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ZIP PROPERTY ON MALCEV-NEUMANN SERIES MODULES

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Let R be a ring, M_R a right R-module, G a totally ordered group, σ a map from G into the group of automorphisms of R which assigns to each $x \in G$ an automorphism $\sigma_x \in \mathrm{Aut}(R)$, τ a map from $G \times G$ to $U(R)$ (the group of unit elements of R) and $M((G;\sigma;\tau))$ the Malcev-Neumann series module. Then, under some certain conditions, we show that M_R is a right zip R-module if and only if $M((G;\sigma;\tau))_{R((G;\sigma;\tau))}$ is a right zip $R((G;\sigma;\tau))$-module, where $R((G;\sigma;\tau))$ is the Malcev-Neumann series ring.

1. Introduction

Throughout this paper R denotes an associative ring with identity. Recall from [3] that R is a *right zip* ring if the right annihilator of a subset $X \subseteq R$ is zero, then $\mathrm{r}_R(X_0) = 0$ for a finite subset X_0 of X, equivalently for a left ideal L of R if $\mathrm{r}_R(L) = 0$, then there exists a finitely generated ideal $L_0 \subseteq L$ such that $\mathrm{r}_R(L_0) = 0$.

The concept of zip rings was initiated by Zelmanowitz [8] where it was not so called zip at that time, however he showed that any ring satisfying the descending chain condition on right annihilator ideals is a right zip ring but the converse is not true.

Extensions of zip rings were studied by several authors. In [1] Beachy and Blair showed that if R is a commutative zip ring, then $R[x]$ is a zip ring.

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In ([4], Theorem 1) Hong et al showed that if R is an Armendariz ring, then R is a right zip ring if and only if $R[x]$ is a right zip ring.

In ([2], Theorem 2.8) Cortes studied skew polynomial extension over zip rings and he showed that, if σ is an automorphism of R and R is σ-Armendariz, then R is a right zip ring if and only if $R[x;\sigma]$ is a right zip ring.

Recall from [9] that a right R-module M_R is called a *right zip* module provided that if the right annihilator of a subset X of M_R is zero, then there exists a finite subset $X_0 \subseteq X$ such that $\mathrm{r}_R(X_0) = 0$.

In the following section we introduce results concerned with the transfer of a right zip property of M_R and a twisted Malcev-Neumann series module extension $M((G;\sigma;\tau))$.

2. Zip Modules over Twisted Malcev-Neumann Series Rings

Let R be a ring, G a totally ordered group, σ a map from G into the group of automorphisms of R which assigns to each $x \in G$ an automorphism $\sigma_x \in \mathrm{Aut}(R)$ where $\sigma_1 = \mathrm{id}_R$ with 1 the identity of group G, and τ a map from $G \times G$ to $U(R)$ (the group of invertible elements of R). Let $A = R((G;\sigma;\tau))$ denote the set of all formal sums $f = \sum\limits_{x\in G} a_x x$ such that $\mathrm{supp}(f) = \{x \in G \mid a_x \neq 0\}$ is a well ordered subset of G, with componentwise addition and the multiplication rule is given by

$$(\sum_{x\in G} a_x x)(\sum_{y\in G} b_y y) = \sum_{z\in G}(\sum_{\{(x,y)|xy=z\}} a_x \sigma_x(b_y)\tau(x,y))z,$$

for each $\sum\limits_{x\in G} a_x x$ and $\sum\limits_{y\in G} b_y y \in A$. In order to ensure the associativity it is necessary that

(i) $\sigma_x(\tau(y,z))\tau(x,yz) = \tau(x,y)\tau(xy,z)$ and

(ii) $\sigma_x\sigma_y = \eta(x,y)\sigma_{xy}$,

where $\eta(x,y)$ denotes the automorphism of R induced by the unit $\tau(x,y)$, for all $x,y,z \in G$, see ([5], Lemma 1.1). It is now routine to check that A is a ring which is called the *ring of Malcev-Neumann series*.

The Malcev-Neumann construction appeared for the first time in the latter part of the 1940 (the Laurent series ring, a particular case of Malcev-Neumann ring, was used before by Hilbert). Using them, Malcev and Neumann independently showed (in 1948 and 1949, respectively.) that the group ring of an ordered group over a division ring can be embedded in a division ring. Since then, the construction has appeared in many papers, mainly in the study of various properties of division rings and related topics.

In [7] Sonin generalized the construction to obtain Malcev-Neumann modules over Malcev-Neumann rings as follows:

If M_R is a right R-module, then the *Malcev-Neumann series module* $B = M((G;\sigma;\tau))$ is the set of all formal sums $\sum_{x\in G} m_x x$ with coefficients in M and well-ordered supports, with pointwise addition and scalar multiplication rule defined by

$$(\sum_{x\in G} m_x x)(\sum_{y\in G} a_y y) = \sum_{z\in G}(\sum_{\{(x,y)|xy=z\}} m_x\sigma_x(a_y)\tau(x,y))z,$$

where $\sum_{x\in G} m_x x \in B$ and $\sum_{y\in G} a_y y \in A$. One can easily check that (i) and (ii) ensure that $M((G;\sigma;\tau))$ is a right A-module.

Let V be a subset of M_R, then $V((G;\sigma;\tau))$ is defined as follows:

$$V((G;\sigma;\tau)) = \{\varphi = \sum_{x\in G} m_x x \in B \,|\, 0 \neq m_x \in V \text{ and } x \in \text{supp}(\varphi)\}.$$

For $\varphi = \sum_{x\in G} m_x x \in B$, let $C_\varphi = \{m_x|\ x \in \text{supp}(\varphi)\}$ and for a subset $V \subseteq B$, we have $C_V = \cup_{\varphi\in V} C_\varphi$.

As usual we shall identify R with the subring $R1_G \subseteq A$, identify G with the subgroup $1_R G$ of invertible elements in A, and identify M_R with the submodule $M1_G \subseteq B$.

In this section, we generalize the results of [6] to the Malcev-Neumann series modules. We start with the following definitions, see [10] and the literature therein for more details.

Definition 2.1. A ring R is called σ-compatible if, for all $a,b \in R$ and $x \in G$, $ab = 0$ if and only if $a\sigma_x(b) = 0$.

Definition 2.2 ([10])**.** A right R-module M_R is called σ-compatible if, for each $m \in M$, $a \in R$ and $x \in G$, $ma = 0$ if and only if $m\sigma_x(a) = 0$.

Definition 2.3. A ring R is called (G,σ)-Armendariz if whenever $fg = 0$ implies $a_x\sigma_x(b_y) = 0$ for each $x \in \text{supp}(f)$ and $y \in \text{supp}(g)$, where $f = \sum_{x\in G} a_x x$ and $g = \sum_{y\in G} b_y y$ be elements of A.

We extend the (G,σ)-Armendariz concept to modules as follows:

Definition 2.4. A right R-module M_R is called (G,σ)-Armendariz if whenever $\varphi f = 0$ implies $m_x\sigma_x(a_y) = 0$ for each $x \in \text{supp}(\varphi)$ and $y \in \text{supp}(f)$, where $\varphi = \sum_{x\in G} m_x x \in B$ and $f = \sum_{y\in G} a_y y \in A$.

It is clear that, R is a (G,σ)-Armendariz and σ-compatible ring if and only if R_R is a (G,σ)-Armendariz and σ-compatible module.

For a subset X of M_R, we define $\mathrm{r}_A(X)$ as the set:

$$\mathrm{r}_A(X) = \{f \in A|\ (x1)f = 0 \text{ for each } x \in X\}.$$

Lemma 2.5. *Let M_R be a right R-module. Then $\mathrm{r}_A(X) = \mathrm{r}_R(X)((G;\sigma;\tau))$, for any subset X of M_R.*

Proof. Let $f = \sum_{g\in G} a_g g \in \mathrm{r}_A(X)$. Then for each $x \in X$ we have $(x1)f = 0$. Thus

$$0 = (x1)(\sum_{g\in G} a_g g) = \sum_{g\in G} x\sigma_1(a_g)\tau(1,g)g = \sum_{g\in G} xa_g\tau(1,g)g,$$

which implies that $xa_g\tau(1,g) = 0$ for each $g \in \mathrm{supp}(f)$. Since $\tau(1,g)$ is invertible, $xa_g = 0$. Hence $a_g \in \mathrm{r}_R(X)$ for each $g \in \mathrm{supp}(f)$. So $f \in \mathrm{r}_R(X)((G;\sigma;\tau))$ and $r_A(X) \subseteq r_R(X)((G;\sigma;\tau))$.

On the other hand, suppose that $f = \sum_{g\in G} a_g g \in \mathrm{r}_R(X)((G;\sigma;\tau))$, then $a_g \in r_R(X)$ for each $g \in \mathrm{supp}(f)$. Thus $xa_g = 0$ for each $x \in X$ and $g \in \mathrm{supp}(f)$. We have $x\sigma_1(a_g) = 0$ and we have that $x\sigma_1(a_g)\tau(1,g) = 0$ for each $x \in X$ and $g \in \mathrm{supp}(f)$. Hence $(x1)f = 0$ for each $x \in X$, and it follows that $f \in \mathrm{r}_A(X)$. So $\mathrm{r}_R(X)((G;\sigma;\tau)) \subseteq \mathrm{r}_A(X)$. Therefore $\mathrm{r}_A(X) = \mathrm{r}_R(X)((G;\sigma;\tau))$. □

For a right R-module M_R, we define

$$\mathrm{r}_R(2^M) = \{\mathrm{r}_R(U)|\ U \subseteq M\},$$
$$\mathrm{r}_A(2^B) = \{\mathrm{r}_A(V)|\ V \subseteq B\}.$$

The above Lemma gives us the map $\psi : \mathrm{r}_R(2^M) \longrightarrow \mathrm{r}_A(2^B)$ defined by $\psi(I) = I((G;\sigma;\tau))$ for every $I \in \mathrm{r}_R(2^M)$. Obviously ψ is an injective map.

In the following Lemma we show that ψ is a bijective map if and only if M_R is (G,σ)-Armendariz.

Lemma 2.6. *Let M_R be a σ-compatible module. The following conditions are equivalent:*
(1) M_R is a (G,σ)-Armendariz module.
(2) $\psi : \mathrm{r}_R(2^M) \longrightarrow \mathrm{r}_A(2^B)$ defined by $\psi(I) = I((G;\sigma;\tau))$ is a bijective map.

Proof. (1)⇒(2)
It is only necessary to show that ψ is surjective. Let $V \subseteq B$ and $T = C_V = \cup_{\varphi\in V} C_\varphi = \cup_{\varphi\in V}\{m_x|\ x \in \mathrm{supp}(\varphi)\}$. We show that

$$\mathrm{r}_A(V) = \psi(\mathrm{r}_R(T)) = \mathrm{r}_R(T)((G;\sigma;\tau))$$

and it is enough to show that $\mathrm{r}_A(\varphi) = \mathrm{r}_R(C_\varphi)((G;\sigma;\tau))$ for each $\varphi = \sum_{x\in G} m_x x \in V$. In fact, let $f = \sum_{y\in G} a_y y \in \mathrm{r}_A(\varphi)$. Then $\varphi f = 0$. Since M_R is a (G,σ)-Armendariz and σ-compatible module, $m_x a_y = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Then $a_y \in \mathrm{r}_R(C_\varphi)$ for each $y \in \mathrm{supp}(f)$. Thus $f \in \mathrm{r}_R(C_\varphi)((G;\sigma;\tau))$ and $\mathrm{r}_A(\varphi) \subseteq \mathrm{r}_R(C_\varphi)((G;\sigma;\tau))$. Now, let $f = \sum_{y\in G} a_y y \in \mathrm{r}_R(C_\varphi)((G;\sigma;\tau))$. Then $a_y \in \mathrm{r}_R(C_\varphi)$ for each $y \in \mathrm{supp}(f)$. Hence $m_x a_y = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Since M_R is σ-compatible, it follows that $m_x \sigma_x(a_y) = 0$, which implies that $m_x \sigma_x(a_y)\tau(x,y) = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Hence

$$0 = \sum_{z\in G}\Big(\sum_{\{(x,y)|xy=z\}} m_x\sigma_x(a_y)\tau(x,y)\Big)z = \varphi f.$$

So $f \in \mathrm{r}_A(\varphi)$ and it follows that $\mathrm{r}_R(C_\varphi)((G;\sigma;\tau)) \subseteq \mathrm{r}_A(\varphi)$. Consequently,

$$\begin{aligned}\mathrm{r}_A(V) &= \cap_{\varphi\in V}\mathrm{r}_A(\varphi) = \cap_{\varphi\in V}\mathrm{r}_R(C_\varphi)((G;\sigma;\tau))\\ &= (\cap_{\varphi\in V}\mathrm{r}_R(C_\varphi))((G;\sigma;\tau))\\ &= \mathrm{r}_R(T)((G;\sigma;\tau)) = \psi(\mathrm{r}_R(T)).\end{aligned}$$

(2)⇒(1)
Let $f = \sum_{y\in G} a_y y \in A$ and $\varphi = \sum_{x\in G} m_x x \in B$ such that $\varphi f = 0$. Then $f \in \mathrm{r}_A(\varphi)$. By assumption $\mathrm{r}_A(\varphi) = T((G;\sigma;\tau))$ for some right ideal T of R. Hence $f \in T((G;\sigma;\tau))$ which implies that $a_y \in T \subseteq \mathrm{r}_A(\varphi)$ for each $y \in \mathrm{supp}(f)$. So, $\varphi(a_y 1) = 0$ and we have that

$$0 = \Big(\sum_{x\in G} m_x x\Big)(a_y 1) = \sum_{x\in G} m_x\sigma_x(a_y)\tau(x,1)x$$

for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Since $\tau(x,1)$ is an invertible element, it follows that $m_x\sigma_x(a_y) = 0$ for each $x \in \mathrm{supp}(\varphi)$ and $y \in \mathrm{supp}(f)$. Therefore M_R is a (G,σ)-Armendariz module. □

Theorem 2.7. *Let M_R be σ-compatible and a (G,σ)-Armendariz module. Then M_R is a right zip R-module if and only if B_A is a right zip A-module.*

Proof. Suppose that B_A is a right zip A-module and $X \subseteq M_R$ such that $\mathrm{r}_R(X) = 0$. Let $Y = \{m1 |\ m \in X\}$ be the embedding of X in B_A. Then, by Lemma 2.5, we have

$$\begin{aligned}\mathrm{r}_A(Y) &= \{f \in A |\ (m1)f = 0, \text{ for all } m \in X\}\\ &= \mathrm{r}_A(X) = \mathrm{r}_R(X)((G;\sigma;\tau)) = 0.\end{aligned}$$

Since B_A is a right zip A-module, for some $m_1, m_2, \ldots, m_n \in X$ there exists a finite set $Y^{'} = \{m_1 1, m_2 1, \ldots, m_n 1\}$ such that $Y^{'} \subseteq Y$ and $\mathrm{r}_A(Y^{'}) = 0$. Let $X^{'} = \{m_1, m_2, \ldots, m_n\}$ which is a nonempty finite subset of X. Then, from Lemma 2.5, we have

$$\begin{aligned} 0 = \mathrm{r}_A(Y^{'}) &= \{f \in A \,|\, (m1)f = 0, \text{ for all } m \in X^{'}\} \\ &= \mathrm{r}_A(X^{'}) = \mathrm{r}_R(X^{'})((G;\sigma;\tau)) \end{aligned}$$

which implies that $\mathrm{r}_R(X^{'}) = 0$. Hence M_R is a right zip R-module.
Conversely, suppose that M_R is a right zip R-module and $Y \subseteq B_A$ such that $\mathrm{r}_A(Y) = 0$. Let

$$T = C_Y = \cup_{\varphi \in Y} C_\varphi = \cup_{\varphi \in Y} \{m_x |\, x \in \mathrm{supp}(\varphi)\}.$$

Then, by Lemma 2.6,

$$0 = \mathrm{r}_A(Y) = \mathrm{r}_R(T)((G;\sigma;\tau))$$

which implies that $\mathrm{r}_R(T) = 0$. Since M_R is a right zip R-module, there exists a finite subset $T_0 \subseteq T$ such that $\mathrm{r}_R(T_0) = 0$. For each $m \in T_0$ there exists $\varphi_m \in Y$ such that for some $x \in \mathrm{supp}(\varphi_m)$, $m_x = m$. Let Y_0 be a minimal subset of Y with respect to inclusion such that $\varphi_m \in Y_0$ for each $m \in T_0$. Then Y_0 is a nonempty finite subset of Y. We consider

$$T_1 = C_{Y_0} = \cup_{\varphi \in Y_0} C_\varphi = \cup_{\varphi \in Y_0} \{m_x |\, x \in \mathrm{supp}(\varphi)\}.$$

Note that $T_0 \subseteq T_1$ and we have that $\mathrm{r}_R(T_1) \subseteq r_R(T_0) = 0$. Thus, by Lemma 2.6, we have

$$\mathrm{r}_A(Y_0) = \mathrm{r}_R(T_1)((G;\sigma;\tau)) = 0.$$

So B_A is a right zip A-module. □

When $M_R = R_R$ we have the following consequence of the last theorem.

Corollary 2.8 ([6], Theorem 2.1)**.** *Suppose that R is σ-compatible and a (G,σ)-Armendariz ring. Then R is a right zip ring if and only if A is a right zip ring.*

Let α be a ring automorphism of R and set $G = \mathbb{Z}$ endowed with the usual order. Define $\sigma : G \longrightarrow \mathrm{Aut}(R)$ via $\sigma(x) = \alpha^x$ for every $x \in \mathbb{Z}$ and $\tau(x,y) = 1$ for any $x, y \in \mathbb{Z}$. Then $M((G;\sigma;\tau))_{R((G;\sigma;\tau))} = M[[x,x^{-1};\alpha]]_{R[[x,x^{-1};\alpha]]}$ the usual skew Laurent power series extension of M_R.

We can introduce the restricted version of (G,σ)-Armendariz condition on skew Laurent power series modules and skew formal power series modules, respectively, as follows:

Definition 2.9. A right R-module M_R is called an α-skew Laurent power serieswise Armendariz (shortly, α-SLPA) module if $\varphi(x)f(x)=0$, where $\varphi(x)=\sum_{i=s}^{\infty} m_i x^i \in M[[x,x^{-1};\alpha]]$ and $f(x)=\sum_{j=t}^{\infty} a_j x^j \in R[[x,x^{-1};\alpha]]$ for $s,t\in\mathbb{Z}$, implies that $m_i\alpha^i(a_j)=0$ for all $i\geq s$ and $j\geq t$.

Definition 2.10. A right R-module M_R is called an α-skew power serieswise Armendariz (shortly, α-SPA) module if $\varphi(x)f(x)=0$, where $\varphi(x)=\sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x)=\sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$, implies that $m_i\alpha^i(a_j)=0$ for all $i\geq 0$ and $j\geq 0$.

It is clear that R is an α-SLPA (resp. α-SPA) ring if and only if R_R is an α-SLPA (resp. α-SPA) module.

Proposition 2.11. *Let α be a ring automorphism of R. Then a right R-module M_R is α-SPA if and only if M_R is α-SLPA.*

Proof. Since $M[[x;\alpha]]\subseteq M[[x,x^{-1};\alpha]]$ and $R[[x;\alpha]]\subseteq R[[x,x^{-1};\alpha]]$, we can easily conclude that: if M_R is α-SLPA, then M_R is α-SPA.

Conversely, assume that M_R is α-SPA and let $\varphi(x)=\sum_{i=-s}^{\infty} m_i x^i \in M[[x,x^{-1};\alpha]]$, $f(x)=\sum_{j=-t}^{\infty} a_j x^j \in R[[x,x^{-1};\alpha]]$, for $s,t\in\mathbb{Z}_{\geq 0}$, be such that $\varphi(x)f(x)=0$. We have

$$\begin{aligned}
0=(\varphi(x)f(x))x^{s+t} &= (\sum_{i=-s}^{\infty} m_i x^i)(\sum_{j=-t}^{\infty} a_j x^j)x^{s+t}\\
&= (\sum_{i=-s}^{\infty} m_i x^i)(\sum_{j=-t}^{\infty} x^s\alpha^{-s}(a_j)x^j x^t)\\
&= (\sum_{i=-s}^{\infty} m_i x^i)(x^s\sum_{j=-t}^{\infty} \alpha^{-s}(a_j)x^{j+t})\\
&= (\sum_{i=-s}^{\infty} m_i x^i x^s)(\sum_{j=-t}^{\infty} \alpha^{-s}(a_j)x^{j+t})\\
&= (\sum_{i=-s}^{\infty} m_i x^{i+s})(\sum_{j=-t}^{\infty} \alpha^{-s}(a_j)x^{j+t}).
\end{aligned}$$

Set $i+s=k$ and $j+t=l$, we get that

$$0=(\varphi(x)f(x))x^{s+t}=(\sum_{k=0}^{\infty} m_{k-s}x^k)(\sum_{l=0}^{\infty}\alpha^{-s}(a_{l-t})x^l)=(\varphi(x)x^s)g(x).$$

Hence, $\varphi(x)x^s = \sum_{k=0}^{\infty} m_{k-s}x^k \in M[[x;\alpha]]$ and $g(x) = \sum_{l=0}^{\infty} \alpha^{-s}(a_{l-t})x^l \in R[[x;\alpha]]$. So,

$$0 = m_{k-s}\alpha^k(\alpha^{-s}(a_{l-t})) = m_{k-s}\alpha^{k-s}(a_{l-t})$$

for all $k \geq 0$ and $l \geq 0$. Hence $m_i\alpha^i(a_j) = 0$ for all $i \geq -s$ and $j \geq -t$, as required. □

From Theorem 2.7, we obtain the following result:

Corollary 2.12. *Let α be a ring automorphism of R, M_R an α-compatible and α-SLPA module. Then M_R is a right zip R-module if and only if $M[[x,x^{-1};\alpha]]_{R[[x,x^{-1};\alpha]]}$ is a right zip $R[[x,x^{-1};\alpha]]$-module.*

Proof. Take $G = \mathbb{Z}$ and $\tau(x,y) = 1$ for any $x,y \in \mathbb{Z}$. For any $x \in \mathbb{Z}$, let $\sigma_x = \alpha^x$. Then the result follows from Theorem 2.7. □

Set $M_R = R_R$ in Corollary 2.12, we get:

Corollary 2.13. *Let α be a ring automorphism of R, R an α-compatible and α-SLPA ring. Then R is a right zip ring if and only if $R[[x,x^{-1};\alpha]]$ is a right zip ring.*

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