbf{E}_q(\bar{j}_q t) - 1)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \mathbf{E}_q \left(\left(\bar{j}_q x \oplus_q \bar{j}_q y \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \bar{i}_q t \right) \frac{t^{1-2k}}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \\
&= \frac{t^{1-2k}}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{l\}_q!} \mathcal{B}_{\text{NWA},\lambda^i,l,q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \\
&\sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{n\}_q!} \mathcal{B}_{\text{NWA},\lambda^j,n,q}^{(k-1)}(\bar{i}_q y).
\end{aligned} \tag{33}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 2.4. A q -analogue of [12, p. 2994], [11, p. 551].

$$\begin{aligned}
&\sum_{|v|=n} \binom{n}{v}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} (\bar{j}_q)^{v_3} \mathcal{B}_{\text{NWA},\lambda^i,v_1,q}^{(k)}(\bar{j}_q x) \mathcal{B}_{\text{NWA},\lambda^j,v_2,q}^{(k-1)}(\bar{i}_q y) s_{\text{NWA},\lambda^i,v_3,q}(i) \\
&= \sum_{|v|=n} \binom{n}{v}_q (\bar{j}_q)^{v_1} (\bar{i}_q)^{v_2} (\bar{i}_q)^{v_3} \mathcal{B}_{\text{NWA},\lambda^i,v_1,q}^{(k)}(\bar{i}_q x) \mathcal{B}_{\text{NWA},\lambda^j,v_2,q}^{(k-1)}(\bar{j}_q y) s_{\text{NWA},\lambda^i,v_3,q}(j)
\end{aligned} \tag{34}$$

Proof. Use the symmetry in $\phi_q(t)$. \square

Theorem 2.5. A q -analogue of [12, p. 2996]. We have

$$\begin{aligned}
&\sum_{v=0}^n \binom{n}{v}_q \sum_{q \ l=0}^{i-1} \sum_{m=0}^{j-1} \lambda^{l+m} (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{B}_{\text{NWA},\lambda,v,q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}l_q}{\bar{i}_q} \right) \\
&\mathcal{B}_{\text{NWA},\lambda,n-v,q}^{(k)} \left(\bar{i}_q y \oplus_q \frac{\bar{i}m_q}{\bar{j}_q} \right) \\
&= \sum_{v=0}^n \binom{n}{v}_q \sum_{q \ l=0}^{j-1} \sum_{m=0}^{i-1} \lambda^{l+m} (\bar{j}_q)^v (\bar{i}_q)^{n-v} \mathcal{B}_{\text{NWA},\lambda,v,q}^{(k)} \left(\bar{i}_q x \oplus_q \frac{\bar{i}l_q}{\bar{j}_q} \right) \\
&\mathcal{B}_{\text{NWA},\lambda,n-v,q}^{(k)} \left(\bar{j}_q y \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right)
\end{aligned} \tag{35}$$

Proof. We can expand the following symmetric function $\phi'_q(t)$ by using the formula for a geometric sequence:

$$\begin{aligned}
 \phi'_q(t) &\equiv \frac{\mathbb{E}_q(\bar{i}\bar{j}_q(x \oplus y)t)(\lambda^i \mathbb{E}_q(\bar{i}\bar{j}_q t) - 1)(\lambda^j \mathbb{E}_q(\bar{i}\bar{j}_q t) - 1)}{(\lambda \mathbb{E}_q(\bar{i}_q t) - 1)^k (\lambda \mathbb{E}_q(\bar{j}_q t) - 1)^k} t^{2k-2} \\
 &= \mathbb{E}_q(\bar{i}\bar{j}_q(x \oplus y)t) \frac{1}{(\bar{i}_q)^{k-1} (\bar{j}_q)^{k-1}} \\
 &\quad \left(\frac{\bar{i}_q t}{\lambda \mathbb{E}_q(\bar{i}_q t) - 1} \right)^{k-1} \left(\frac{\bar{j}_q t}{\lambda \mathbb{E}_q(\bar{j}_q t) - 1} \right)^{k-1} \left(\frac{\lambda^i \mathbb{E}_q(\bar{i}\bar{j}_q t) - 1}{\lambda \mathbb{E}_q(\bar{j}_q t) - 1} \right) \left(\frac{\lambda^j \mathbb{E}_q(\bar{i}\bar{j}_q t) - 1}{\lambda \mathbb{E}_q(\bar{i}_q t) - 1} \right) \\
 &= \frac{1}{(\bar{i}_q)^{k-1} (\bar{j}_q)^{k-1}} \sum_{l=0}^{i-1} \sum_{m=0}^{j-1} \lambda^{l+m} \left(\frac{\bar{i}_q t}{\lambda \mathbb{E}_q(\bar{i}_q t) - 1} \right)^{k-1} \left(\frac{\bar{j}_q t}{\lambda \mathbb{E}_q(\bar{j}_q t) - 1} \right)^{k-1} \\
 &\quad \mathbb{E}_q \left(\left(\bar{j}_q x \oplus_q \frac{\bar{j}l_q}{\bar{i}_q} \right) \bar{i}_q t \right) \mathbb{E}_q \left(\left(\bar{i}_q y \oplus_q \frac{\bar{i}m_q}{\bar{j}_q} \right) \bar{j}_q t \right) \\
 &= \frac{1}{(\bar{i}_q)^{k-1} (\bar{j}_q)^{k-1}} \left(\sum_{l=0}^{i-1} \lambda^l \sum_{v_1=0}^{\infty} \frac{(\bar{i}_q)^{v_1} t^{v_1}}{\{v_1\}_q!} \mathcal{B}_{\text{NWA}, \lambda, v_1, q}^{(k-1)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}l_q}{\bar{i}_q} \right) \right) \\
 &\quad \left(\sum_{m=0}^{j-1} \lambda^m \sum_{v_2=0}^{\infty} \frac{(\bar{j}_q)^{v_2} t^{v_2}}{\{v_2\}_q!} \mathcal{B}_{\text{NWA}, \lambda, v_2, q}^{(k-1)} \left(\bar{i}_q y \oplus_q \frac{\bar{i}m_q}{\bar{j}_q} \right) \right).
 \end{aligned} \tag{36}$$

The theorem follows by using the symmetry in $\phi'_q(t)$ and changing $k-1$ to k . \square

Theorem 2.6. A q -analogue of [12, p. 2997]. We have

$$\begin{aligned}
 &\sum_{v=0}^n \binom{n}{v} \sum_{q|l=0}^{i-1} (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{B}_{\text{NWA}, \lambda, n-v, q}^{(k)}(\bar{i}_q y) \\
 &\quad \sum_{m=0}^{j-1} \lambda^{l+m} \mathcal{B}_{\text{NWA}, \lambda, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}l_q}{\bar{i}_q} \oplus_q \bar{m}_q \right) \\
 &= \sum_{v=0}^n \binom{n}{v} \sum_{q|l=0}^{j-1} (\bar{j}_q)^v (\bar{i}_q)^{n-v} \mathcal{B}_{\text{NWA}, \lambda, n-v, q}^{(k)}(\bar{j}_q y) \\
 &\quad \sum_{m=0}^{i-1} \lambda^{l+m} \mathcal{B}_{\text{NWA}, \lambda, v, q}^{(k)} \left(\bar{i}_q x \oplus_q \frac{\bar{i}l_q}{\bar{j}_q} \oplus_q \bar{m}_q \right).
 \end{aligned} \tag{37}$$

Proof. Similar to above. \square

Theorem 2.7. A q -analogue of [11, p. 552]. We have

$$\begin{aligned}
& \frac{1}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \sum_{m=0}^n \binom{n}{m}_q (\bar{i}_q)^m (\bar{j}_q)^{n-m} \\
& \mathcal{B}_{\text{NWA}, \lambda^j, n-m, q}^{(k-1)} (\bar{i}_q y) \sum_{l=0}^{i-1} \lambda^{jl} \mathcal{B}_{\text{NWA}, \lambda^i, m, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j} \bar{l}_q}{\bar{i}_q} \right) \\
& = \frac{1}{(\bar{j}_q)^k (\bar{i}_q)^{k-1}} \sum_{m=0}^n \binom{n}{m}_q (\bar{j}_q)^m (\bar{i}_q)^{n-m} \\
& \mathcal{B}_{\text{NWA}, \lambda^i, n-m, q}^{(k-1)} (\bar{j}_q y) \sum_{l=0}^{j-1} \lambda^{il} \mathcal{B}_{\text{NWA}, \lambda^j, m, q}^{(k)} \left(\bar{i}_q x \oplus_q \frac{\bar{i} \bar{l}_q}{\bar{j}_q} \right).
\end{aligned} \tag{38}$$

Proof. We can expand the following symmetric function $\psi_q(t)$ by using the formula for a geometric sequence:

$$\begin{aligned}
\psi_q(t) & \equiv \frac{\mathbb{E}_q(\bar{i} \bar{j}_q(x \oplus y)t) (\lambda^{ij} \mathbb{E}_q(\bar{i} \bar{j}_q t) - 1)}{(\lambda^i \mathbb{E}_q(\bar{i}_q t) - 1)^k (\lambda^j \mathbb{E}_q(\bar{j}_q t) - 1)^k} t^{2k-1} = \mathbb{E}_q(\bar{i} \bar{j}_q(x \oplus y)t) \frac{1}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \\
& \left(\frac{\bar{i}_q t}{\lambda^i \mathbb{E}_q(\bar{i}_q t) - 1} \right)^k \left(\frac{\bar{j}_q t}{\lambda^j \mathbb{E}_q(\bar{j}_q t) - 1} \right)^{k-1} \left(\frac{\lambda^{ij} \mathbb{E}_q(\bar{i} \bar{j}_q t) - 1}{\lambda^j \mathbb{E}_q(\bar{j}_q t) - 1} \right) \\
& = \frac{1}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \left(\frac{\bar{i}_q t}{\lambda^i \mathbb{E}_q(\bar{i}_q t) - 1} \right)^k \left(\frac{\bar{j}_q t}{\lambda^j \mathbb{E}_q(\bar{j}_q t) - 1} \right)^{k-1} \sum_{l=0}^{i-1} \lambda^{lj} \\
& \mathbb{E}_q \left(\left(\bar{j}_q x \oplus_q \frac{\bar{j} \bar{l}_q}{\bar{i}_q} \right) \bar{i}_q t \right) \mathbb{E}_q \left((\bar{i}_q y) \bar{j}_q t \right) \\
& = \frac{1}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \left(\sum_{l=0}^{i-1} \lambda^{jl} \sum_{v_1=0}^{\infty} \frac{(\bar{i}_q)^{v_1} t^{v_1}}{\{v_1\}_q!} \mathcal{B}_{\text{NWA}, \lambda^i, v_1, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j} \bar{l}_q}{\bar{i}_q} \right) \right) \\
& \sum_{v_2=0}^{\infty} \frac{(\bar{j}_q)^{v_2} t^{v_2}}{\{v_2\}_q!} \mathcal{B}_{\text{NWA}, \lambda^j, v_2, q}^{(k-1)} (\bar{i}_q y) = \frac{1}{(\bar{i}_q)^k (\bar{j}_q)^{k-1}} \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} \sum_{q, l=0}^{i-1} \right. \\
& \left. \lambda^{jl} (\bar{i}_q)^m (\bar{j}_q)^{n-m} \mathcal{B}_{\text{NWA}, \lambda^i, m, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j} \bar{l}_q}{\bar{i}_q} \right) \mathcal{B}_{\text{NWA}, \lambda^j, n-m, q}^{(k-1)} (\bar{i}_q y) \right) \frac{t^n}{\{n\}_q!}.
\end{aligned} \tag{39}$$

The theorem follows by using the symmetry in $\psi_q(t)$. \square

3. Mixed formulas

This is a continuation of the very similar computations in [4], to which we will refer.

Corollary 3.1. *A q -analogue of [10, (31) p. 314]. If i is even then*

$$\begin{aligned}
 & \sum_{m=0}^1 \lambda^{im} \mathcal{F}_{\text{NWA}, \lambda^2, n-1, q} \left(\bar{i}_q x \oplus_q \frac{\bar{i}m_q}{\bar{2}_q} \right) \\
 &= -\frac{2}{\{n\}_q (\bar{2}_q)^{n-1}} \sum_{k=0}^n \binom{n}{k}_q \frac{(\bar{i}_q)^k}{i} (\bar{2}_q)^{n-k} \mathcal{B}_{\text{NWA}, \lambda^i, k, q} (\bar{2}_q x) \sigma_{\text{NWA}, \lambda^2, n-k, q}(i) \\
 &= \frac{1}{(\bar{2}_q)^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\bar{2}_q)^k (\bar{i}_q)^{n-k-1} \mathcal{F}_{\text{NWA}, \lambda^2, k, q} (\bar{i}_q x) s_{\text{NWA}, \lambda^i, n-k-1, q}(2) \\
 &= -\frac{2}{\{n\}_q (\bar{2}_q)^{n-1}} \frac{(\bar{i}_q)^n}{i} \sum_{m=0}^{i-1} (-1)^m \lambda^{2m} \mathcal{B}_{\text{NWA}, \lambda^i, n, q} \left(\bar{2}_q x \oplus_q \frac{\bar{2}m_q}{\bar{i}_q} \right).
 \end{aligned} \tag{40}$$

Proof. Put $j = 2$ in formula (56) [4], and multiply by $-\frac{2}{\{n\}_q (\bar{2}_q)^{n-1}}$. \square

Corollary 3.2. *A q -analogue of [10, (32) p. 314].*

$$\begin{aligned}
 & \sum_{m=0}^1 (-1)^{m+1} \lambda^m \mathcal{B}_{\text{NWA}, \lambda, n, q} \left(x \oplus_q \frac{\bar{2}m_q}{\bar{2}_q} \right) \\
 &= \frac{\{n\}_q (\bar{2}_q)^{n-1}}{(\bar{2}_q)^n} \sum_{m=0}^1 \lambda^m \mathcal{F}_{\text{NWA}, \lambda, n-1, q} \left(x \oplus_q \frac{\bar{2}m_q}{\bar{2}_q} \right).
 \end{aligned} \tag{41}$$

Proof. Put $i = 2$ in formula (40), replace x and λ^2 by $\frac{x}{\bar{2}_q}$ and λ , and multiply by $\frac{\{n\}_q (\bar{2}_q)^{n-1}}{(\bar{2}_q)^n}$. \square

Corollary 3.3. *A q -analogue of [10, (33) p. 314].*

$$\begin{aligned}
 & \sum_{m=0}^1 (-1)^m \lambda^{jm} \mathcal{B}_{\text{NWA}, \lambda^2, n, q} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{2}_q} \right) = -\frac{\{n\}_q}{(\bar{2}_q)^n} \sum_{k=0}^{n-1} \binom{n-1}{k}_q \\
 & (\bar{j}_q)^k (\bar{2}_q)^{n-k-1} \mathcal{F}_{\text{NWA}, \lambda^j, k, q} (\bar{2}_q x) s_{\text{NWA}, \lambda^2, n-k-1, q}(j) \\
 &= -\frac{\{n\}_q}{(\bar{2}_q)^n} (\bar{j}_q)^{n-1} \sum_{m=0}^{j-1} \lambda^{2m} \mathcal{F}_{\text{NWA}, \lambda^j, n-1, q} \left(\bar{2}_q x \oplus_q \frac{\bar{2}m_q}{\bar{j}_q} \right).
 \end{aligned} \tag{42}$$

Proof. Put $i = 2$ in formula (56) [4], and multiply by $\frac{2}{(\bar{2}_q)^n}$. \square

The following formula is a generalization of [4, (57)].

Theorem 3.1. *A q -analogue of [5, (3.9) p. 3356].*

$$\begin{aligned}
& \sum_{|v|=n} \binom{n}{\bar{v}}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} \mathcal{B}_{\text{NWA}, \lambda^i, v_1, q}^{(k)}(\bar{j}_q x) \mathcal{F}_{\text{NWA}, \lambda^j, v_2, q}^{(k-1)}(\bar{i}_q y) \sigma_{\text{NWA}, \lambda^j, v_3, q}(i) (\bar{j}_q)^{v_3} \\
&= \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{F}_{\text{NWA}, \lambda^j, n-v, q}^{(k-1)}(\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \\
& \mathcal{B}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right).
\end{aligned} \tag{43}$$

Proof. Define the following function

$$\begin{aligned}
g_q(t) &\equiv \frac{\mathbf{E}_q(\bar{i}\bar{j}_q(x \oplus y)t) ((-1)^{i+1} \lambda^{ij} \mathbf{E}_q(\bar{i}\bar{j}_q t) + 1)}{(\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1)^k (\lambda^j \mathbf{E}_q(\bar{j}_q t) + 1)^k} = \frac{2^{1-k}}{(\bar{i}_q t)^k} \mathbf{E}_q(\bar{i}\bar{j}_q(x \oplus y)t) \\
& \left(\frac{\bar{i}_q t}{\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1} \right)^k \left(\frac{2}{\lambda^j \mathbf{E}_q(\bar{j}_q t) + 1} \right)^{k-1} \left(\frac{(-1)^{i+1} \lambda^{ij} \mathbf{E}_q(\bar{i}\bar{j}_q t) + 1}{\lambda^j \mathbf{E}_q(\bar{j}_q t) + 1} \right).
\end{aligned} \tag{44}$$

By using the formula for a geometric sequence, we can expand $g_q(t)$ in two ways:

$$\begin{aligned}
g_q(t) &\stackrel{\text{by(26)}}{=} \frac{2^{1-k}}{(\bar{i}_q t)^k} \left(\sum_{v=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^i, v, q}^{(k)}(\bar{j}_q x) \frac{(\bar{i}_q t)^v}{\{v\}_q!} \right) \left(\sum_{m=0}^{\infty} \sigma_{\text{NWA}, \lambda^j, m, q}(i) \frac{(\bar{j}_q t)^m}{\{m\}_q!} \right) \\
& \left(\sum_{l=0}^{\infty} \mathcal{F}_{\text{NWA}, \lambda^j, l, q}^{(k-1)}(\bar{i}_q y) \frac{(\bar{j}_q t)^l}{\{l\}_q!} \right) = \frac{2^{1-k}}{(\bar{i}_q t)^k} \left(\frac{\bar{i}_q t}{\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1} \right)^k \\
& \frac{2^{k-1}}{(\lambda^j \mathbf{E}_q(\bar{j}_q t) + 1)^{k-1}} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \mathbf{E}_q \left(\left(\bar{j}_q x \oplus_q \bar{j}_q y \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \bar{i}_q t \right) \\
&= \frac{2^{1-k}}{(\bar{i}_q t)^k} \sum_{m=0}^{i-1} (-1)^m \lambda^{jm} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{l\}_q!} \mathcal{B}_{\text{NWA}, \lambda^i, l, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \\
& \sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{n\}_q!} \mathcal{F}_{\text{NWA}, \lambda^j, n, q}^{(k-1)}(\bar{i}_q y).
\end{aligned} \tag{45}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 3.2. *A q -analogue of [5, p. 3353]. Under the assumption that i is even, we have*

$$\begin{aligned} & \sum_{|\mathbf{v}|=n} \binom{n}{\bar{\mathbf{v}}}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} \mathcal{B}_{\text{NWA},\lambda^i,v_1,q}^{(k)}(\bar{j}_q x) \mathcal{F}_{\text{NWA},\lambda^j,v_2,q}^{(k-1)}(\bar{i}_q y) s_{\text{NWA},\lambda^i,v_3,q}(i) (\bar{j}_q)^{v_3} \\ &= -\frac{\{n\}_q (\bar{i}_q)^k}{2(\bar{i}_q)^{k-1}} \sum_{|\mathbf{v}|=n-1} \binom{n-1}{\bar{\mathbf{v}}}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} (\bar{j}_q)^{v_3} \mathcal{B}_{\text{NWA},\lambda^i,v_1,q}^{(k-1)}(\bar{j}_q y) \\ & \mathcal{F}_{\text{NWA},\lambda^j,v_2,q}^{(k)}(\bar{i}_q x) s_{\text{NWA},\lambda^i,v_3,q}(j). \end{aligned} \tag{46}$$

Proof. We can write $g_q(t)$ as follows:

$$\begin{aligned} g_q(t) & \stackrel{\text{by(25),(44)}}{=} \frac{2^{1-k}}{(\bar{i}_q t)^k} \left(\sum_{v=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,v,q}^{(k)}(\bar{j}_q x) \frac{(\bar{i}_q t)^v}{\{\mathbf{v}\}_q!} \right) \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^j,m,q}(i) \frac{(\bar{j}_q t)^m}{\{m\}_q!} \right) \\ & \left(\sum_{l=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^j,l,q}^{(k-1)}(\bar{i}_q y) \frac{(\bar{j}_q t)^l}{\{l\}_q!} \right) = -\frac{2^{-k}}{(\bar{i}_q t)^{k-1}} E_q(\bar{i} \bar{j}_q (x \oplus y) t) \left(\frac{\bar{i}_q t}{\lambda^i E_q(\bar{i}_q t) - 1} \right)^{k-1} \\ & \left(\frac{2}{\lambda^j E_q(\bar{j}_q t) + 1} \right)^k \left(\frac{\lambda^i E_q(\bar{i} \bar{j}_q t) - 1}{\lambda^i E_q(\bar{i}_q t) - 1} \right) \stackrel{\text{by(25)}}{=} \\ & -\frac{2^{-k}}{(\bar{i}_q t)^{k-1}} \left(\sum_{v=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^j,v,q}^{(k)}(\bar{i}_q x) \frac{(\bar{j}_q t)^v}{\{\mathbf{v}\}_q!} \right) \\ & \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^i,m,q}(j) \frac{(\bar{j}_q t)^m}{\{m\}_q!} \right) \left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^i,l,q}^{(k-1)}(\bar{j}_q y) \frac{(\bar{i}_q t)^l}{\{l\}_q!} \right). \end{aligned} \tag{47}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

Theorem 3.3. *A q -analogue of [5, p. 3353]. Under the assumption that i is*

even,

$$\begin{aligned}
& \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{F}_{\text{NWA}, \lambda^j, n-v, q}^{(k-1)}(\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} (-1)^m \\
& \mathcal{B}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \\
& = -\frac{\{n\}_q (\bar{i}_q)^k}{2(\bar{i}_q)^{k-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (\bar{i}_q)^{n-k-1} (\bar{j}_q)^k \mathcal{B}_{\text{NWA}, \lambda^i, n-k-1, q}^{(k-1)}(\bar{j}_q y) \sum_{m=0}^{j-1} \lambda^{im} \\
& \mathcal{F}_{\text{NWA}, \lambda^j, k, q}^{(k)} \left(\bar{i}_q x \oplus_q \frac{\bar{i}m_q}{\bar{j}_q} \right).
\end{aligned} \tag{48}$$

Proof. We can expand $g_q(t)$ as follows:

$$\begin{aligned}
g_q(t) & \stackrel{\text{by(44)}}{=} -\frac{2^{-k}}{(\bar{i}_q t)^{k-1}} \mathbf{E}_q(\bar{i} \bar{j}_q (x \oplus y) t) \\
& \left(\frac{\bar{i}_q t}{\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1} \right)^{k-1} \left(\frac{2}{\lambda^j \mathbf{E}_q(\bar{j}_q t) + 1} \right)^k \left(\frac{\lambda^{ij} \mathbf{E}_q(\bar{i} \bar{j}_q t) - 1}{\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1} \right). \\
& = -\frac{2^{-k}}{(\bar{i}_q t)^{k-1}} \left(\frac{\bar{i}_q t}{\lambda^i \mathbf{E}_q(\bar{i}_q t) - 1} \right)^{k-1} \\
& \frac{2^k}{(\lambda^j \mathbf{E}_q(\bar{j}_q t) + 1)^k} \sum_{m=0}^{j-1} \lambda^{im} \mathbf{E}_q \left(\left(\bar{i}_q x \oplus_q \frac{\bar{i}m_q}{\bar{j}_q} \right) \bar{j}_q t \right) \mathbf{E}_q(\bar{i} \bar{j}_q y t) \\
& = -\frac{2^{-k}}{(\bar{i}_q t)^{k-1}} \sum_{m=0}^{j-1} \lambda^{im} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{l\}_q!} \mathcal{B}_{\text{NWA}, \lambda^i, l, q}^{(k-1)}(\bar{j}_q y) \\
& \sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{n\}_q!} \mathcal{F}_{\text{NWA}, \lambda^j, n, q}^{(k)}(\bar{i}_q x \oplus_q \frac{\bar{i}m_q}{\bar{j}_q}).
\end{aligned} \tag{49}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

Theorem 3.4.

$$\begin{aligned}
& \sum_{|v|=n} \binom{n}{v}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} (\bar{j}_q)^{v_3} \mathcal{F}_{\text{NWA}, \lambda^i, v_1, q}^{(k)}(\bar{j}_q x) \mathcal{B}_{\text{NWA}, \lambda^j, v_2, q}^{(k-1)}(\bar{i}_q y) s_{\text{NWA}, \lambda^j, v_3, q}(i) \\
& = \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{B}_{\text{NWA}, \lambda^j, n-v, q}^{(k-1)}(\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{F}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right)
\end{aligned} \tag{50}$$

Proof. Define the following function

$$\Psi_q(t) \equiv \frac{E_q(\bar{i}\bar{j}_q(x \oplus y)t)(\lambda^{ij}E_q(\bar{i}\bar{j}_qt) - 1)}{(\lambda^i E_q(\bar{i}_qt) + 1)^k (\lambda^j E_q(\bar{j}_qt) - 1)^k} t^{k-1} = E_q(\bar{i}\bar{j}_q(x \oplus y)t) \left(\frac{2}{(\lambda^i E_q(\bar{i}_qt) + 1)} \right)^k \left(\frac{\bar{j}_qt}{\lambda^j E_q(\bar{j}_qt) - 1} \right)^{k-1} \left(\frac{\lambda^{ij} E_q(\bar{i}\bar{j}_qt) - 1}{\lambda^j E_q(\bar{j}_qt) - 1} \right) \frac{2^{-k}}{(\bar{j}_q)^{k-1}}. \quad (51)$$

By using the formula for a geometric sequence, we can expand $\Psi_q(t)$ in two ways:

$$\begin{aligned} \Psi_q(t) &\stackrel{\text{by(25)}}{=} \left(\sum_{v=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda^i,v,q}^{(k)}(\bar{j}_q x) \frac{(\bar{i}_qt)^v}{\{v\}_q!} \right) \left(\sum_{m=0}^{\infty} s_{\text{NWA},\lambda^j,m,q}(i) \frac{(\bar{j}_qt)^m}{\{m\}_q!} \right) \\ &\left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA},\lambda^j,l,q}^{(k-1)}(\bar{i}_q y) \frac{(\bar{j}_qt)^l}{\{l\}_q!} \right) \frac{2^{-k}}{(\bar{j}_q)^{k-1}} = \frac{2^k}{(\lambda^i E_q(\bar{i}_qt) - 1)^k} \\ &\frac{(\bar{j}_qt)^{k-1}}{(\lambda^j E_q(\bar{j}_qt) - 1)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} E_q \left(\left(\bar{j}_q x \oplus_q \bar{j}_q y \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \bar{i}_q t \right) \frac{2^{-k}}{(\bar{j}_q)^{k-1}} \\ &= \sum_{m=0}^{i-1} \frac{2^{-k} \lambda^{jm}}{(\bar{j}_q)^{k-1}} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{l\}_q!} \mathcal{F}_{\text{NWA},\lambda^i,l,q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \\ &\sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{n\}_q!} \mathcal{B}_{\text{NWA},\lambda^j,n,q}^{(k-1)}(\bar{i}_q y). \end{aligned} \quad (52)$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. □

The following example illustrates that similar formulas with \mathcal{H} polynomials can easily be constructed.

Theorem 3.5.

$$\begin{aligned} &\sum_{|v|=n} \binom{n}{v}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} (\bar{j}_q)^{v_3} \mathcal{H}_{\text{NWA},\lambda^i,v_1,q}^{(k)}(\bar{j}_q x) \mathcal{B}_{\text{NWA},\lambda^j,v_2,q}^{(k-1)}(\bar{i}_q y) s_{\text{NWA},\lambda^j,v_3,q}(i) \\ &= \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{B}_{\text{NWA},\lambda^j,n-v,q}^{(k-1)}(\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{H}_{\text{NWA},\lambda^i,v,q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \end{aligned} \quad (53)$$

Proof. Use $\Psi_q(t)$ again. □

Theorem 3.6. A q -analogue of [5, (3.11) p. 3356].

$$\begin{aligned}
& \sum_{|v|=n} \binom{n}{v}_q (\bar{i}_q)^{v_1} (\bar{j}_q)^{v_2} (\bar{j}_q)^{v_3} \mathcal{F}_{\text{NWA}, \lambda^i, v_1, q}^{(k)} (\bar{j}_q x) \mathcal{B}_{\text{NWA}, \lambda^j, v_2, q}^{(k-1)} (\bar{i}_q y) n_{\text{NWA}, \lambda^j, v_3, q}(i) \\
&= \sum_{v=0}^n \binom{n}{v}_q (\bar{i}_q)^v (\bar{j}_q)^{n-v} \mathcal{B}_{\text{NWA}, \lambda^j, n-v, q}^{(k-1)} (\bar{i}_q y) \sum_{m=0}^{i-1} \lambda^{jm} \mathcal{F}_{\text{NWA}, \lambda^i, v, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right)
\end{aligned} \tag{54}$$

Proof. Define the following function

$$\begin{aligned}
f_q(t) &\equiv \frac{\mathbb{E}_q(\bar{i}\bar{j}_q(x \oplus y)t)(\lambda^{ij}\mathbb{E}_q(\bar{i}\bar{j}_qt) - 1)}{(\lambda^i\mathbb{E}_q(\bar{i}_qt) + 1)^k(\lambda^j\mathbb{E}_q(\bar{j}_qt) - 1)^k} t^k = \mathbb{E}_q(\bar{i}\bar{j}_q(x \oplus y)t) \\
&\left(\frac{2}{\lambda^i\mathbb{E}_q(\bar{i}_qt) + 1} \right)^k \left(\frac{\bar{j}_qt}{\lambda^j\mathbb{E}_q(\bar{j}_qt) - 1} \right)^{k-1} \left(\frac{\lambda^{ij}\mathbb{E}_q(\bar{i}\bar{j}_qt) - 1}{\lambda^j\mathbb{E}_q(\bar{j}_qt) - 1} \right) \frac{1}{2^k(\bar{j}_q)^{k-1}}.
\end{aligned} \tag{55}$$

By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$\begin{aligned}
f_q(t) &\stackrel{\text{by(25)}}{=} \left(\sum_{v=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^i, v, q}^{(k)} (\bar{j}_q x) \frac{(\bar{i}_qt)^v}{\{v\}_q!} \right) \left(\sum_{m=0}^{\infty} s_{\text{NWA}, \lambda^j, m, q}(i) \frac{(\bar{j}_qt)^m}{\{m\}_q!} \right) \\
&\left(\sum_{l=0}^{\infty} \mathcal{B}_{\text{NWA}, \lambda^j, l, q}^{(k-1)} (\bar{i}_q y) \frac{(\bar{j}_qt)^l}{\{l\}_q!} \right) \frac{1}{2^k(\bar{j}_q)^{k-1}} = \frac{2^k}{(\lambda^i\mathbb{E}_q(\bar{i}_qt) + 1)^k} \\
&\frac{(\bar{j}_qt)^{k-1}}{(\lambda^j\mathbb{E}_q(\bar{j}_qt) - 1)^{k-1}} \sum_{m=0}^{i-1} \lambda^{jm} \mathbb{E}_q \left(\left(\bar{j}_q x \oplus_q \bar{j}_q y \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \bar{i}_q t \right) \frac{1}{2^k(\bar{j}_q)^{k-1}} \\
&= \sum_{m=0}^{i-1} \frac{\lambda^{jm}}{2^k(\bar{j}_q)^{k-1}} \sum_{l=0}^{\infty} \frac{(\bar{i}_q)^l t^l}{\{l\}_q!} \mathcal{F}_{\text{NWA}, \lambda^i, l, q}^{(k)} \left(\bar{j}_q x \oplus_q \frac{\bar{j}m_q}{\bar{i}_q} \right) \\
&\sum_{n=0}^{\infty} \frac{(\bar{j}_q)^n t^n}{\{n\}_q!} \mathcal{B}_{\text{NWA}, \lambda^j, n, q}^{(k-1)} (\bar{i}_q y).
\end{aligned} \tag{56}$$

The theorem follows by equating the coefficients of $\frac{t^n}{\{n\}_q!}$. \square

4. Multiplication formulas

We will now define two quite general q -Appell polynomials, which have some similarities with the Appell polynomials in [9]. The names are chosen to resemble the Euler and Bernoulli polynomials.

Definition 4.1. A q -analogue of Lu, Luo [6, p. 4]. The generating function for the generalized NWA q -Apostol \mathcal{E} polynomials of degree v and order n , $\mathcal{E}_{\text{NWA},\lambda,\mu,\theta;v,q}^{(n)}(x)$, is given by

$$\left(\frac{2^\mu t^\theta}{\lambda E_q(t) + 1}\right)^n E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v}{\{v\}_q!} \mathcal{E}_{\text{NWA},\lambda,\mu,\theta;v,q}^{(n)}(x), \theta \in \mathbb{N}. \quad (57)$$

Several q -Appell polynomials in this article are special cases of these polynomials, e.g. the q -Euler polynomial is the case $\theta = 0, \mu = 1$.

Theorem 4.1. A q -analogue of [6, (2.3) p. 5], first multiplication formula for q -Apostol- \mathcal{E} polynomials

$$\mathcal{E}_{\text{NWA},\lambda,\mu,\theta;v,q}^{(n)}(\bar{m}_q x) = \frac{(\bar{m}_q)^v}{((\bar{m}_q)^\theta)^n} \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{E}_{\text{NWA},\lambda^m,\mu,\theta;v,q}^{(n)}\left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}\right), \quad (58)$$

where $k = j_1 + 2j_2 + \dots + (m-1)j_{m-1}$, m odd.

Proof.

$$\begin{aligned} \sum_{v=0}^{\infty} \mathcal{E}_{\text{NWA},\lambda,\mu,\theta;v,q}^{(n)}(\bar{m}_q x) \frac{t^v}{\{v\}_q!} &= \frac{(2^\mu t^\theta)^n}{(\lambda E_q(t) + 1)^n} E_q(\bar{m}_q xt) \\ &= \frac{(2^\mu t^\theta)^n}{(\lambda^m E_q(\bar{m}_q t) + 1)^n} \left(\sum_{i=0}^{m-1} (-\lambda)^i E_q(\bar{i}_q t)\right)^n E_q(\bar{m}_q xt) \\ &\stackrel{\text{by(6)}}{=} \left(\frac{2^\mu t^\theta \bar{m}_q^\theta}{(\lambda^m E_q(\bar{m}_q t) + 1)}\right)^n \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} (-\lambda)^k E_q\left(\left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}\right) \bar{m}_q t\right) \frac{1}{((\bar{m}_q)^\theta)^n} \\ &= \sum_{v=0}^{\infty} \left(\frac{(\bar{m}_q)^v}{((\bar{m}_q)^\theta)^n} \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} (-\lambda)^k \mathcal{E}_{\text{NWA},\lambda^m,\mu,\theta;v,q}^{(n)}\left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}\right)\right) \frac{t^v}{\{v\}_q!}. \end{aligned} \quad (59)$$

The theorem follows by equating the coefficients of $\frac{t^v}{\{v\}_q!}$. □

The following formula only applies for special values of the integers.

Theorem 4.2. A q -analogue of [6, (2.4) p. 5], second multiplication formula for q -Apostol- \mathcal{E} polynomials.

$$\begin{aligned} \mathcal{E}_{\text{NWA},\lambda,\mu,\theta;v,q}^{(n)}(\bar{m}_q x) &= \frac{(-1)^n 2^{\mu n} (\bar{m}_q)^{v+(1-\theta)n}}{\{v+1\}_{(1-\theta)n,q} (\bar{m}_q)^n} \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{B}_{\text{NWA},\lambda^m,v+(1-\theta)n,q}^{(n)}\left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}\right), \end{aligned} \quad (60)$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m even, $\mathbf{v} \geq (\theta - 1)n$.

Proof.

$$\begin{aligned}
\sum_{\mathbf{v}=0}^{\infty} \mathcal{E}_{\text{NWA}, \lambda, \mu, \theta; \mathbf{v}, q}^{(n)}(\bar{m}_q x) \frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!} &= \frac{(2^\mu t^\theta)^n}{(\lambda E_q(t) + 1)^n} E_q(\bar{m}_q x t) \\
&= \frac{(2^\mu t^\theta)^n}{(1 - \lambda^m E_q(\bar{m}_q t))^n} \left(\sum_{i=0}^{m-1} (-\lambda)^i E_q(i_q t) \right)^n E_q(\bar{m}_q x t) \\
&= \left(\frac{2^\mu t \bar{m}_q}{1 - \lambda^m E_q(\bar{m}_q t)} \right)^n \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} (-\lambda)^k E_q \left((x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}) \bar{m}_q t \right) \frac{t^{(\theta-1)n}}{(\bar{m}_q)^n} \\
&= (-1)^n t^{(\theta-1)n} 2^{\mu n} \sum_{\mathbf{v}=0}^{\infty} \left(\frac{(\bar{m}_q)^{\mathbf{v}}}{(\bar{m}_q)^n} \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} (-\lambda)^k \mathcal{B}_{\text{NWA}, \lambda^m, \mathbf{v}; q}^{(n)} \left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q} \right) \right) \frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!}
\end{aligned} \tag{61}$$

The theorem follows by equating the coefficients of $\frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!}$. \square

Corollary 4.2. A q -analogue of [8, (2.1) p. 49], [6, p. 7], first multiplication formula for generalized q - \mathcal{H} polynomials.

$$\mathcal{H}_{\text{NWA}, \lambda, \mathbf{v}, q}^{(n)}(\bar{m}_q x) = \frac{(\bar{m}_q)^{\mathbf{v}}}{(\bar{m}_q)^n} \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{H}_{\text{NWA}, \lambda^m, \mathbf{v}, q}^{(n)} \left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q} \right), \tag{62}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m odd.

Corollary 4.3. A q -analogue of [8, (2.2) p. 49], [6, p. 7], second multiplication formula for generalized q - \mathcal{H} polynomials.

$$\mathcal{H}_{\text{NWA}, \lambda, \mathbf{v}, q}^{(n)}(\bar{m}_q x) = \frac{(-2)^n (\bar{m}_q)^{\mathbf{v}}}{(\bar{m}_q)^n} \sum_{|\vec{j}|=n} (-\lambda^k) \binom{n}{\vec{j}} \mathcal{B}_{\text{NWA}, \lambda^m, \mathbf{v}, q}^{(n)} \left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q} \right), \tag{63}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m even.

Theorem 4.3. A q -analogue of [8, p. 51], an explicit formula for the multiple alternating q -power sum:

$$\begin{aligned}
\sigma_{\text{NWA}, \lambda, \mathbf{v}, q}^{(l)}(n) &= 2^{-l} \sum_{j=0}^l \binom{l}{j} \frac{(-1)^{jn} \lambda^{(n-1)j+l}}{\{\mathbf{v}+1\}_{l, q}} \\
\sum_{m=0}^{\mathbf{v}+l} \binom{\mathbf{v}+l}{m}_q \mathcal{H}_{\text{NWA}, \lambda, m, q}^{(j)} \left(\overline{(n-1)j+l}_q \right) \mathcal{H}_{\text{NWA}, \lambda, \mathbf{v}+l-m, q}^{(l-j)},
\end{aligned} \tag{64}$$

Proof. We use the generating function technique. Put $k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0, 1 \leq i \leq n-1$. All zeros are neglected.

$$\begin{aligned}
 & \sum_{\mathbf{v}=0}^{\infty} \sigma_{\text{NWA},\lambda,\mathbf{v},q}^{(l)}(n) \frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!} \stackrel{\text{by(17)}}{=} (-1)^l \sum_{\mathbf{v}=0}^{\infty} \left(\sum_{|\vec{j}|=l} \binom{l}{\vec{j}} (-\lambda)^k (\bar{k}_q)^{\mathbf{v}} \right) \frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!} \\
 & = (\lambda E_q(t) - \lambda^2 E_q(2qt) + \cdots + (-1)^n \lambda^{n-1} E_q(\overline{n-1}qt))^l \\
 & = \left(\frac{(-\lambda)^n E_q(\bar{n}qt)}{\lambda E_q(t) + 1} + \frac{\lambda E_q(t)}{\lambda E_q(t) + 1} \right)^l \\
 & = \sum_{j=0}^l \binom{l}{j} \left(\frac{(-\lambda)^n E_q(\bar{n}qt)}{\lambda E_q(t) + 1} \right)^j \left(\frac{\lambda E_q(t)}{\lambda E_q(t) + 1} \right)^{l-j} \\
 & \stackrel{\text{by(6)}}{=} (2t)^{-l} \sum_{j=0}^l \binom{l}{j} (-1)^{jn} \lambda^{(n-1)j+l} \sum_{m=0}^{\infty} \mathcal{H}_{\text{NWA},\lambda,m,q}^{(j)} \left(\overline{(n-1)j+l} \right) \frac{t^m}{\{m\}_q!} \\
 & \sum_{i=0}^{\infty} \mathcal{H}_{\text{NWA},\lambda,i,q}^{(l-j)} \frac{t^i}{\{i\}_q!} = \sum_{\mathbf{v}=0}^{\infty} \left[2^{-l} \sum_{j=0}^l \binom{l}{j} \frac{(-1)^{jn} \lambda^{(n-1)j+l}}{\{\mathbf{v}+1\}_{l,q}} \right. \\
 & \left. \sum_{m=0}^{\mathbf{v}+l} \binom{\mathbf{v}+l}{m} \mathcal{H}_{\text{NWA},\lambda,m,q}^{(j)} \left(\overline{(n-1)j+l} \right) \mathcal{H}_{\text{NWA},\lambda,\mathbf{v}+l-m,m,q}^{(l-j)} \right] \frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!}.
 \end{aligned} \tag{65}$$

The theorem follows by equating the coefficients of $\frac{t^{\mathbf{v}}}{\{\mathbf{v}\}_q!}$. \square

Theorem 4.4. For m odd, we have the following recurrence relation for q -Apostol- \mathcal{E} -numbers.

$$\mathcal{E}_{\text{NWA},\lambda,\mu,\theta;n,q}^{(l)} = (-1)^l \sum_{j=0}^n \binom{n}{j}_q \frac{(\bar{m}_q)^n}{((\bar{m}_q)^\theta)^l (\bar{m}_q)^{n-j}} \mathcal{E}_{\text{NWA},\lambda^m,\mu,\theta;j,q}^{(l)} \sigma_{\text{NWA},\lambda,n-j,q}^{(l)}(m), \tag{66}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$ in $\sigma_{\text{NWA},\lambda,n-j,q}^{(l)}(m)$.

Proof.

$$\begin{aligned}
 & \mathcal{E}_{\text{NWA},\lambda,\mu,\theta;n,q}^{(l)} \stackrel{\text{by(58)}}{=} \frac{(\bar{m}_q)^n}{((\bar{m}_q)^\theta)^l} \sum_{|\vec{\mathbf{v}}|=l} (-\lambda)^k \binom{l}{\vec{\mathbf{v}}} \mathcal{E}_{\text{NWA},\lambda^m,\mu,\theta;n,q}^{(l)} \left(\frac{\bar{k}_q}{\bar{m}_q} \right) \\
 & = \frac{(\bar{m}_q)^n}{((\bar{m}_q)^\theta)^l} \sum_{|\vec{\mathbf{v}}|=l} (-\lambda)^k \binom{l}{\vec{\mathbf{v}}} \sum_{j=0}^n \binom{n}{j}_q \mathcal{E}_{\text{NWA},\lambda^m,\mu,\theta;j,q}^{(l)} \left(\frac{\bar{k}_q}{\bar{m}_q} \right)^{n-j} \\
 & = \sum_{j=0}^n \binom{n}{j}_q \frac{(\bar{m}_q)^n}{(\bar{m}_q)^{n-j} ((\bar{m}_q)^\theta)^l} \mathcal{E}_{\text{NWA},\lambda^m,\mu,\theta;j,q}^{(l)} \sum_{|\vec{\mathbf{v}}|=l} (-\lambda)^k \binom{l}{\vec{\mathbf{v}}} (\bar{k}_q)^{n-j} \stackrel{\text{by(17)}}{=} \text{LHS}.
 \end{aligned} \tag{67}$$

□

Definition 4.4. The generating function for the generalized NWA q -Apostol \mathcal{C} polynomials of degree v and order n , $\mathcal{C}_{\text{NWA},\lambda,\theta;v,q}^{(n)}(x)$, is given by

$$\left(\frac{t^\theta}{\lambda E_q(t) - 1}\right)^n E_q(xt) = \sum_{v=0}^{\infty} \frac{t^v}{\{v\}_q!} \mathcal{C}_{\text{NWA},\lambda,\theta;v,q}^{(n)}(x), \theta \in \mathbb{N}. \quad (68)$$

Theorem 4.5. *Multiplication formula for q -Apostol- \mathcal{C} polynomials*

$$\mathcal{C}_{\text{NWA},\lambda,\theta;v,q}^{(n)}(\bar{m}_q x) = \frac{(\bar{m}_q)^v}{((\bar{m}_q)^\theta)^n} \sum_{|\vec{j}|=n} \lambda^k \binom{n}{\vec{j}} \mathcal{C}_{\text{NWA},\lambda^m,\theta;v,q}^{(n)}\left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}\right), \quad (69)$$

where $k = j_1 + 2j_2 + \dots + (m-1)j_{m-1}$.

Proof.

$$\begin{aligned} \sum_{v=0}^{\infty} \mathcal{C}_{\text{NWA},\lambda,\theta;v,q}^{(n)}(\bar{m}_q x) \frac{t^v}{\{v\}_q!} &= \frac{t^{\theta n}}{(\lambda E_q(t) - 1)^n} E_q(\bar{m}_q x t) \\ &= \frac{t^{\theta n}}{(\lambda^m E_q(\bar{m}_q t) - 1)^n} \left(\sum_{i=0}^{m-1} \lambda^i E_q(\bar{i}_q t) \right)^n E_q(\bar{m}_q x t) \\ &\stackrel{\text{by (6)}}{=} \left(\frac{t^\theta \bar{m}_q^\theta}{(\lambda^m E_q(\bar{m}_q t) - 1)} \right)^n \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} \lambda^k E_q\left((x \oplus_q \frac{\bar{k}_q}{\bar{m}_q}) \bar{m}_q t \right) \frac{1}{((\bar{m}_q)^\theta)^n} \\ &= \sum_{v=0}^{\infty} \left(\frac{(\bar{m}_q)^v}{((\bar{m}_q)^\theta)^n} \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} \lambda^k \mathcal{C}_{\text{NWA},\lambda^m,\theta;v,q}^{(n)}\left(x \oplus_q \frac{\bar{k}_q}{\bar{m}_q} \right) \right) \frac{t^v}{\{v\}_q!}. \end{aligned} \quad (70)$$

The theorem follows by equating the coefficients of $\frac{t^v}{\{v\}_q!}$. □

5. Discussion

This was the first multiplication formula for a q -Appell polynomial of general form; the Ward numbers replace the integers in the function argument. Certainly there are other general q -Appell polynomials with similar expansion and multiplication formulas.

Many of the proofs use the formula for a geometric sequence in q -form and the generating function for the q -Appell polynomials and the power sums. The integers i and j are crucial for the formulas; by the generating function, if λ^i, v appears as index in a polynomial, certainly the factor $(\bar{i}_q)^v$ will also appear. If the orders of two polynomials in a formula are k and $k-1$, the last one with

index λ^j , and argument $\bar{i}_q y$, surely a function $\sigma_{\text{NWA}, \lambda^j, m, q}(i)$ or $s_{\text{NWA}, \lambda^j, m, q}(i)$, together with $(\bar{j}_q)^m$ will appear. If a polynomial has λ^i, v as index, it will have (\bar{j}_q) in the function argument, and vice versa.

These considerations also hold for the case $q = 1$. Even if the reader is not interested in q -calculus, this paper is a good summary of the recent trends on Apostol type Appell polynomials; just put $q = 1$.

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