LE MATEMATICHE
Vol. LXX (2015) – Fasc. I, pp. 191–211
doi: 10.4418/2015.70.1.14

POSETS OF h-VECTORS OF STANDARD DETERMINANTAL SCHEMES

MATEY MATEEV

We study the combinatorial structure of the poset $\mathcal{H}_s^{(t,c)}$ consisting of h-vectors of length s of codimension c standard determinantal schemes, defined by the maximal minors of a $t \times (t+c-1)$ homogeneous, polynomial matrix. We show that $\mathcal{H}_s^{(t,c)}$ obtains a natural stratification, where each strata contains a maximum h-vector. Moreover, we prove that any h-vector in $\mathcal{H}_s^{(t,c)}$ is bounded from above by a h-vector of the same length and which corresponds to a codimension c level standard determinantal scheme. Furthermore, we show that the only strata in which there exists also a minimum h-vector is the one consisting of h-vectors of level standard determinantal schemes.

1. Introduction

A scheme $X \subseteq \mathbb{P}^n$ of codimension c is called standard determinantal if its defining ideal is generated by the maximal minors of a homogeneous polynomial $t \times (t+c-1)$ matrix. Classical examples of such objects are rational normal curves, rational normal scrolls and some Segre varieties. Due to their important role, both in commutative algebra and algebraic geometry, the standard determinantal schemes have been an active research area and received considerable attention in the literature. We refer the reader to the books of W. Bruns and U.

Entrato in redazione: 25 maggio 2014

AMS 2010 Subject Classification: Primary: 13C40,13D40; Secondary: 05E40, 06A07
Keywords: Standard determinantal scheme, h-vector, stratification

Vetter [3], of R. M. Miró Roig [13], and of C. Baetica [2] for overviews of this subject.

The degree matrix of a standard determinantal scheme X is an integer matrix whose entries are the degrees of the polynomials in the matrix which generates the defining ideal of X. The defining ideal of a standard determinantal scheme X is a Cohen-Macaulay ideal and its graded minimal free resolution is given by the Eagon-Northcott complex (see [7]). It follows from [7] that the degree matrix of X determines its graded Betti numbers and thus also its Hilbert function and h-vector. Using this fact, we study the combinatorial structure of the poset (short for partially ordered set) $\mathcal{H}_s^{(t,c)}$ consisting of h-vectors of length s of codimension c standard determinantal schemes, having degree matrices of size $t \times (t+c-1)$ for some $t \geq 1$. Hilbert functions of determinantal ideals have been studied in a combinatorial context among many others by S. Abhyankar [1], S. Ghorpade [8, 9], A. Conca and J. Herzog [5], and D. Kulkarni [12].

The paper is organized as follows. In section 2 we provide the necessary background results and fix some notation as well. The starting point of Section 3 is the observation that grouping all degree matrices of fixed size $t \times (t+c-1)$ by the number of equal rows counted from top to bottom and considering the posets consisting of the corresponding h-vectors gives a natural stratification on the poset $\mathcal{H}_s^{(t,c)} = \bigcup_{r=1}^{t} \mathcal{H}_s^{(t,r,c)}$. We prove that each strata and $\mathcal{H}_s^{(t,c)}$ itself contains a maximum, which we construct explicitly (Proposition 3.4, Proposition 3.14 and Corollary 3.20). Moreover, we show that the h-vector of any standard determinantal scheme is bounded by the h-vector of some level standard determinantal scheme of the same codimension (Theorem 3.18).

According to [6, Theorem 3.2] any element in the strata $\mathcal{H}_s^{(t,t,c)}$ consisting of h-vectors of level standard determinantal schemes is a pure O-sequence, i.e. it is the h-vector of some artinian monomial level algebra. We show that the only strata in $\mathcal{H}_s^{(t,c)}$ where the existence of a minimum h-vector is granted is $\mathcal{H}_s^{(t,t,c)}$ and construct the minimum explicitly (Proposition 3.4).

Many of the results in this paper have been suggested using intensive computer experiments done with CoCoA (see [4]) .

2. Preliminaries

Let $S = K[X_0, \dots, X_n]$ be a polynomial ring over an infinite field K.

For any two integers $t, c \geq 1$, a matrix M of size $t \times (t+c-1)$, with polynomial entries, is called *homogeneous* if and only if all its minors are homogeneous polynomials (if and only if all its entries and 2×2 minors are homogeneous).

An ideal $I \subseteq S$ of height c is *standard determinantal* if it is generated by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix $M = [f_{i,j}]$, with $f_{i,j} \in S$ homogeneous polynomials of degree $a_j - b_i$. The matrix M is called *the defining matrix of I*. Without loss of generality we can assume that the defining matrix M of I does not contain invertible elements, i.e. $f_{i,j} = 0$ for all i, j with $a_j = b_i$. Clearly whenever $a_j < b_i$ we have $f_{i,j} = 0$. To the matrix M we assign a matrix of integers $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$, where $a_{i,j} = a_j - b_i$, which is called *the degree matrix of the ideal I*. Without loss of generality we will assume that $a_1 \leq \cdots \leq a_t$ and $b_1 \leq \cdots \leq b_{t+c-1}$, so the entries of A increase from left to right and from bottom to the top. Since $a_{i,i} \leq 0$ implies that all the minors containing the first i columns are zero, we have $a_{i,i} > 0$ for all i.
From now on, r will denote *the number of maximal equal rows* in a degree matrix. Accordingly to the assumed ordering on A we have

$$r = \max\{i \mid a_{1,1} = \cdots = a_{i,1}\}.$$

When $r = t$ we will say that A has *equal rows*. According to [6, Proposition 3.3] it appears that any level standard determinantal ideal I (that is the last free module in the minimal free resolution of I is of the form $S(-a)^b$) has a degree matrix with equal rows.
Abusing language we will call any matrix of integers $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ a *degree matrix* if it is the degree matrix of some standard determinantal ideal. The matrices of integers that are also degree matrices can be characterized in the following way:

Proposition 2.1. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a matrix of integers. Then A is a degree matrix if and only if it is homogeneous (i.e. $a_{i,j} + a_{k,l} = a_{i,l} + a_{k,j}$ for all $i,k = 1,\dots,t$ and $j,l = 1,\dots,t+c-1$) and $a_{i,i} > 0$, for all $i = 1,\dots,t$.*

For the proof see e.g. [10, Proposition 1.2].

A *standard determinantal scheme $X \subseteq \mathbb{P}^n$* of codimension c is a scheme whose defining ideal I_X is standard determinantal. Every standard determinantal scheme is arithmetically Cohen-Macaulay. More precisely in codimension 1 or 2 the family of standard determinantal schemes is equal to the family of arithmetically Cohen-Macaulay schemes. In codimension 3 or higher the inclusion is strict, i.e. there are arithmetically Cohen-Macaulay schemes that are not standard determinantal.

Let $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a numerical function. We define its first difference by $\triangle h(i) = h(i) - h(i-1)$ and its higher differences by $\triangle^d h = \triangle(\triangle^{d-1} h)$. We also make the convention $\triangle^0(h) = h$.

Definition 2.2. (A) Let $X \subseteq \mathbb{P}^n$ be an arithmetically Cohen-Macaulay projective scheme of dimension $d-1$ with defining ideal I_X.

Let $\mathfrak{a}_X = (I_X + (L_1, \dots, L_d)) \subseteq S$, where $L_i \in S_1$ is a linear form such that $L_i \in NZD_S(S/(I_X + (L_1, \dots, L_{i-1})))$ for all $i = 1, \dots, d$.
The ring $S/\mathfrak{a}_X \cong R/J_X$, where $R = K[X_1, \dots, X_c] \cong S/(L_1, \dots, L_d)$ and $J_X \cong I_X(S/(L_1, \dots, L_d))$, is called *an artinian reduction of* X (or of its coordinate ring S/I_X). It has Krull dimension 0 and for his Hilbert function holds:

$$HF_{R/J_X}(i) = \triangle^d HF_{S/I_X}(i).$$

Furthermore, as $[R/J_X]_n = 0$ for $n \gg 0$ the Hilbert function of R/J_X is a finite sequence of integers $1, h_1, h_2, \dots, h_s, 0$. The sequence
$h^X = (1, h_{1,}, \dots, h_s)$ is called *the h-vector of* X.

(B) The series $HS_X(z) = \sum_{i \geq 0} HF_X(i) z^i$ is called *the Hilbert series of* X. It is well known, that it can be written in a rational form as

$$HS_X(z) = \frac{\mathrm{hp}(z)}{(1-z)^d},$$

where $\dim(S/I_X) = d$. The numerator

$$\mathrm{hp}(z) = 1 + h_1 z + h_2 z^2 + \cdots + h_s z^s,$$

with $h_s \neq 0$ is called *h-polynomial of* X (or of S/I_X) and its coefficients form the h-vector of X, $h^X = (1, h_1, \dots, h_s)$.

Clearly $\mathrm{hp}(1) = h_0 + \cdots + h_s = \deg(X) = e_0(S/I_X)$, where we set $h_0 = 1$.

The number h_1 is called *the embedding codimension of* X, i.e. h_1 is the codimension of X inside the smallest linear space containing it.
We denote by $\tau(h^X)$ the degree of the h-polynomial.

The degree matrix A of a standard determinantal scheme X determines the graded Betti-numbers of its homogeneous coordinate ring S/I_X and thus also $\mathrm{hp}(z)$ and h^X. We will write therefore $\mathrm{hp}^A(z)$ and h^A instead of $\mathrm{hp}(z)$ and h^X.

Definition 2.3. The defining ideal I_X of any projective subscheme $X \subseteq \mathbb{P}^n$ of codimension c is generated by at least c forms. If I_X has exactly c minimal generators, then we will call X a complete intersection scheme.

Remark 2.4. Recall that the h-polynomial of a complete intersection scheme $X \subseteq \mathbb{P}^n$ generated in degrees $(a_1, \dots, a_c)$ is given by the formula

$$\mathrm{hp}^{(a_1, \dots, a_c)}(z) = \prod_{i=1}^{c} (1 + z + \cdots + z^{a_i - 1}).$$

Complete intersection schemes are the simplest example for a standard determinantal scheme. An algorithm for computing the h-polynomial of any standard determinantal scheme can be found in [6, Proposition 2.1].

For any matrix A and positive integers k and l we use the following notation: $A^{(k,l)}$ is the matrix obtained from A by deleting the k-th row and l-th column. By convention, $A^{(k,0)}$ (respectively $A^{(0,l)}$) means that only the k-th row (respectively the l-th column) has been deleted. With this notation the h-polynomial of any standard determinantal scheme can be recursively described as follows:

Lemma 2.5 ([6], Lemma 1.1)**.** *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix. For any $k = 1,\dots,t$ and $l = 1,\dots,t+c-1$ such that $a_{k,l} \geq 0$, we have:*

$$\mathrm{hp}^{A}(z) = z^{a_{k,l}}\mathrm{hp}^{A^{(k,l)}}(z) + (1+\cdots+z^{a_{k,l}-1})\mathrm{hp}^{A^{(0,l)}}(z).$$

Remark 2.6. Lemma 2.5 implies the following recursive formula for the h-vector of A:

$$h_i^A = h_{i-a_{k,l}}^{A^{(k,l)}} + \sum_{k=0}^{a_{k,l}-1} h_{i-k}^{A^{(0,l)}}.$$

In particular, if A has some entry $a_{k,l} = 0$, then $h^A = h^{A^{(k,l)}}$.

The degree and the leading coefficient of the h-polynomial of a standard determinantal scheme can be precisely computed.

Lemma 2.7 ([6], Lemma 2.2)**.** *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and let $h^A = (h_0,\dots,h_{\tau(h^A)})$. Then:*

(1) $\tau(h^A) = a_{1,1}+\cdots+a_{1,c}+a_{2,c+1}+\cdots+a_{t,t+c-1}-c$,

(2) $h_{\tau(h^A)} = \binom{r+c-2}{c-1}$, where $r = \max\{i \mid a_{1,1} = \cdots = a_{i,1}\}$.

Notice that since a degree matrix is homogeneous Lemma 2.7 is telling us how to obtain all h-vectors of standard determinantal schemes with prescribed length and last entry.

3. Posets of h-vectors

We would like to stress that the degree matrices we will deal with during this section are allowed to have zero entries.

A *poset* $(P,\leq)$ (short for *partially ordered set*) is a set P equipped with a binary relation " $\leq$ " that is reflexive (i.e. $a \leq a$ for all $a \in P$), antisymmetric ($a \leq b \leq a$ implies $a = b$) and transitive ($a \leq b \leq c$ implies $a \leq c$).
For two degree matrices A and B we will write $h^A \leq h^B$ if $h_i^A \leq h_i^B$ for all i.
If $h^A \leq h^B$ we will write also $A \leq_h B$. With this order, the set

$$M^{(c)} := \bigcup_{t\geq 1} M^{(t,c)},$$

where

$$M^{(t,c)} := \left\{A \in \mathbb{Z}^{t\times(t+c-1)} | A \text{ is a degree matrix}\right\},$$

becomes a poset for any fixed integer $c \geq 1$. For an integer $s \geq 1$ we define

$$N_s^{(c)} := \left\{A \in M^{(c)} | \, \tau(h^A) = s\right\}.$$

To $N_s^{(c)}$ we assign the poset

$$\mathcal{H}_s^{(c)} := \left\{h^A | A \in N_s^{(c)}\right\}.$$

Notice that the degree matrices in $N_s^{(c)}$ are not of fixed size. This implies in particular together with Remark 2.6 that the map

$$N_s^{(c)} \longrightarrow \mathcal{H}_s^{(c)}, A \longmapsto h^A$$

is surjective but certainly not bijective.

Definition 3.1. Let $(P, \leq)$ be a poset.

(1) An element $x \in P$ is called a maximal element in P if there exists no $z \in P$ such that $x \leq z$.

(2) An element $y \in P$ is called a minimal element in P if there exists no $z \in P$ such that $z \leq y$.

(3) A maximal element x which satisfies $x \geq y$ for any $y \in P$ is called maximum.

(4) A minimal element x which satisfies $x \leq y$ for any $y \in P$ is called minimum.

For totally ordered sets, the notions of maximal element and maximum, respectively minimal element and minimum coincide.

The existence of a minimum and maximum h-vector in the poset $\mathcal{H}_s^{(c)}$ can be easily shown.

Lemma 3.2. *There exist h-vectors $h^{\min}, h^{\max} \in \mathcal{H}_s^{(c)}$ such that*

$$h^{\min} \leq h \leq h^{\max} \textit{ for all } h \in \mathcal{H}_s^{(c)}.$$

Proof. Let A be the degree matrix $A = [1, \dots, 1, s] \in \mathbb{Z}^{1\times c}$. Then clearly $h^A \le h^B$ for all $B \in N_s^{(c)}$, so that $h^{\min} = h^A = (1, \dots, 1)$.

Let $C = [c_{i,j}] \in \mathbb{Z}^{(s+1)\times(s+c)}$ be a degree matrix with $c_{i,j} = 1, \forall i, j$. We claim that $h^{\max} = h^C$. Choose $A \in N_s^{(c)}$ and let $X \subseteq \mathbb{P}^n$ be a standard determinantal scheme with degree matrix A. Let $J_X \subseteq R = K[X_1, \dots, X_c]$ be the artinian reduction of the defining ideal I_X of X. Since $h^A = (h_0, \dots, h_s)$ and $h_i^A = HF_{R/J_X}(i)$ for all i, we have $[J_X]_i = R_i, \forall i \ge s+1$, so that $J_X \supseteq R_+^{s+1}$. On the other hand the ideal R_+^{s+1} is standard determinantal with defining matrix

$$\begin{bmatrix} X_1 & \cdots & X_c & 0 & \cdots & 0 & \\ 0 & & & & & & \\ \vdots & & & \ddots & & \ddots & \vdots \\ 0 & & \cdots & 0 & X_1 & \cdots & X_c \end{bmatrix} \in R^{(s+1)\times(s+c)}$$

and degree matrix C. We obtain therefore

$$h_i^A = HF_{R/J_X}(i) \le HF_{R/R_+^{s+1}}(i) = h_i^C, \text{ for all } i$$

and the assertion follows. □

As we just have seen it is not difficult to determine the minimum and the maximum in $\mathcal{H}_s^{(c)}$. The situation changes quickly if we study only subsets of $M^{(c)}$ and $N_s^{(c)}$ consisting of degree matrices of fixed size.
Consider the following subset of $M^{(t,c)}$:

$$N_s^{(t,c)} := \{A \in M^{(t,c)} \mid \tau(h^A) = s\},$$

for an integer $s \ge t-1$. We denote by

$$\mathcal{H}_s^{(t,c)} := \left\{h^A | A \in N_s^{(t,c)}\right\}$$

the corresponding set of h-vectors. For an integer $1 \le r \le t$ we define

$$N_s^{(t,r,c)} = \left\{A \in N_s^{(t,c)} | a_{1,1} = \cdots = a_{r,1} > a_{r+1,1}\right\}$$

and

$$\mathcal{H}_s^{(t,r,c)} := \left\{h^A | A \in N_s^{(t,r,c)}\right\}.$$

We obtain a natural stratification on $N_s^{(t,c)}$ and on $\mathcal{H}_s^{(t,c)}$, namely

$$N_s^{(t,c)} = N_s^{(t,1,c)} \cup \ldots \cup N_s^{(t,r,c)} \cup \ldots \cup N_s^{(t,t,c)}$$

and

$$\mathcal{H}_s^{(t,c)} = \mathcal{H}_s^{(t,1,c)} \cup \ldots \cup \mathcal{H}_s^{(t,r,c)} \cup \ldots \cup \mathcal{H}_s^{(t,t,c)}.$$

By [6, Proposition 3.3] any element in the poset $\mathcal{H}_s^{(t,t,c)}$ is the h-vector of some codimension c level standard determinantal scheme. Furthermore, by [6, Theorem 3.2] any such h-vector is a pure O-sequence, i.e. the h-vector of some artinian monomial level algebra. On the other hand [6, Proposition 3.3] shows also that for $1 \le r \le t-1$ any element in $\mathcal{H}_s^{(t,r,c)}$ is the h-vector of some non-level standard determinantal scheme.

3.1. Posets of h-vectors of level standard determinantal schemes

We will show next that in $\mathcal{H}_s^{(t,t,c)}$ there are a minimum and a maximum h-vector.

We introduce the following notation:

$$\mathbb{N}^{\deg}_{(t,c,s)} := \left\{ (a_1, \ldots, a_{t+c-1}) \in \mathbb{N}^{t+c-1} | \sum_{i=1}^{t+c-1} a_i = s+c, a_1 \le \cdots \le a_{t+c-1} \right\}.$$

Thus an element $\mathbf{a} \in \mathbb{N}^{\deg}_{(t,c,s)}$ is a partition of $s+c$ ordered in an increasing way. Obviously, there exists one to one correspondence $\mathbb{N}^{\deg}_{(t,c,s)} \longleftrightarrow N_s^{(t,t,c)}$, given by $\mathbf{a} \longmapsto A$, where each row of A is equal to $\mathbf{a}$.

For two elements $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$ we will write $\mathbf{a} \lhd \mathbf{b}$ if and only if $\mathbf{a} = \mathbf{b}$ or there exist $i < j \in \mathbb{N}$ such that

$$\mathbf{a} = (b_1, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_{j-1}, b_j + 1, b_{j+1}, \ldots, b_{t+c-1}).$$

If there exist elements $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{N}^{\deg}_{(t,c,s)}$ such that $\mathbf{a} = \mathbf{a}_1 \lhd \cdots \lhd \mathbf{a}_m = \mathbf{b}$, we will use the notation $\mathbf{a} < \mathbf{b}$.

Obviously the relation $\lhd$ does not define a partial order on $\mathbb{N}^{\deg}_{(t,c,s)}$, since it is not transitive. If $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c}$ and $\mathbf{a} \lhd \mathbf{b} \lhd \mathbf{c}$, it does not hold $\mathbf{a} \lhd \mathbf{c}$. So, in order to make $\mathbb{N}^{\deg}_{(t,c,s)}$ to a poset, we have to take the transitive closure of $\lhd$, which is given by $<$. It is easy to verify that $\mathbf{a} < \mathbf{b} < \mathbf{a}$ implies $\mathbf{a} = \mathbf{b}$.

We will show next, that the correspondence $\mathbb{N}^{\deg}_{(t,c,s)} \longleftrightarrow N_s^{(t,t,c)}$ preserves the partial order.

Lemma 3.3. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$ *and let* $A, B \in N_s^{(t,t,c)}$ *be the corresponding degree matrices with rows equal to* $\mathbf{a}$, *respectively* $\mathbf{b}$. *If* $\mathbf{a} \lhd \mathbf{b}$, *then* $h^A \le h^B$.

Proof. We may assume that $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a} \lhd \mathbf{b}$, hence that $\mathbf{b} = (b_1, \ldots, b_{t+c-1})$ and $\mathbf{a} = (b_1, \ldots, b_i - 1, \ldots, b_j + 1, \ldots, b_{t+c-1})$. We will prove the claim by induction on t and c. For $c = 1$ the claim is trivial.

Let $c > 1$. For $t = 1$, by Remark 2.4 it holds that:

$$\begin{aligned}\mathrm{hp}^A(z) &= \mathrm{hp}^{(b_1,\dots,b_i-1,\dots,b_j+1,\dots,b_c)}(z)\\ &= \mathrm{hp}^{(b_1,\dots,b_{i-1},b_{i+1},\dots,b_{j-1},b_{j+1},\dots,b_c)}(z)\cdot \mathrm{hp}^{(b_i-1,b_j+1)}(z).\end{aligned}$$

Since for any $c,d \in \mathbb{N}$ we have

$$\begin{aligned}h^{(c,d)} &= (1,\dots,\underbrace{c-1,c,\dots,c,c-1}_{d-c+3},\dots,1),\\ h^{(c-1,d+1)} &= (1,\dots,\underbrace{c-1,\dots,c-1}_{d-c+3},\dots,1)\end{aligned}$$

and $\tau(h^{(c,d)}) = \tau(h^{(c-1,d+1)})$ (i.e. $\deg(\mathrm{hp}^{(c,d)}(z)) = \deg(\mathrm{hp}^{(c-1,d+1)}(z))$), it holds that $h^{(c,d)} \geq h^{(c-1,d+1)}$, and therefore $h^A \leq h^B$ as claimed.

Let $t > 1$. We assume that $j < t+c-1$. The case $j = t+c-1$ is proved similarly. Applying Remark 2.6 for b_{t+c-1} on B and A we have:

$$h^B = h^{B^{(t,t+c-1)}} + \sum_{k=0}^{b_{t+c-1}-1} h_{i-k}^{B^{(0,t+c-1)}} \text{ and } h^A = h^{A^{(t,t+c-1)}} + \sum_{k=0}^{b_{t+c-1}-1} h_{i-k}^{A^{(0,t+c-1)}}.$$

As by induction it holds $h^{B^{(t,t+c-1)}} \geq h^{A^{(t,t+c-1)}}$ and $h^{B^{(0,t+c-1)}} \geq h^{A^{(0,t+c-1)}}$, we conclude. □

Lemma 3.3 provides the tool needed for showing the existence of the minimum and the maximum h-vector in the poset $\mathcal{H}_s^{(t,t,c)}$.

Proposition 3.4. *For any integer $s \geq t-1$ there exist h-vectors $h^{\min}$ and $h^{\max}$ in $\mathcal{H}_s^{(t,t,c)}$ such that $h^{\min} \leq h \leq h^{\max}$, for all $h \in \mathcal{H}_s^{(t,t,c)}$.*

Proof. Fix $s \geq t-1$ and let $\mathbf{a} = (1,\dots,1,s-t+2) \in \mathbb{N}^{\deg}_{(t,c,s)}$. To prove that $h^A = h^{\min}$, by Lemma 3.3 it suffices to show that $\mathbf{a} < \mathbf{b}$ for all $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$. Let $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}, \mathbf{b} \neq \mathbf{a}$. We can find integers i and j such that $b_i > 1$ and $b_j < s-t+2$. It follows that

$$\mathbf{b} \triangleright \mathbf{b}' = (b_1,\dots,b_k-1,\dots,b_l+1,\dots,b_{t+c-1}),$$

where $k = \min\{i | b_i > 1\}$ and $l = \max\{j | b_j < s-t+2\}$. If $\mathbf{a} = \mathbf{b}'$, we are done, otherwise we can repeat the process with $\mathbf{b}'$ instead of $\mathbf{b}$. Clearly after finitely many steps we will obtain $\mathbf{a}$, since the result of each step is a non-decreasing partition of $s+c$, where the difference between the entries at the positions k and l increases by 2.

Let $\mathbf{c} = (c_1, \dots, c_{t+c-1}) \in \mathbb{N}^{\deg}_{(t,c,s)}$, where $c_1 = \cdots = c_k = d$ and $c_{k+1} = \cdots = c_{t+c-1} = d+1$, for some $d \in \mathbb{N}$. According to Lemma 3.3 in order to show that $h^C = h^{\min}$ it is enough to show that $\mathbf{c} > \mathbf{b}$ for any $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$. Let $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$. Since $\mathbf{b} \neq \mathbf{c}$ there exist indexes $i < j$, such that $b_j - b_i \geq 2$. Therefore

$$\mathbf{b} \lhd \mathbf{b}' = (b_1, \dots, b_k + 1, \dots, b_l - 1, \dots, b_{t+c-1}),$$

where $k = \max\{m | \, b_m = b_i\}$ and $l = \min\{n | \, b_n = b_j\}$. If $\mathbf{b}' = \mathbf{c}$ we are done, otherwise we repeat the process with $\mathbf{b}'$ instead of $\mathbf{b}$. After finitely many steps $\mathbf{c}$ will be reached, since the result of each step is a non-decreasing partition of $s+c$, where the difference between the entries at the positions k and l decreases by 2. □

Remark 3.5. O. Greco, M. Mateev and C. Söger showed (see [11]) that a similar result holds also for the poset of h-vectors of the union of two sets of points in $\mathbb{P}^2$. More precisely the existence of a minimum in this poset was proved and the existence of a maximum, conjectured.

Definition 3.6. Let $(P, \leq)$ be a poset and let $x, y \in P$. We say that x covers y (in the poset P) if $x \neq y$ and $y \leq x$, and there does not exist $z \in P \setminus \{x, y\}$ such that $y \leq z \leq x$.

A useful tool for dealing with finite posets is the Hasse diagram.

Definition 3.7. Starting with a poset $(P, \leq)$, we define a directed graph with vertex set P by the rule that (x, y) is an edge if x covers y in P. The digraph H is called a Hasse digraph for P. When it is drawn in the plane with edges as straight lines going from the lower endpoint to the upper endpoint it is called a Hasse diagram.

Using Hasse diagrams we can easily visualize the structure of the posets $\mathcal{H}_s^{(t,r,c)}$, as illustrated in the following example:

Example 3.8. Consider the poset $\mathcal{H}_7^{(2,2,3)}$. Computing the possible partitions of 10, we can obtain the Hasse diagram of $\mathbb{N}^{\deg}_{(2,3,7)}$ by drawing an edge for any $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\deg}_{(2,3,7)}$ with $\mathbf{a} \lhd \mathbf{b}$.

$$
\begin{tikzcd}
 & (2,2,3,3)=\mathbf{a_1} \arrow[dl] \arrow[dr] & \\
\mathbf{a_2}=(1,3,3,3) \arrow[d] & & (2,2,2,4)=\mathbf{a_3} \arrow[dll] \arrow[d] \\
\mathbf{a_4}=(1,2,3,4) \arrow[rr] \arrow[d] \arrow[dr] & & (1,2,2,5)=\mathbf{a_5} \arrow[dl] \arrow[d] \\
\mathbf{a_6}=(1,1,4,4) \arrow[r] & \mathbf{a_7}=(1,1,3,5) \arrow[r] & (1,1,2,6)=\mathbf{a_8} \arrow[d] \\
 & & (1,1,1,7)=\mathbf{a_9}
\end{tikzcd}
$$

In the notation of Lemma 3.3 the corresponding h-vectors are

$$
\begin{aligned}
h^{A_1} &= (1,3,6,10,14,14,9,3), & h^{A_6} &= (1,3,5,7,9,7,5,3),\\
h^{A_2} &= (1,3,6,10,12,12,7,3), & h^{A_7} &= (1,3,5,7,7,7,5,3),\\
h^{A_3} &= (1,3,6,10,12,12,9,3), & h^{A_8} &= (1,3,5,5,5,5,5,3),\\
h^{A_4} &= (1,3,6,9,11,10,7,3), & h^{A_9} &= (1,3,3,3,3,3,3,3),\\
h^{A_5} &= (1,3,6,8,8,8,7,3). & &
\end{aligned}
$$

We can easily see that $\mathcal{H}_7^{(2,2,3)}$ has the same Hasse diagram as $\mathbb{N}^{\deg}_{(2,3,7)}$ and the minimum, respectively maximum h-vector corresponds to $\mathbf{a_9}$, respectively $\mathbf{a_1}$.

3.2. Posets of h-vectors of non-level standard determinantal schemes

Having seen that $\mathcal{H}_s^{(t,t,c)}$ contains a minimum and a maximum h-vector, it is natural to ask whether the same is true for any $\mathcal{H}_s^{(t,r,c)}$, $1 \le r \le t-1$. As the following example shows if $r \le t-1$ the existence of a minimum is in general not granted.

Example 3.9. Consider the set $N_7^{(4,3,3)}$. By Lemma 2.7 it consists of the following degree matrices

$$A_1 = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 4 \\ 1 & 1 & 1 & 2 & 2 & 4 \\ 1 & 1 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 3 & 3 & 3 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

In particular it follows that $\mathcal{H}_7^{(4,3,3)}$ consists of the h-vectors

$$\begin{aligned} h^{A_1} &= (1,3,6,10,15,21,18,6), & h^{A_4} &= (1,3,6,10,14,16,12,6), \\ h^{A_2} &= (1,3,6,10,15,18,15,6), & h^{A_5} &= (1,3,6,10,12,15,9,6). \\ h^{A_3} &= (1,3,6,10,13,13,12,6), \end{aligned}$$

The corresponding Hasse diagram shows that in $\mathcal{H}_7^{(4,3,3)}$ there is no minimum and is given by:

$$\begin{tikzcd} & h^{A_1} \arrow[d] & \\ & h^{A_2} \arrow[d] & \\ & h^{A_4} \arrow[dl] \arrow[dr] & \\ h^{A_5} & & h^{A_3} \end{tikzcd}$$

Remark 3.10. Notice that for any matrix $B \in N_s^{(t,r,c)}$, we have

$$b_{1,1} + \cdots + b_{1,t+c-1} \geq s + c + (t-r).$$

This follows directly from Lemma 2.7 since

$$s+c=b_{1,1}+\cdots+b_{1,c}+b_{2,c}+b_{r,c+(r-1)}+b_{r+1,c+r}+\cdots+b_{t,t+c-1}$$

and

$$\begin{aligned} b_{i,c+(i-1)} &= b_{1,c+(i-1)}, \quad \text{for all} \quad i=2,\dots,r \\ b_{i,c+(i-1)} &\le b_{1,c+(i-1)}-1, \quad \text{for all} \quad i=r+1,\dots,t. \end{aligned}$$

We will show next that for any $1\le r\le t-1$, the poset $\mathcal{H}_s^{(t,r,c)}$ contains a maximum h-vector, which correspond to a matrix of the form

$$A=\begin{bmatrix} a_1 & \cdots & a_{t+c-1} \\ \vdots & & \vdots \\ a_1 & \cdots & a_{t+c-1} \\ a_1-1 & \cdots & a_{t+c-1}-1 \\ \vdots & & \vdots \\ a_1-1 & \cdots & a_{t+c-1}-1 \end{bmatrix}, \tag{1}$$

where $a_1+\cdots+a_{t+c-1}=s+c+(t-r)$ and the matrix $A'\in\mathbb{Z}^{r\times(t+c-1)}$ consisting of the first r equal rows of A satisfies $h^{A'}=h^{\max}\in\mathcal{H}_s^{(r,r,c+(t-r))}$.

We introduce the following notation

$$\begin{aligned} L_s^{(t,r,c)} &= \left\{B\in N_s^{(t,r,c)} | a_{1,1}-a_{i,1}=1, \forall i\ge r+1\right\} \\ R_s^{(t,r,c)} &= \{B\in N_s^{(t,r,c)} | B\notin L_s^{(t,r,c)}\} = N_s^{(t,r,c)}\setminus L_s^{(t,r,c)}. \end{aligned}$$

Obviously, it holds $N_s^{(t,r,c)}=L_s^{(t,r,c)}\cup R_s^{(t,r,c)}$. Furthermore, by definition any $B=[b_{i,j}]\in L_s^{(t,r,c)}$ is of the form

$$B=\begin{bmatrix} b_1 & \cdots & b_{t+c-1} \\ \vdots & & \vdots \\ b_1 & \cdots & b_{t+c-1} \\ b_1-1 & \cdots & b_{t+c-1}-1 \\ \vdots & & \vdots \\ b_1-1 & \cdots & b_{t+c-1}-1 \end{bmatrix}, \tag{2}$$

and it holds $b_1+\cdots+b_{t+c-1}=s+c+(t-r)$.

Lemma 3.11. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and assume that there exist indexes $1 \le i < j \le t+c-1$ such that $a_{1,j} - a_{1,i} \ge 2$.*
Let $k = \max\{m| \, a_{1,i} = \cdots = a_{1,m}\}$ and $l = \min\{n| \, a_{1,n} = \cdots = a_{1,j}\}$. If B is the degree matrix obtained from A by adding 1 to the k-th column and subtracting 1 from the l-th column, then $h^A \le h^B$.

Proof. We prove the claim by induction on t and c. For $t = 1$ the claim follows from Lemma 3.3. Let $t > 1$, for $c = 1$ the claim is trivial, so let $c > 1$.

Without loss of generality we may apply Remark 2.6 for $a_{t,t+c-1}$, assuming that $B^{(t,t+c-1)}$ and $B^{(0,t+c-1)}$ contain the modified columns of A. It follows then by induction that $h^{A^{(t,t+c-1)}} \le h^{B^{(t,t+c-1)}}$ and $h^{A^{(0,t+c-1)}} \le h^{B^{(0,t+c-1)}}$, so we conclude. □

Lemma 3.12. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix which satisfies the following conditions:*

(1) A has $r \le t-1$ maximal rows, i.e. $a_{1,1} = \cdots = a_{r,1}$,

(2) there exists an index $1 \le j \le t+c-1$ such that $a_{1,j} \ge 2$ and $a_{1,j} - a_{1,j-1} \ge 1$,

(3) there exists an index $r+1 \le i \le t$ such that $a_{i-1,1} - a_{i,1} \ge 1$ and if $i = r+1$, then $a_{r,1} - a_{r+1,1} \ge 2$.

Let B be the matrix obtained from A by adding 1 to the i-th row and subtracting 1 from the j-th column. Then $h^A \le h^B$.

Proof. We proceed by induction on t and c. For $c = 1$ and $t \ge 1$ the claim is trivial. Let $c > 1$, $t = 2$ and let

$$B = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j}-1 & a_{1,j+1} & \cdots & a_{1,t+c-1} \\ a_{2,1}+1 & \cdots & a_{2,j-1}+1 & a_{2,j} & a_{2,j+1}+1 & \cdots & a_{2,t+c-1}+1 \end{bmatrix}$$

be the matrix obtained from A by adding 1 to the second row and subtracting 1 from the j-th column. We assume that $j < t+c-1$. The computation for $j = t+c-1$ is analogous. Applying Remark 2.6 for $a_{1,t+c-1}$ we have

$$h_i^B = h_{i-a_{1,t+c-1}}^{(a_{2,1}+1,\dots,a_{2,j},\dots,a_{2,t+c-2}+1)} + \sum_{k=0}^{a_{1,t+c-1}-1} h_{i-k}^{B^{(0,t+c-1)}}.$$

By the inductive hypothesis on c it holds $h^{B^{(0,t+c-1)}} \ge h^{A^{(0,t+c-1)}}$. Since we obviously have $h^{(a_{2,1}+1,\dots,a_{2,j},\dots,a_{2,t+c-2}+1)} \ge h^{(a_{2,1},\dots,a_{2,j},\dots,a_{2,t+c-2})}$, the claim follows.

Let $t > 2$. Obviously there is an entry $a_{k,l} > 0$ which remains unchanged by performing the operation described in the statement and such that $B^{(k,l)}$ and $B^{(0,l)}$ contain the modified row and column of A. By the inductive hypothesis on

t and c we have $h^{B^{(k,l)}} \geq h^{A^{(k,l)}}$, and $h^{B^{(0,l)}} \geq h^{A^{(0,l)}}$. The assertion follows therefore from Remark 2.6 applied for the indexes (k,l). □

Remark 3.13. Notice that the operation defined in Lemma 3.12 does not change the number of equal rows or the length of the corresponding h-vector.

Proposition 3.14. *Let r,t,c be positive integers, where $t \geq 2$ and $r \leq t-1$. There exists a h-vector $h^{\max} \in \mathcal{H}_s^{(t,r,c)}$, such that $h \leq h^{\max}$ for all $h \in \mathcal{H}_s^{(t,r,c)}$. Moreover, it holds $h^{\max} = h^A$, where A is the degree matrix described in (1).*

Proof. Let $C = [c_{i,j}] \in N_s^{(t,r,c)}$. We can assume that $C \in L_s^{(t,r,c)}$, as for any $C \in R_s^{(t,r,c)}$ repeated application of Lemma 3.12, will produce a matrix $B = [b_{i,j}] \in L_s^{(t,r,c)}$, which has the form described in (2). Furthermore, for the corresponding h-vectors it holds that $h^C \leq h^B$. Notice that since by each step the entries in a certain non-maximal row increase by one, only finitely many steps are needed to obtain B. If $C \in L_s^{(t,r,c)}$ is not equal to the matrix A defined in (1), then there have to be entries $c_{1,i} < c_{1,j}$, such that $c_{1,j} - c_{1,i} \geq 2$. Applying Lemma 3.11 on C will produce a degree matrix $A' \in L_s^{(t,r,c)}$, such that $h^{A'} \geq h^C$. If $A = A'$ we have found the maximal h-vector, otherwise we apply Lemma 3.11 on A'. Since each time we lower the difference between a pair of columns of the matrix, after finitely many steps we will reach the matrix A. □

The next example illustrates the operations described in Lemma 3.11 and Lemma 3.12.

Example 3.15. Consider again the set $N_7^{(4,3,3)}$ from Example 3.9. We have

$$L_7^{(4,3,3)} = \{A_1, A_2, A_3, A_4\} \text{ and } R_7^{(4,3,3)} = \{A_5\}.$$

Writing $A \xrightarrow[(i,j)]{(+,-)} B$, respectively $A \xrightarrow[(i,j)]{\binom{+}{-}} B$ for the degree matrix B obtained from A by applying Lemma 3.11 on the columns i and j or respectively applying Lemma 3.12 on the i-th row and j-th column of A, we have

$$\begin{tikzcd}
A_5 \arrow[r, "\binom{+}{-}", "(4{,}4)"'] & A_4 \arrow[r, "(+{,}-)", "(3{,}4)"'] & A_2 \arrow[r, "(+{,}-)", "(2{,}6)"'] & A_1 \\
& A_3 \arrow[u, "(5{,}6)", "(+{,}-)"'] & &
\end{tikzcd}$$

3.3. Maximum h-vector

The next problem we will approach is whether there exists a maximum h-vector in the poset $\mathcal{H}_s^{(t,c)}$. We will show in this section that there is one and it is equal to the maximum h-vector $h^{\max}$ in $\mathcal{H}_s^{(t,t,c)}$. We start with some preparatory lemmas.

Lemma 3.16. *Given* $\mathbf{a} = (a_1, \ldots . a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *two integer sequences, such that* $a_i \leq b_i$ *for all* $i = 1, \ldots, n$. *Let* $\mathbf{c} = (c_1, \ldots, c_n)$ *and* $\mathbf{d} = (d_1, \ldots, d_n)$ *be two permutations of* $\mathbf{a}$, *respectively* $\mathbf{b}$ *such that* $c_1 \leq \cdots \leq c_n$ *and* $d_1 \leq \cdots \leq d_n$. *Then* $c_i \leq d_i$ *for all* $i = 1, \ldots, n$.

Proof. We will prove the claim by induction on n. For $n = 2$ we have the following possibilities

- $a_1 \leq a_2$ and $b_1 \leq b_2$,
- $a_1 \geq a_2$ and $b_1 \geq b_2$,
- $a_1 \leq a_2$ and $b_1 \geq b_2$,
- $a_1 \geq a_2$ and $b_1 \leq b_2$.

Obviously in the first two cases there is nothing to show. The inequalities in the third and the fourth case imply $a_1 \leq a_2 \leq b_2 \leq b_1$ respectively $a_2 \leq a_1 \leq b_1 \leq b_2$ and the claim follows.

Let $n \geq 2$ and

$$a_i = \min\{a_k | k = 1, \ldots, n\}, \, b_j = \min\{b_k | k = 1, \ldots, n\}.$$

We have the sequences

$$(a_i, a_1, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$(b_j, b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_{j-1} \widehat{b_j}, b_{j+1}, \ldots, b_n),$$

where $a_i \leq a_j \leq b_j \leq b_i$. Let

$$\mathbf{a}' = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$\mathbf{b}' = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{j-1}, b_i, b_{j+1}, \ldots, b_n),$$

be the sequences obtained from $\mathbf{a}$ and $\mathbf{b}$ after removing a_i and b_j and moving b_i to the j-th position in $\mathbf{b}'$. If we denote by $\widetilde{\mathbf{a}}$, respectively $\widetilde{\mathbf{b}}$ the nondecreasing reordering of $\mathbf{a}'$, respectively $\mathbf{b}'$, then as $a_j \le b_j \le b_i$, it holds by induction $\widetilde{a}_i \le \widetilde{b}_i$ for any i and the assertion follows. □

The next result gives us a direct way how to compare h-vectors corresponding to degree matrices with equal rows.

Lemma 3.17. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ and $B = [b_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be two degree matrices with equal rows, such that $a_{i,j} \le b_{i,j}$ for all $i = 1,\dots,t$ and $j = 1,\dots,t+c-1$. Then $h^A \le h^B$.*

Proof. The claim follows directly from [6, Theorem 2.4] □

Next, using Lemma 3.16 and Lemma 3.17 we show that the h-vector of any standard determinantal scheme is bounded from above by the h-vector of a level standard determinantal scheme.

Theorem 3.18. *To any standard determinantal scheme $X \subseteq \mathbb{P}^n$ there exists a level standard determinantal scheme $Y \subseteq \mathbb{P}^n$ of the same codimension such that*

$$h^X \le h^Y \text{ and } \tau(h^X) = \tau(h^Y).$$

Proof. Let A be the degree matrix of X. Without loss of generality we can assume that $\tau(h^X) = s$ and $A \in N_s^{(t,c)}$. To prove the claim it suffices to show that there is a degree matrix $B \in N_s^{(t,t,c)}$ such that $h^A \le h^B$. We will show this by induction on t and c.

When $t = 1$ the claim is trivial, so let $t > 1$ and $c = 1$.
Let $B \in N_s^{(t,t,1)}$ be the degree matrix, whose rows are equal to a nondecreasing reordering $a_{\sigma(1),\sigma(1)} \le \cdots \le a_{\sigma(t),\sigma(t)}$ of the diagonal elements of A. We have then obviously $h^A = h^B$.
Let $c > 1$ and $a_{i_0,j_0} = \min\{a_{1,1},\dots,a_{1,c},a_{2,c+1},\dots,a_{t,t+c-1}\}$. If $(b_1,\dots,b_{t+c-1})$ is a nondecreasing reordering of $(a_{1,1},\dots,a_{1,c},a_{2,c+1},\dots,a_{t,t+c-1})$ and $B \in N_s^{(t,t,c)}$ is the matrix whose rows are equal to $(b_1,\dots,b_{t+c-1})$, then $b_1 = a_{i_0,j_0}$ and by Remark 2.6, applied on A for the indices (i_0, j_0) and on B for $(1,1)$, we have

$$h_i^A = h_{i-a_{i_0,j_0}}^{A^{(i_0,j_0)}} + \sum_{k=0}^{a_{i_0,j_0}-1} h_{i-k}^{A^{(0,j_0)}} \text{ and } h_i^B = h_{i-b_1}^{B^{(1,1)}} + \sum_{k=0}^{b_1-1} h_{i-k}^{B^{(0,1)}}.$$

We distinguish the following cases:

Case 1: $i_0 \in \{1,t\}$. Since the proof of the claim for $i_0 = 1$ is the same as for $i_0 = t$, we will show it only for $i_0 = 1$ (notice that $i_0 = 1$ implies $j_0 = 1$). Consider first $A^{(1,1)}$. As

$$\begin{array}{ccccccc} a_{1,2} & \cdots & a_{1,c} & a_{2,c+1} & \cdots & a_{t,t+c-1} \\ \vee\!| & & \vee\!| & \| & & \| \\ a_{2,2} & \cdots & a_{2,c} & a_{2,c+1} & \cdots & a_{t,t+c-1} \end{array}$$

by Lemma 3.16 we have

$$\begin{array}{ccccc} b_2 & \leq & \cdots & \leq & b_{t+c-1} \\ \vee\!| & & & & \vee\!| \\ d_2 & \leq & \cdots & \leq & d_{t+c-1} \end{array},$$

where $(d_2,\cdots,d_{t+c-1})$ is the nondecreasing reordering of $(a_{2,2},\ldots,a_{2,c},a_{2,c+1},\ldots,a_{t,t+c-1})$. If D is the matrix with rows equal to $(d_2,\cdots,d_{t+c-1})$, then Lemma 3.17 together with the inductive hypothesis shows that $h^{A^{(1,1)}} \leq h^D \leq h^{B^{(1,1)}}$. On the other hand, for $A^{(0,1)}$, as $(b_2,\ldots,b_{t+c-1})$ is the nondecreasing reordering of $(a_{1,2},\ldots,a_{1,c},a_{2,c+1},\ldots a_{t,t+c-1})$, it holds by induction that $h^{A^{(0,1)}} \leq h^{B^{(0,1)}}$ and we can conclude.

Case 2: $2 \leq i_0 \leq t-1$. Looking at the matrix $A^{(0,j_0)}$ we obtain the inequalities

$$\begin{array}{cccccccccc} a_{1,1} & \cdots & a_{1,c-1} & a_{1,c} & \cdots & a_{i_0-1,j_0-1} & a_{i_0+1,j_0+1} & \cdots & a_{t,t+c-1} \\ \| & & \| & \vee\!| & & \vee\!| & \| & & \| \\ a_{1,1} & \cdots & a_{1,c-1} & a_{2,c} & \cdots & a_{i_0,j_0-1} & a_{i_0+1,j_0+1} & \cdots & a_{t,t+c-1} \end{array},$$

which according to Lemma 3.16 imply the following inequalities on the corresponding nondecreasing reorderings:

$$\begin{array}{ccccc} b_2 & \leq & \cdots & \leq & b_{t+c-1} \\ \vee\!| & & & & \vee\!| \\ f_2 & \leq & \cdots & \leq & f_{t+c-1} \end{array}.$$

By the induction hypothesis and Lemma 3.17 we have $h^{A^{(0,j_0)}} \leq h^F \leq h^{B^{(0,1)}}$, where F is the matrix whose rows are equal to $(f_2,\ldots,f_{t+c-1})$.
Considering the matrix $A^{(i_0,j_0)}$ and using the fact that the nondecreasing reordering of $(a_{1,1},\ldots,a_{1,c},a_{2,c+1},\ldots,a_{i_0-1,j_0-1},a_{i_0+1,j_0+1},\ldots a_{t,t+c-1})$ is $(b_2,\ldots,b_{t+c-1})$ we have by induction $h^{A^{(i_0,j_0)}} \leq h^{B^{(1,1)}}$ and the claim follows from Remark 2.6 applied on A and B for the indices (i_0,j_0), respectively $(1,1)$. $\square$

Example 3.19. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & 1 & 2 & 3 & 3 \\ -1 & 0 & 1 & 2 & 2 \end{bmatrix} \in N_8^{(3,3)}$,
with corresponding h-vector $h^A = (1,3,6,9,10,9,6,3,1)$. The nondecreasing reordering of $(1,2,3,3,2)$ is $(1,2,2,3,3)$, therefore we obtain the matrix

$$B = \begin{bmatrix} 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \end{bmatrix},$$

whose corresponding h-vector is $h^B = (1,3,6,10,15,20,21,15,6)$ and $h^A \le h^B$.

Theorem 3.18 provides the tool needed for showing the existence of the maximum in $\mathcal{H}_s^{(t,c)} = \mathcal{H}_s^{(t,1,c)} \cup \ldots \cup \mathcal{H}_s^{(t,t,c)}$.

Corollary 3.20. *For any positive integers $t,c,s \in \mathbb{N}$ and $s \ge t-1$ there exists a h-vector $H^{\max} \in \mathcal{H}_s^{(t,c)}$, such that $h \le H^{\max}$ for all $h \in \mathcal{H}_s^{(t,c)}$. Furthermore $H^{\max} = h^{\max} \in \mathcal{H}_s^{(t,t,c)}$.*

Proof. Let $A \in N_s^{(t,c)}$. By Theorem 3.18 there is a matrix $B \in N_s^{(t,t,c)}$ such that $h^A \le h^B$. By Proposition 3.4 there exists a degree matrix $C \in N_s^{(t,t,c)}$ such that $h^C = h^{\min} \in \mathcal{H}_s^{(t,t,c)}$. We have therefore $h^A \le h^B \le h^C$ and the claim follows. □

According to Corollary 3.20, $\mathcal{H}_s^{(t,c)} = \mathcal{H}_s^{(t,1,c)} \cup \ldots \cup \mathcal{H}_s^{(t,t,c)}$ contains a maximum h-vector, which is the maximum in the stratum $\mathcal{H}_s^{(t,t,c)}$. Therefore it is natural to ask whether there exists a minimum h-vector in $\mathcal{H}_s^{(t,c)}$. Obviously if there exist one, then by Lemma 2.7 it has to come from $\mathcal{H}_s^{(t,1,c)}$. As we have seen in the previous section the poset $\mathcal{H}_s^{(t,1,c)}$ does not have a minimum in general (see also Example 3.21). It turns out that the same is true also for $\mathcal{H}_s^{(t,c)}$. The following example is a good illustration for this fact.

Example 3.21. Consider the poset $\mathcal{H}_7^{(2,3)} = \mathcal{H}_7^{(2,1,3)} \cup \mathcal{H}_7^{(2,2,3)}$. Then the strata $\mathcal{H}_7^{(2,1,3)}$ consist of the h-vectors:

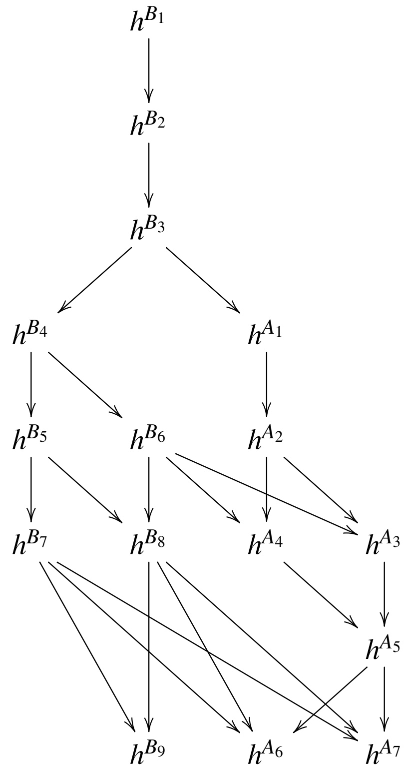
$h^{A_1} = (1,3,6,10,12,9,4,1)$, $h^{A_2} = (1,3,6,9,10,8,4,1)$,
$h^{A_3} = (1,3,6,8,8,6,3,1)$, $h^{A_4} = (1,3,6,7,7,7,4,1)$,
$h^{A_5} = (1,3,5,7,7,5,3,1)$, $h^{A_6} = (1,3,4,5,4,4,2,1)$,
$h^{A_7} = (1,3,4,4,4,4,3,1)$.

For $\mathcal{H}_7^{(2,2,3)}$ we have

$h^{B_1} = (1,3,6,10,14,14,9,3)$, $h^{B_2} = (1,3,6,10,12,12,9,3)$,
$h^{B_3} = (1,3,6,10,12,12,7,3)$, $h^{B_4} = (1,3,6,9,11,10,7,3)$,
$h^{B_5} = (1,3,5,7,9,7,5,3)$, $h^{B_6} = (1,3,6,8,8,8,7,3)$,
$h^{B_7} = (1,3,5,7,7,7,5,3)$, $h^{B_8} = (1,3,5,5,5,5,5,3)$,
$h^{B_9} = (1,3,3,3,3,3,3,3)$.

The Hasse diagram corresponding to $\mathcal{H}_7^{(2,3)}$ can be seen in the next page: there exist more than one minimal h-vector.

Notice that in the poset $\mathcal{H}_7^{(2,1,3)}$ there are two minimal elements (so, there is no minimum in $\mathcal{H}_7^{(2,1,3)}$) and none of them is comparable to the minimum h-vector in $\mathcal{H}_7^{(2,2,3)}$.



REFERENCES

[1] S. Abhyankar, *Determinantal loci and enumerative combinatorics of Young tableaux*, Algebraic geometry and commutative algebra Vol. I, 1–26, Kinokuniya, Tokyo, 1988.

[2] C. Baetica, *Combinatorics of determinantal ideals*, Nova Science Publishers Inc., Hauppauge, NY, 2006.

[3] W. Bruns - U. Vetter, *Determinantal rings*, Monografías de Matemática [Mathematical Monographs], vol. 45, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1988.

[4] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*, http://cocoa.dima.unige.it.

[5] A. Conca - J. Herzog, *On the Hilbert function of determinantal rings and their canonical module*, Proc. Amer. Math. Soc. 122 (3) (1994), 677–681.

[6] A. Constantinescu - M. Mateev, *Determinantal Schemes and Pure O-sequences*, http://arxiv.org/abs/1403.1172 (2014).

[7] J. A. Eagon - D. G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. 269 (1962), 188–204.

[8] S. Ghorpade, *Young multitableaux and higher-dimensional determinants*, Adv. Math. 121 (2) (1996), 167–195.

[9] S. Ghorpade, *Hilbert functions of ladder determinantal varieties*, Formal powerseries and algebraic combinatorics (Barcelona,1999) 246 (1-3) (2002), 131–175.

[10] E. Gorla, *The general hyperplane section of a curve.*, Transactions of the american mathematical society 358 (2) (2005), 819–869.

[11] O. Greco - M. Mateev - C. Söger, *The h-vector of the union of two sets of points in the projective plane*, Le Mathematiche 67 (1) (2012), 197–222.

[12] D. Kulkarni, *Counting of paths and coefficients of the Hilbert polynomial of a determinantal ideal*, Discrete Mathematics 154 (1996), 141–151.

[13] R. M. Miró-Roig, *Determinantal ideals*, Progress in Mathematics, vol. 264, Birkhäuser Verlag, Basel, 2008.

MATEY MATEEV
Mathematics Department
Basel University
e-mail: matey.mateev@unibas.ch