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SOME SUBORDINATION AND SUPERORDINATION RESULTS WITH AN INTEGRAL OPERATOR

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In this article, we obtain some subordination and superordination preserving properties of meromorphic univalent functions in the punctured open unit disk associated with an integral operator. Some Sandwich-type results are also presented.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $n \in \mathbb{N} = \{1, 2, \dots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

Let f and g be members of $\mathcal{H}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$.
In such a case, we write

$$f \prec g \quad (z \in \mathbb{U}) \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

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If the function g is univalent in $\mathbb{U}$, then we have (cf. [5]),

$$f \prec g \ \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \ \text{ and } \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.1 ([5])**.** Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the differential subordination:

$$\phi(p(z); zp'(z)) \prec h(z) \ \ (z \in \mathbb{U}), \tag{1}$$

then $p(z)$ is called a solution of the differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant.

Definition 1.2 ([6])**.** Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If $p(z)$ and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination:

$$h(z) \prec \varphi(p(z), zp'(z)) \ (z \in \mathbb{U}), \tag{2}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

Definition 1.3 ([5])**.** Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that

$$q'(\zeta) \neq 0 \ \ (\zeta \in \partial\mathbb{U} \backslash E(q)).$$

Further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0) \equiv \mathcal{F}_0$ and $\mathcal{F}(1) \equiv \mathcal{F}_1$.

Definition 1.4 ([6])**.** A function $L(z,t)$ $(z \in \mathbb{U}, t \geq 0)$ is said to be a subordination chain if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z, .)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$ and $L(z, t_1) \prec L(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{3}$$

which are analytic in the punctured open unit disk $\mathbb{U}^*$. For functions $f \in \Sigma$ given by (3), and $g \in \Sigma$ given by

$$g(z) := \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f*g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g*f)(z).$$

Analogous to the integral operator defined by Jung et al. [1], Lashin [2] introduced and investigated the following integral operator

$$Q_{\alpha,\beta} : \Sigma \to \Sigma, \tag{4}$$

defined in terms of the familiar Gamma function by

$$Q_{\alpha,\beta} f(z) = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^{\beta} (1-\frac{t}{z})^{\alpha-1} f(t)dt$$

$$= \frac{1}{z} + \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)} a_k z^k \ \ (\alpha > 0;\ \beta > 0;\ z \in \mathbb{U}^*).$$

By setting

$$f_{\alpha,\beta}(z) := \frac{1}{z} + \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+\alpha+1)}{\Gamma(k+\beta+1)} z^k \ \ (\alpha > 0;\ \beta > 0;\ z \in \mathbb{U}^*), \tag{5}$$

Wang et al. [8] defined and studied an integral operator $Q^{\lambda}_{\alpha,\beta} : \Sigma \to \Sigma$ which is defined as follows:

Let $f^{\lambda}_{\alpha,\beta}(z)$ be defined such that

$$f_{\alpha,\beta}(z) * f^{\lambda}_{\alpha,\beta}(z) = \frac{1}{z(1-z)^{\lambda}} \ (\alpha > 0;\ \beta > 0;\ \lambda > 0;\ z \in \mathbb{U}^*). \tag{6}$$

Then

$$Q^{\lambda}_{\alpha,\beta} f(z) := f^{\lambda}_{\alpha,\beta}(z) * f(z) \ (z \in \mathbb{U}^*, f \in \Sigma). \tag{7}$$

From (5), (6) and (7) it follows that

$$Q^{\lambda}_{\alpha,\beta} f(z) = \frac{1}{z} + \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{(\lambda)_{k+1}\Gamma(k+\beta+1)}{(k+1)!\Gamma(k+\beta+\alpha+1)} a_k z^k \ \ (z \in \mathbb{U}^*), \tag{8}$$

where $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k = \left\{ \begin{smallmatrix} 1, & k=0 \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & k\in\mathbb{N}:=\{1,2,\ldots\} \end{smallmatrix} \right\}. \tag{9}$$

Clearly, we know that

$$Q^1_{\alpha,\beta} = Q_{\alpha,\beta}.$$

It is readily verified from (8) that

$$z(Q^\lambda_{\alpha,\beta} f)'(z) = \lambda Q^{\lambda+1}_{\alpha,\beta} f(z) - (\lambda+1) Q^\lambda_{\alpha,\beta} f(z), \tag{10}$$

$$z(Q^\lambda_{\alpha,\beta} f)'(z) = (\beta+\alpha-1) Q^\lambda_{\alpha-1,\beta} f(z) - (\beta+\alpha) Q^\lambda_{\alpha,\beta} f(z). \tag{11}$$

2. A Set of Lemmas

The following lemmas will be required in our present investigation.

Lemma 2.1 ([7]). *The function $L(z,t) : \mathbb{U}\times[0,\infty) \longrightarrow \mathbb{C}$ of the form: $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0,\ t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if*

$$\Re\left\{ \frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

Lemma 2.2 ([3]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition:*

$$\Re\{H(is,t)\} \leq 0$$

for all real s, and

$$t \leq -n(1+s^2)/2 \ (n \in \mathbb{N}).$$

If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ is analytic in $\mathbb{U}$ and

$$\Re\{H(p(z), zp'(z))\} > 0 \ \ (z \in \mathbb{U})$$

then

$$\Re\{p(z)\} > 0 \ \ (z \in \mathbb{U}).$$

Lemma 2.3 ([4]). *Let $k,\ \gamma \in \mathbb{C}$ with $k \neq 0$ and $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If*

$$\Re\{kh(z) + \gamma\} > 0 \ \ (z \in \mathbb{U}),$$

then the solution of the following differential equation

$$q(z) + \frac{zq'(z)}{kq(z)+\gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies the inequality

$$\Re\{kq(z) + \gamma\} > 0 \ \ (z \in \mathbb{U}).$$

Lemma 2.4 ([5]). *Let $p \in \mathcal{F}(a)$ and let*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in $\mathbb{U}$ with

$$q(z) \neq a \text{ and } n \geq 1.$$

If q is not subordinate to p, then there exist two points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \text{ and } \zeta_0 \in \partial\mathbb{U} \backslash E(q),$$

such that

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), \; q(z_0) = p(\zeta_0) \text{ and } z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Lemma 2.5 ([6]). *Let $q \in \mathcal{H}[a,1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z) \qquad (z \in \mathbb{U}).$$

If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{F}(a)$, then

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{F}(a)$, then q is the best subordinant.

In this paper, we aim to prove some subordination and superordination-preserving properties associated with the integral operator $Q^{\lambda}_{\alpha,\beta}$. Sandwich-type results involving this operator is also derived.

3. Main Results

We begin with proving the following subordination theorem involving the operator $Q^{\lambda}_{\alpha,\beta} f$ defined by (8).

Theorem 3.1. *Let $f,\ g \in \Sigma$ and*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \left(\phi(z) = \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu};\ z \in \mathbb{U}\right), \tag{12}$$

$$(\lambda > 0;\ \alpha > 1;\ \beta > 0;\ \mu > 0),$$

where δ is given by

$$\delta = \frac{1+\mu^2(\beta+\alpha-1)^2 - \left|1-\mu^2(\beta+\alpha-1)^2\right|}{4\mu(\beta+\alpha-1)} \quad (z \in \mathbb{U}). \tag{13}$$

Then the subordination condition

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \tag{14}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \tag{15}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by

$$F(z) := \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \text{ and } \ G(z) := \left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \ (z \in \mathbb{U}). \tag{16}$$

We first show that if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{17}$$

then

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

From (11) and the definition of functions G and ϕ, we obtain that

$$\phi(z) = G(z) + \frac{zG'(z)}{\mu(\beta+\alpha-1)}. \tag{18}$$

Differentiating both sides of (18) with respect to z yields

$$\phi'(z) = \left(1 + \frac{1}{\mu(\beta+\alpha-1)}\right)G'(z) + \frac{zG''(z)}{\mu(\beta+\alpha-1)}. \tag{19}$$

Combining (17) and (19), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{\mu(\beta+\alpha-1)+q(z)} = h(z) \ (z \in \mathbb{U}). \tag{20}$$

It follows from (12) and (20) that

$$\Re\{h(z)+\mu(\beta+\alpha-1)\}>0 \quad (z\in\mathbb{U}). \tag{21}$$

Moreover, by using Lemma 2.3, we conclude the differential equation (20) has a solution $q(z)\in\mathcal{H}(\mathbb{U})$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+\mu(\beta+\alpha-1)}+\delta, \tag{22}$$

where δ is given by (13). From (20) and (21) we obtain

$$\Re\{H(q(z),zq'(z))\}>0 \quad (z\in\mathbb{U}).$$

To verify the condition

$$\Re\{H(iv,t)\}\leq 0 \quad \left(v\in\mathbb{R};\ t\leq -\frac{1}{2}(1+v^2)\right), \tag{23}$$

we proceed as follows:

$$\Re\{H(iv,t)\}=\Re\left\{iv+\frac{t}{\mu(\beta+\alpha-1)+iv}+\delta\right\}$$

$$=\frac{t\mu(\beta+\alpha-1)}{|\mu(\beta+\alpha-1)+iv|^2}+\delta\leq -\frac{E_\delta(v)}{2\,|\mu(\beta+\alpha-1)+iv|^2},$$

where

$$E_\delta(v):=[\mu(\beta+\alpha-1)-2\delta]\,v^2-\mu(\beta+\alpha-1)\,[2\delta\mu(\beta+\alpha-1)-1]. \tag{24}$$

For δ given by (13), we can prove easily that the expression $E_\delta(v)$ given by (24) is greater than or equal to zero. Hence, from (22), we see that (23) holds true. Thus, using Lemma 2.2, we conclude that

$$\Re\{q(z)\}>0 \quad (z\in\mathbb{U}).$$

Moreover, we see that the condition $G'(0)\neq 0$ is satisfied. Hence, the function G defined by (16) is convex (univalent) in $\mathbb{U}$.
Next, we prove that the subordination condition (14) implies that

$$F(z)\prec G(z) \qquad (z\in\mathbb{U}),$$

for the functions F and G defined by (16). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and

$$G'(\zeta)\neq 0 \ \ (\zeta\in\partial\mathbb{U}).$$

For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(1+t)}{\mu(\beta+\alpha-1)} zG'(z), \tag{25}$$
$$(0 \le t < \infty;\ z \in \mathbb{U};\ \alpha > 1;\ \beta > 0;\ \mu > 0).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{(1+t)}{\mu(\beta+\alpha-1)}\right) \neq 0,$$
$$(0 \le t < \infty;\ z \in \mathbb{U};\ \alpha > 1;\ \beta > 0;\ \mu > 0).$$

This shows that the function

$$L(z,t) = a_1(t)z + \dots$$

satisfies the condition $a_1(t) \neq 0$ $(0 \le t < \infty)$. Furthermore, we have

$$\Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \Re\left\{\mu(\beta+\alpha-1) + (1+t)(1 + \frac{zG''(z)}{G'(z)})\right\} > 0.$$

Therefore, by using of Lemma 2.1, we deduce that $L(z,t)$ is a subordination chain, since

$$\phi(z) = G(z) + \frac{zG'(z)}{\mu(\beta+\alpha-1)} = L(z,0),$$

it follows from the definition of subordinations chains

$$L(z,0) \prec L(z,t)\ (0 \le t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U})\ (\zeta \in \partial\mathbb{U};\ 0 \le t < \infty). \tag{26}$$

Now, suppose that F is not subordinate to G, then by Lemma 2.4, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, such that

$$F(z_0) = G(\zeta_0) \text{ and } z_0F'(z_0) = (1+t)\zeta_0 G'(\zeta_0)\ \ (0 \le t < \infty). \tag{27}$$

Hence, by using (16), (25), (27) and (14), we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{(1+t)}{\mu(\beta+\alpha-1)}\zeta_0 G'(\zeta_0) = F(z_0) + \frac{1}{\mu(\beta+\alpha-1)} z_0 F'(z_0)$$
$$= \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z_0)}{Q^{\lambda}_{\alpha,\beta}(f)(z_0)}\right)\left(z_0 Q^{\lambda}_{\alpha,\beta}(f)(z_0)\right)^{\mu} \in \phi(\mathbb{U}).$$

This contradicts (26). Thus, we deduce that $F \prec G$.
Considering $F = G$, we see that the function G is the best dominant. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $f,\ g \in \Sigma$ and*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta\left(\phi(z)=\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu};z\in\mathbb{U}\right), \tag{28}$$

$$(\lambda>0;\ \alpha>0;\ \beta>0;\ \mu>0),$$

where δ is given by

$$\delta=\frac{1+\lambda^2\mu^2-\left|1-\lambda^2\mu^2\right|}{4\mu\lambda\mu}\qquad(z\in\mathbb{U}). \tag{29}$$

Then the subordination condition

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}\prec\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \tag{30}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}\prec\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \tag{31}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by (16). Taking the logarithmic differentiation on both sides of the second equation in (16) and using the equation (10), the proof is similar to that of Theorem 3.1. $\square$

Theorem 3.3. *Let $f,g\in\Sigma$ and*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta,\left(\phi(z)=\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu};z\in\mathbb{U}\right), \tag{32}$$

$$(\lambda>0;\ \alpha>1;\ \beta>0;\ \mu>0),$$

where δ is given by (13). If the function

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}\in\mathcal{F}$, then the superordination condition

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}\prec\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \tag{33}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}, \tag{34}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$ *is the best subordinant.*

Proof. Suppose that the function F, G and q are defined by (16) and (17), respectively. By applying similar method as in the proof of Theorem 3.1, we get

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Next to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ is defined by (25). Since G is convex, by applying a similar method as in Theorem 3.1, we deduce that $L(z,t)$ is a subordination chain. Therefore, by using Lemma 2.5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{zG^{'}(z)}{\mu(\beta+\alpha-1)} = \phi(G(z), G'(z))$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 3.3. $\square$

Theorem 3.4. *Let* $f,\ g \in \Sigma$ *and*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \left(\phi(z) = \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}; z \in \mathbb{U}\right), \tag{35}$$

$$(\lambda > 0;\ \alpha > 0;\ \beta > 0;\ \mu > 0),$$

where δ *is given by (29). If the function*

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ *and* $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, *then the superordination condition*

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}, \tag{36}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \tag{37}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$ *is the best subordinant.*

Proof. The proof is similar to that of Theorem 3.3. □

Combining the above-mentioned subordination and superordination results involving the operator $Q^{\lambda}_{\alpha,\beta}$ the following "Sandwich-type result" is derived.

Theorem 3.5. *Let* $f,\ g_j \in \Sigma\ (j=1,2)$ *and*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta,$$

where

$$\phi_j(z) = \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g_j)(z)}{Q^{\lambda}_{\alpha,\beta}(g_j)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_j)(z)\right)^{\mu},$$

$$(j=1,2; z\in\mathbb{U};\ \lambda>0;\ \alpha>1;\ \beta>0;\ \mu>0),$$

and δ *is given by (13). If the function*

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ *and* $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, *then the condition*

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g_1)(z)}{Q^{\lambda}_{\alpha,\beta}(g_1)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

$$\prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g_2)(z)}{Q^{\lambda}_{\alpha,\beta}(g_2)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu} \tag{38}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}, \tag{39}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu}$ *and* $\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}$ *are respectively, the best subordinant and the best dominant.*

Theorem 3.6. *Let* $f,\ g_j \in \Sigma\ (j=1,2)$ *and*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta, \left(\phi_j(z) = \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g_j)(z)}{Q^{\lambda}_{\alpha,\beta}(g_j)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_j)(z)\right)^{\mu}; z\in\mathbb{U}\right),$$

$$(\lambda > 0;\ \alpha > 0;\ \beta > 0;\ \mu > 0),$$

where δ is given by (29). If the function

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, then the condition

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g_1)(z)}{Q^{\lambda}_{\alpha,\beta}(g_1)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$
$$\prec \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g_2)(z)}{Q^{\lambda}_{\alpha,\beta}(g_2)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu} \tag{40}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}, \tag{41}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}$ are respectively, the best subordinant and the best dominant.

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