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REGULARITY OF TOR FOR WEAKLY STABLE IDEALS

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It is proved that if I and J are weakly stable ideals in a polynomial ring $R = k[x_1,\dots,x_n]$, with k a field, then the regularity of $\operatorname{Tor}_i^R(R/I,R/J)$ has the expected upper bound. We also give a bound for the regularity of $\operatorname{Ext}_R^i(R/I,R)$ for I a weakly stable ideal.

1. Introduction

Let k be a field. Let $R = k[x_1,\dots,x_n]$ be a graded polynomial ring over k with $|x_i| = 1$ for every i. Let M and N be finitely generated graded R-modules. In [6] it is shown that if $\dim \operatorname{Tor}_1^R(M,N) \le 1$ then

$$\operatorname{reg}_R \operatorname{Tor}_i^R(M,N) \le \operatorname{reg}_R M + \operatorname{reg}_R N + i \quad \text{for every } i. \tag{1}$$

In general this bound may not hold. Indeed, assume it holds for $M = N = R/I$ where I is an homogeneous ideal in R and set $T_1 = \operatorname{Tor}_1^R(R/I,R/I)$. It is clear that $T_1 \cong I/I^2$; hence using the exact sequence

$$0 \to I^2 \to I \to T_1 \to 0$$

we deduce from 2.2 that

$$\operatorname{reg}_R I^2 \le \max\{\operatorname{reg}_R I, \operatorname{reg}_R T_1 + 1\}.$$

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Since $\operatorname{reg}_R R/I = \operatorname{reg}_R I - 1$, it follows

$$\operatorname{reg}_R I^2 \leq 2 \operatorname{reg}_R I.$$

Hence, every ideal not satisfying the previous inequality gives an example where (1) does not hold. There are many such examples; see for instance [5].

Although (1) does not hold in general, it is natural to look for classes of modules where the bound holds without the dimension assumption.

We prove that if I and J are weakly stable ideals then

$$\operatorname{reg}_R \operatorname{Tor}_i^R(R/I, R/J) \leq \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i \quad \text{for every } i,$$

see Theorem 3.7 and see Section 3 for the definition of weakly stable ideals.

The last section is concerned with the regularity of $\operatorname{Ext}^i_R(R/I, R)$ with I weakly stable ideal.

2. Background

Throughout the paper $R = k[x_1, \dots, x_n]$, with k a field, denotes a graded polynomial ring with $|x_i| = 1$ for every i. Let M and N be finitely generated graded R-modules. We denote by M_i the i-th graded component of M. The supremum and infimum of a graded module M are defined as

$$\sup M = \sup\{i \mid M_i \neq 0\}$$

$$\inf M \; = \inf\{i \mid M_i \neq 0\}.$$

We define the graded R-module $M(-a)$ by $M(-a)_d = M_{a+d}$, the shift of M up by a degrees. Let $\mathfrak{m}$ denote the ideal $(x_1, \dots, x_n)$. The $\mathfrak{m}$-torsion functor on the category of graded R-modules is defined by

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M : \mathfrak{m}^t x = 0 \quad \text{for some } t\}.$$

The i-th local cohomology module of M, denoted $H^i_{\mathfrak{m}}(M)$, is the i-th right derived functor of $\Gamma_{\mathfrak{m}}(\_)$ in the category of graded R-modules, and morphisms of degree 0.

We set $a_i(M) = \sup(H^i_{\mathfrak{m}}(M))$; by [1, 3.5.4] $a_i(M)$ is finite unless $H^i_{\mathfrak{m}}(M) = 0$ where we set $a_i(M) = -\infty$. The Castelnuovo-Mumford regularity of M is then

$$\operatorname{reg}_R M = \sup_i \{a_i(M) + i\}.$$

Regularity can also be computed with a minimal graded free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to 0$$

of M. Recall that $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$, so β_{ij} is the number of copies of $R(-j)$ in position i in the resolution. The number

$$t_i(M) = \sup\{j : \beta_{ij} \neq 0\},$$

is the largest degree of an element in the basis of F_i; it is easily seen that

$$t_i(M) = \sup(\operatorname{Tor}_i^R(M,k)).$$

It is proved, for example in [4, 2.2], that

$$\operatorname{reg}_R M = \sup_i \{t_i(M) - i\}.$$

Remark 2.1. If M is an R-module then $\operatorname{reg}_R M(-a) = \operatorname{reg}_R M + a$ for any $a \in \mathbb{Z}$. This can be checked by computing the regularity with a free resolution.

If M has finite length then $\operatorname{reg}_R M = \sup M$. This follows by computing regularity with local cohomology.

Remark 2.2. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of graded R-modules. Then

1. $\operatorname{reg}_R M \leq \max\{\operatorname{reg}_R L, \operatorname{reg}_R N\}$
2. $\operatorname{reg}_R L \leq \max\{\operatorname{reg}_R M, \operatorname{reg}_R N + 1\}$
3. $\operatorname{reg}_R N \leq \max\{\operatorname{reg}_R M, \operatorname{reg}_R L - 1\}$.

This follows from the induced long exact sequence in local cohomology.

The next lemma is a straightforward consequence of the previous inequalities.

Lemma 2.3. *If $K \xrightarrow{f} M \xrightarrow{t} N \xrightarrow{g} C$ is an exact sequence of graded R-modules, K and C have finite length then*

$$\operatorname{reg}_R M \leq \max\{\operatorname{reg}_R K, \operatorname{reg}_R N, \operatorname{reg}_R C + 1\}.$$

Proof. The exact sequence induces exact sequences of R-modules

$$0 \to \operatorname{Im} f \to M \to \operatorname{Im} t \to 0, \qquad 0 \to \operatorname{Im} t \to N \to \operatorname{Im} g \to 0.$$

By 2.2 these exact sequences give the following inequalities:

$$\operatorname{reg}_R M \leq \max\{\operatorname{reg}_R \operatorname{Im} f, \operatorname{reg}_R \operatorname{Im} t\} \qquad \operatorname{reg}_R \operatorname{Im} t \leq \max\{\operatorname{reg}_R N, \operatorname{reg}_R \operatorname{Im} g + 1\},$$

and hence an inequality

$$\operatorname{reg}_R M \leq \max\{\operatorname{reg}_R \operatorname{Im} f, \operatorname{reg}_R N, \operatorname{reg}_R \operatorname{Im} g + 1\}.$$

Since K and C have finite length $\operatorname{reg}_R \operatorname{Im} f \leq \operatorname{reg}_R K$ and $\operatorname{reg}_R \operatorname{Im} g \leq \operatorname{reg}_R C$. $\square$

Remark 2.4. Note that

$$\operatorname{reg}_R M = \max\{\operatorname{reg}_R(\Gamma_{\mathfrak{m}}(M)), \operatorname{reg}_R(M/\Gamma_{\mathfrak{m}}(M))\}.$$

This follows from the definition of regularity, since $H^0_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(M)$.

The following result is well-known.

Lemma 2.5. *If M has finite length then* $\operatorname{reg}_R \operatorname{Tor}_i^R(M,N) \le \operatorname{reg}_R M + t_i(N)$.
In particular,

$$\operatorname{reg}_R \operatorname{Tor}_i^R(M,N) \le \operatorname{reg}_R M + \operatorname{reg}_R N + i.$$

Proof. Write $M = \bigoplus_{i=a}^{b} M_i$ with $a = \inf M$ and $b = \sup M$. We use induction on $b-a$. If $b = a$ then $M = k(-a)^m$, and therefore,

$$\begin{aligned}
\operatorname{reg}_R \operatorname{Tor}_i^R(M,N) &= \operatorname{reg}_R \operatorname{Tor}_i^R(k(-a),N) \\
&= \operatorname{reg}_R \operatorname{Tor}_i^R(k,N)(-a) \\
&= \operatorname{reg}_R \operatorname{Tor}_i^R(k,N) + a \\
&= \operatorname{reg}_R \operatorname{Tor}_i^R(k,N) + \operatorname{reg}_R M \\
&= t_i(N) + \operatorname{reg}_R(M).
\end{aligned}$$

Now assume $b-a > 0$. Denote by $M_{>a}$ the module $\bigoplus_{i=a+1}^{b} M_i$. The short exact sequence

$$0 \to M_{>a} \to M \to k(-a)^m \to 0$$

induces, for each i, an exact sequence

$$\operatorname{Tor}_i^R(M_{>a},N) \to \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(k,N)^m(-a).$$

By induction and Lemma 2.3

$$\begin{aligned}
\operatorname{reg}_R \operatorname{Tor}_i^R(M,N) &\le \max\{\operatorname{reg}_R \operatorname{Tor}_i^R(M_{>a},N), \operatorname{reg}_R \operatorname{Tor}_i^R(k,N)(-a)\} \\
&\le \max\{\operatorname{reg}_R M_{>a} + t_i(N), a + t_i(N)\} \le \operatorname{reg}_R M + t_i(N).
\end{aligned}$$

The last assertion follows as $\operatorname{reg}_R N = \sup\{t_i(N) - i\}$. □

3. Regularity of Tor for weakly stable ideals

We study weakly stable ideals. Let I be a monomial ideal, for a monomial $u \in I$ we let $m(u)$ be the maximum index of a variable appearing in u and we let $l(u)$ be the highest power of $x_{m(u)}$ dividing u.

Definition 3.1. A monomial ideal I is *weakly stable* provided the following "exchange property" is satisfied; for any monomial $u \in I$ and for any $j < m(u)$ there exists a k such that $x_j^k u / x_{m(u)}^{l(u)} \in I$.

Remark 3.2. It is an easy exercise to prove that I is weakly stable if and only if the "exchange property" is verified only for the generators of I.

Remark 3.3. There is also an algebraic characterization of weakly stable ideals. In [2, 4.1.5] Caviglia proved that a monomial ideal I is weakly stable if and only if $\operatorname{Ass} I \subseteq \{(x_1, \ldots, x_t) \mid t = 0, 1, \ldots, n\}$.

Example 3.4. Let $I = (x_1^2, x_1x_2, x_1x_3, x_2^2)$. Clearly the 'exchange property' holds for x_1^2 and x_2^2. We have $m(x_1x_2) = 2$ and $l(x_1x_2) = 1$. For $j = 1$ we take $k = 1$ and we can see that $x_1x_1x_2/x_2$ is in I. The remaining generator is similar. The ideal I is primary and the radical of I is the ideal (x_1, x_2).

Remark 3.5. If I is a weakly stable ideal and J is a monomial ideal, then $(I : J)$ is a weakly stable ideal; see [2, 4.1.4(2)].

Lemma 3.6. *Suppose I is a weakly stable ideal of R and set*

$$I' = \bigcup_{m=1}^{\infty} (I : x_n^m).$$

Then I' is weakly stable and $\Gamma_{\mathfrak{m}}(R/I) = I'/I$.

Proof. Notice that I' is the ideal of R generated by the monomials obtained by setting $x_n = 1$ in the generators of I. First we show I' is weakly stable. We may assume $x_n | m$ for some $m \in G(I)$ where $G(I)$ denotes the set of minimal generators of I. Notice that if

$$i = \max\{ j \mid x_n^j \text{ divides some } u \in G(I)\}$$

then $I' = (I : x_n^i)$ and this ideal is weakly stable by Remark 3.5.

It is clear that $\Gamma_{\mathfrak{m}}(R/I) = \bigcup_i (I : \mathfrak{m}^i)/I$. We claim $\bigcup_i (I : \mathfrak{m}^i) = \bigcup_i (I : x_n^i)$. Take $f \in \bigcup_i (I : x_n^i)$ a monomial so that $f x_n^i \in I$ for some i. Since I is weakly stable we can choose a k such that $f x_j^k \in I$ for every j; hence, $f \in (I : \mathfrak{m}^{kn})$. The other inclusion is obvious. □

We are now ready to prove the main theorem.

Theorem 3.7. *If I and J are weakly stable ideals then*

$$\operatorname{reg}_R \operatorname{Tor}_i^R(R/I, R/J) \leq \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i \quad \textit{for every } i.$$

Proof. Consider the following set

$$\mathfrak{F} = \{(I,J) \mid I,J \text{ are weakly stable ideals and}$$
$$\operatorname{reg}_R \operatorname{Tor}_i^R(R/I,R/J) > \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i \text{ for some } i\}.$$

This set is partially ordered as follows: $(I,J) \leq (I',J')$ if $I \subseteq I'$ and $J \subseteq J'$. Assume that $\mathfrak{F} \neq \emptyset$, we seek a contradiction. Since R is noetherian there exists a maximal element (I,J).

We may assume $x_n|m$ for some $m \in G(I) \cup G(J)$. Otherwise, we let $S = k[x_1,\dots,x_{n-1}]$, then

$$\operatorname{Tor}_i^R(R/I,R/J) \cong \operatorname{Tor}_i^S(S/I\cap S, S/J\cap S) \otimes_S R \quad \text{for every } i.$$

Regularity does not change under faithfully flat extensions; hence it is enough to prove the theorem for S. Moreover, as Tor is symmetric we can assume that $x_n|m$ for some $m \in G(I)$.

By Lemma 3.6, $\Gamma_{\mathfrak{m}}(R/I) = I'/I$, so there is an exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(R/I) \to R/I \to R/I' \to 0$$

which induces, for each i, an exact sequence

$$\cdots \to \operatorname{Tor}_i^R(\Gamma_{\mathfrak{m}}(R/I),R/J) \to \operatorname{Tor}_i^R(R/I,R/J) \to \operatorname{Tor}_i^R(R/I',R/J) \to$$
$$\to \operatorname{Tor}_{i-1}^R(\Gamma_{\mathfrak{m}}(R/I),R/J).$$

The outside terms have finite length, since $\Gamma_{\mathfrak{m}}(R/I)$ has finite length, and therefore by Lemma 2.3

$$\begin{aligned}\operatorname{reg}_R \operatorname{Tor}_i^R(R/I,R/J) \leq \max\{ &\operatorname{reg}_R \operatorname{Tor}_i^R(\Gamma_{\mathfrak{m}}(R/I),R/J),\\ &\operatorname{reg}_R \operatorname{Tor}_i^R(R/I',R/J),\\ &\operatorname{reg}_R \operatorname{Tor}_{i-1}^R(\Gamma_{\mathfrak{m}}(R/I),R/J)+1\}.\end{aligned}$$

We examine the terms on the right hand side. By 2.5 and 2.4 we have

$$\begin{aligned}\operatorname{reg}_R \operatorname{Tor}_i^R(\Gamma_{\mathfrak{m}}(R/I),R/J) &\leq \operatorname{reg}_R \Gamma_{\mathfrak{m}}(R/I) + \operatorname{reg}_R R/J + i\\ &\leq \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i\end{aligned}$$

and

$$\begin{aligned}\operatorname{reg}_R \operatorname{Tor}_{i-1}^R(\Gamma_{\mathfrak{m}}(R/I),R/J)+1 &\leq \operatorname{reg}_R \Gamma_{\mathfrak{m}}(R/I) + \operatorname{reg}_R R/J + i - 1 + 1\\ &\leq \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i.\end{aligned}$$

By 3.6 we know I' is weakly stable. As $I \subsetneq I'$ and the pair (I,J) is maximal in $\mathfrak{F}$

$$\begin{aligned}\operatorname{reg}_R \operatorname{Tor}_i^R(R/I',R/J) &\leq \operatorname{reg}_R R/I' + \operatorname{reg}_R R/J + i \\ &\leq \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i.\end{aligned}$$

The final inequality follows by 2.4 since

$$R/I' = \frac{R/I}{\Gamma_{\mathfrak{m}}(R/I)}.$$

Putting all these inequalities together gives us

$$\operatorname{reg}_R \operatorname{Tor}_i^R(R/I,R/J) \leq \operatorname{reg}_R R/I + \operatorname{reg}_R R/J + i \quad \text{for every } i.$$

This is a contradiction since $(I,J) \in \mathfrak{F}$. □

Remark 3.8. The inequality in Theorem 3.7 is useful because Caviglia gives a formula for the regularity of weakly stable ideals (see [2, 4.1.10]).

4. Regularity of Ext for weakly stable ideals

Let M be an R-module of dimension d.

Regularity of $\operatorname{Ext}_R^i(M,R)$ was studied, for example, in [8]; here we study it in the case $M = R/I$ with I a weakly stable ideal.

Lemma 4.1. *Let $R = k[x_1,\dots,x_n]$. If M is an R-module of finite length then $\operatorname{Ext}_R^i(M,R) = 0$ for $i < n$ and*

$$\operatorname{reg}_R \operatorname{Ext}_R^n(M,R) = -n - \inf M.$$

Proof. By graded local duality, see [1, Theorem 3.6.19], there is the following isomorphism:

$$\operatorname{Hom}_R(H_{\mathfrak{m}}^i(M),E) \cong \operatorname{Ext}_R^{n-i}(M,R(-n)) \cong \operatorname{Ext}_R^{n-i}(M,R)(-n),$$

where E is the injective hull of k. Since M has finite length all the local cohomology modules are zero for $i > 0$ and $H_{\mathfrak{m}}^0(M) = M$. This gives $\operatorname{Ext}_R^i(M,R) = 0$ for $i < n$. The last assertion follows since

$$\operatorname{reg}_R \operatorname{Hom}_R(M,E) = \sup \operatorname{Hom}_R(M,E) = -\inf M. \qquad \square$$

Theorem 4.2. *If I is a weakly stable ideal then*

$$\operatorname{reg}_R \operatorname{Ext}_R^i(R/I,R) \leq -i \quad \text{for every } i.$$

Proof. Set

$$\mathfrak{F} = \{I \mid I \text{ is a weakly stable ideal such that } \\ \operatorname{reg}_R \operatorname{Ext}^i_R(R/I,R) > -i \text{ for some } i\}.$$

Notice that $\mathfrak{F}$ is partially ordered by inclusion of ideals.

The theorem asserts that $\mathfrak{F}$ is empty, so we assume it is not and argue by contradiction. Since R is noetherian there exists $I \in \mathfrak{F}$ a maximal element.

We may assume $x_n|m$ for some $m \in G(I)$; otherwise, let $S = k[x_1,\dots,x_{n-1}]$. Then

$$\operatorname{Ext}^i_R(R/I,R) \cong \operatorname{Ext}^i_S(S/I\cap S,S)\otimes_S R \quad \text{for every } i$$

and regularity does not change under faithfully flat extensions. Hence, it is enough to prove the theorem for S.

By Lemma 3.6 we have $\Gamma_{\mathfrak{m}}(R/I) = I'/I$, with I' weakly stable and $I \subsetneq I'$ so by maximality the assertion holds for I'.

The short exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(R/I) \to R/I \to R/I' \to 0$$

induces, for each i, an exact sequence

$$\operatorname{Ext}^{i-1}_R(\Gamma_{\mathfrak{m}}(R/I),R) \to \\ \to \operatorname{Ext}^i_R(R/I',R) \to \operatorname{Ext}^i_R(R/I,R) \to \operatorname{Ext}^i_R(\Gamma_{\mathfrak{m}}(R/I),R).$$

If $i < n$ then, since $\Gamma_{\mathfrak{m}}(R/I)$ has finite length, the outside terms are zero, giving the isomorphism $\operatorname{Ext}^i_R(R/I',R) \cong \operatorname{Ext}^i_R(R/I,R)$; hence, the assertion holds for I and $i < n$. If $i = n$ we get a short exact sequence

$$0 \to \operatorname{Ext}^n_R(R/I',R) \to \operatorname{Ext}^n_R(R/I,R) \to \operatorname{Ext}^n_R(\Gamma_{\mathfrak{m}}(R/I),R) \to 0.$$

As the bound holds for I' and $\Gamma_{\mathfrak{m}}(R/I)$ has finite length

$$\operatorname{reg}_R \operatorname{Ext}^n_R(R/I,R) \le \max\{\operatorname{reg}_R \operatorname{Ext}^n_R(R/I',R), \\ \operatorname{reg}_R \operatorname{Ext}^n_R(\Gamma_{\mathfrak{m}}(R/I),R)\} \le -n$$

Thus the bound holds for I and for every i; this is the desired contradiction. □

The previous result can be also deduced from a result of Hoa and Hyry. They prove (see [8, Proposition 22]) that if M is a sequentially Cohen-Macaulay module (see [7] for the definition) then

$$\operatorname{reg}_R(\operatorname{Ext}^i_R(M,R)) \le -i - \inf M \quad \text{for every } i.$$

Caviglia and Sbarra proved (see [3, 1.10]) that if I is weakly stable then R/I is sequentially CM, hence Hoa and Hyry's inequality reduces to

$$\operatorname{reg}_R(\operatorname{Ext}^i_R(R/I,R)) \le -i \quad \text{for every } i.$$

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REFERENCES

[1] W. Bruns - J. Herzog. *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, 1998.

[2] G. Caviglia, *Koszul Algebras, Castelnuovo-Mumford Regularity, and Generic Initial Ideals*, PhD thesis, University of Kansas, 2004.

[3] G. Caviglia - E. Sbarra, *Zero-generic initial ideals*, preprint arXiv: 1303.5373, 2013.

[4] M. Chardin, *Some results and questions on Castelnuovo-Mumford regularity*, In Syzygies and Hilbert functions, Lect. Notes Pure Appl. Math. 254, 1–40. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[5] A. Conca; *Regularity jumps for powers of ideals*, Commutative Algebra, 21–32, Lect. Notes Pure Appl. Math. 244, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[6] D. Eisenbud - C. Huneke - B. Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. 128 (3) (2006), 573–605.

[7] J. Herzog - E. Sbarra, *Sequentially Cohen-Macaulay modules and local cohomology*, in Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry, (R. Parimala, ed.), Tata Institute of Fundamental Research, Bombay. Stud. Math., Tata Inst. Fundam. Res. 16 (2002), 327–340.

[8] L. T. Hoa - E. Hyry, *Castelnuovo-Mumford regularity of canonical and deficiency modules*, Journal of Algebra 305 (2) (2006), 877–900.

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