LE MATEMATICHE
Vol. LXXI (2016) – Fasc. I, pp. 135–147
doi: 10.4418/2016.71.1.11

REDUCED SECOND ZAGREB INDEX OF BICYCLIC GRAPHS WITH PENDENT VERTICES

SÜLEYMAN EDIZ

Reduced second Zagreb index has been defined recently. In this paper we characterized extremal bicyclic graphs with pendent vertices with respect to this novel index.

1. Introduction

Let G be a simple connected graph with n vertices and m edges. d_v is the number of edges incident to the vertex v. A vertex of degree one is said to be a pendent vertex. Unicyclic graphs are connected graphs with n vertices and n edges. Bicyclic graphs are connected graphs with n vertices and $n+1$ edges.We write Δ and δ for the largest and the smallest of all degrees of vertices of G, respectively. The first Zagreb and the second Zagreb index of the graph G are defined as:

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_v^2 \tag{1}$$

and

$$M_2 = M_2(G) = \sum_{uv \in E(G)} d_u d_v \tag{2}$$

Entrato in redazione: 31 marzo 2015

AMS 2010 Subject Classification: 05C07, 05C38.
Keywords: Degree based topological index, Reduced second Zagreb index, Bicyclic graph.

respectively. In formula (2), uv denotes an edge connecting the vertices u and v. In 1972, the quantities M_1 and M_2 were found to occur within certain approximate expressions for the total π-electron energy [19]. The first Zagreb index satisfies the identity

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) \tag{3}$$

where the notation is same as in equation(2) [20]. In view of the extensive research on the two Zagreb indices, in particular, their difference $M_2 - M_1$, the difference between the equations (2) and (3), has been never examined. In [17], Furtula *et al.* examined $M_2 - M_1$ and proposed a new degree based topological index, named it 'Reduced second Zagreb index' and characterized the maximum trees with respect to reduced second Zagreb index. Reduced second Zagreb index is defined [17] as follows;

$$RM_2 = RM_2(G) = \sum_{uv \in E(G)} (d_u - 1)(d_v - 1) = M_2(G) - M_1(G) + m \tag{4}$$

where m denotes the number of edges. Zagreb indices of bicyclic graphs are investigated in [2–4, 9]. For other topological indices of bicyclic graphs see in [1, 5–8, 10–16]. RM_2 index of unicyclic graphs were investigated in [18]. In this paper we investigate maximum and minimum bicyclic graphs with respect to RM_2 index.

2. Minimum and maximum RM_2 index of bicyclic graphs

Let DC denote all bicyclic graphs with n vertex , $n+1$ edges and k pendent vertices (here, DC stands for double cycle). The arrangement of cycles of DC has at most three possible cases.

Case 1: $DC_{a,b}(k_1, k_2, \ldots, k_a, s_2, s_3, \ldots, s_b)$ is the set of $G \in DC$ in which the cycles C_a and C_b have only one common vertex. Here,

$$k_1, k_2, \ldots, k_a, s_2, s_3, \ldots, s_b$$

denote the number of pendent vertices of corresponding

$$v_1, v_2, \ldots, v_a, v_2{}', v_3{}', \ldots, v_b{}'$$

vertices. See Figure 1.

Case 2: $DC^l_{a,b}(k_1, k_2, \ldots, k_a, r_1, r_2, \ldots, r_l, s_1, s_2, \ldots, s_b)$ is the set of $G \in DC$ in which the cycles C_a and C_b have no common vertex for $l \geq 0$. Here

$$k_1, k_2, \ldots, k_a, r_1, r_2, \ldots, r_l, s_1, s_2, \ldots, s_b$$

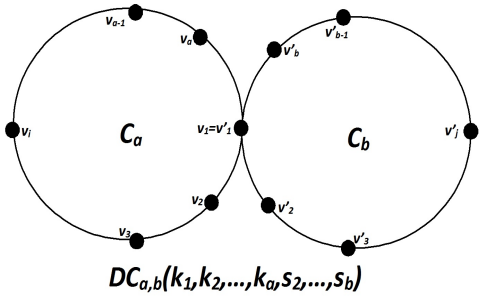


Figure 1: The first class of bicyclic graphs with k pendent vertices: $DC_{a,b}\left(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b\right)$

denote the number of pendent vertices of corresponding

$$v_1,v_2,\ldots,v_a,n_1,n_2,\ldots,n_l,u_1,u_2,\ldots,u_b$$

vertices. See Figure 2.

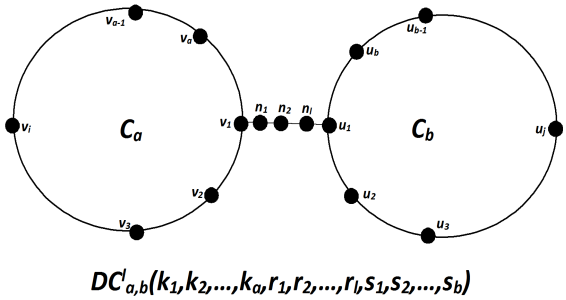


Figure 2: The second class of bicyclic graphs with k pendent vertices: $DC^l_{a,b}\left(k_1,k_2,\ldots,k_a,r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_b\right)$

Case 3: $DC^l_{a+b}\left(k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_2,\ldots,s_{b-l-1}\right)$ is the set of $G\in DC$ in which the cycles C_a and C_b have a common path of length $l+1$ for $l\geq 0$. Here

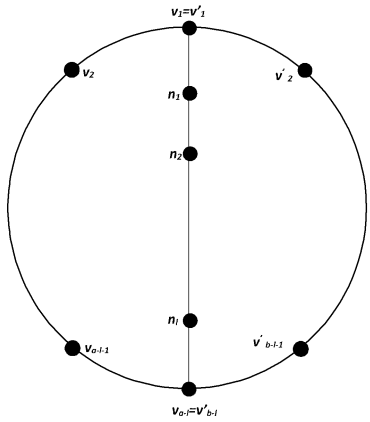
$$k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_2,\ldots,s_{b-l-1}$$

denote the number of pendent vertices of corresponding

$$v_1,v_2,\ldots,v_{a-l},n_1,n_2,\ldots,n_l,{v_2}',{v_3}',\ldots,v'_{b-l-1}$$

vertices. See Figure 3.

With direct calculations, we get the following propositions.



$DC^l_{a+b}(k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_2,s_3,\ldots,s_{b-l-1})$

Figure 3: The third class of bicyclic graphs with k pendent vertices: $DC^l_{a+b}(k_1,k_2,\ldots,k_a,r_1,r_2,\ldots,r_l,s_2,\ldots,s_{b-l-1})$

Proposition 2.1. *Let $DC_{a,b}(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b)$ be the set of $G \in DC$. Then*

$$\begin{aligned} RM_2(G) &= (k_1+3)(k_2+k_a+s_2+s_b+4)+(k_2+1)(k_3+1)+\ldots \\ &\quad +(k_{a-1}+1)(k_a+1)+(s_2+1)(s_3+1)+\cdots+(s_{b-1}+1)(s_b+1). \end{aligned}$$

Proposition 2.2. *Let $DC^l_{a,b}(k_1,k_2,\ldots,k_a,r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_b)$ be the set of $G \in DC$ then*

$$(a) \qquad \begin{aligned} RM_2(G) &= (k_a+k_2+2)+(k_1+2)(s_1+2)+(k_2+1)(k_3+1)+\ldots \\ &\quad +(k_{a-1}+1)(k_a+1)+(s_2+1)(s_3+1)+\cdots+(s_{b-1}+1)(s_b+1) \end{aligned}$$

for $l=0$.

$$(b) \qquad \begin{aligned} RM_2(G) &= (k_1+2)(k_2+k_a+r_1+3)+(k_2+1)(k_3+1)+\ldots \\ &\quad +(k_{a-1}+1)(k_a+1)+(r_1+1)(r_2+1)+\cdots+(r_{l-1}+1)(r_l+1) \\ &\quad +(s_1+2)(s_2+s_b+r_l+3)+(s_2+1)(s_3+1)+\cdots+(s_{b-1}+1)(s_b+1) \end{aligned}$$

for $l \geq 1$.

Proposition 2.3. *Let $DC^l_{a+b}(k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_2,\ldots,s_{b-l-1})$ be the*

set of $G \in DC$ then

$$(a)\quad RM_2(G) = (k_1+2)(k_2+s_2+r_1+3) + (k_{a-l}+2)(k_{a-l-1}+s_{b-l-1}+r_l+3) + (k_2+1)(k_3+1)+\cdots+(k_{a-l-2}+1)(k_{a-l-1}+1)+ (s_2+1)(s_3+1)+\cdots+(s_{b-l-2}+1)(s_{b-l-1}+1) + (r_1+1)(r_2+1)+\ldots.+(r_{l-1}+1)(r_l+1)$$

for $1 \le l \le n-4$.

$$(b)\quad RM_2(G) = (k_1+2)(k_2+s_2+2)+(k_2+1)(k_3+1)+\ldots + (k_{a-2}+1)(k_{a-1}+1)+(k_a+2)(k_{a-1}+s_{b-1}+2) + (s_2+1)(s_3+1)+\cdots+(s_{b-2}+1)(s_{b-1}+1)+(k_1+2)(k_a+2)$$

for $l = 0$.

Lemma 2.4. *Let $G_1 \in DC_{a,b}$ with no pendent vertex and n vertices. Let $G_2 \in DC_{a,b}$ with $k \ge 1$ pendent vertices and n vertices. Then $RM_2(G_1) < RM_2(G_2)$.*

Proof. Let $k = 1$. Let uvl be a path of G_1 where all degrees are 2. Then, we obtain G_2 from G_1 by taking u attached to v as a pendent vertex. In this case $RM_2(G_2) - RM_2(G_1) = 1 > 0$. On the other hand, let uvl be a path of G_1 and $v = v_1$ so that $d_v = 4$. Then, we obtain G_2 from G_1 by taking u attached to v as a pendent vertex. In this case $RM_2(G_2) - RM_2(G_1) = 6$. The other cases for $k \ge 2$ are similar. □

Now, we give the following lemmas whose proofs are similar to that of Lemma 2.4.

Lemma 2.5. *Let $G_1 \in DC^l_{a,b}$ with no pendent vertex and n vertices. Let $G_2 \in DC^l_{a,b}$ with $k \ge 1$ pendent vertices and n vertices. Then $RM_2(G_1) < RM_2(G_2)$.*

Lemma 2.6. *Let $G_1 \in DC^l_{a+b}$ with no pendent vertex and n vertices. Let $G_2 \in DC^l_{a+b}$ with $k \ge 1$ pendent vertices and n vertices. Then $RM_2(G_1) < RM_2(G_2)$.*

Corollary 2.7. *(a) Let $G \in DC_{a,b}(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b)$. Then the minimum RM_2 index of G is $DC_{a,b}(0,0,\ldots,0)$.*
(b) Let $G \in DC^l_{a,b}(k_1,k_2,\ldots,k_a,r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_b)$. Then the minimum RM_2 index of G is $DC^l_{a,b}(0,0,\ldots,0)$.
(c) Let $G \in DC^l_{a+b}(k_1,k_2,\ldots,k_{a-l},r_1,r_2,\ldots,r_l,s_1,s_2,\ldots,s_{b-l-1})$. Then the minimum RM_2 index of G is $DC^l_{a+b}(0,0,\ldots,0)$.
Notice that in all three cases G has no pendent vertices.

Proposition 2.8. *Let $G \in DC_{a,b}(0,0,\ldots,0)$. Then the minimum RM_2 index is $RM_2(G) = n+9$.*

Proof. From Proposition 2.1 and Corollary 2.7, $RM_2(G) = a+b+8$. Since $a+b = n+1$, the desired result is acquired. □

Proposition 2.9. *Let $G \in DC^l_{a,b}(0,0,\ldots,0)$. Then the minimum RM_2 index is $RM_2(DC^l_{a,b}) = n+7$ for $l \geq 1$.*

Proof. From Proposition 2.2b and Corollary 2.7, $RM_2(G) = a+b+l+7$. Since $n = a+b+l$, the desired result is acquired. □

Proposition 2.10. *Let $G \in DC^l_{a+b}(0,0,\ldots,0)$. Then the minimum RM_2 index is $RM_2(DC^l_{a+b}) = n+7$ for $l \geq 1$.*

Proof. From Proposition 2.3a and Corollary 2.7, $RM_2(G) = a+b-l+5$. Since $a+b-l = n+2$, the desired result is acquired. □

Definition 2.11. Let Ξ be a family of the set $DC_{3,3}(k_1,k_2,k_3,s_2,s_3)$ such that $s_2 = s_3 = 0$ and $k_i - k_j = 0$ or $k_i - k_j = 1$ for $1 \leq i \leq j \leq 3$. Or by symmetry, $k_2 = k_3 = 0$ and $s_i - s_j = 0$ or $s_i - s_j = 1$ for $1 \leq i \leq j \leq 3$. See Figure 4 .

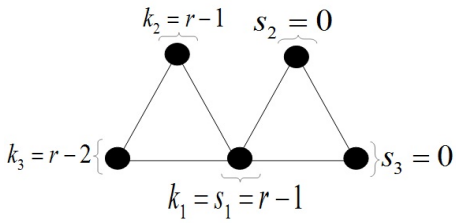


Figure 4: $G = DC_{3,3}(r-1, r-1, r-2, 0, 0) \in \Xi$ for $n = 3r+1$, $r \geq 2$.

Proposition 2.12. *Let $G \in \Xi$ with n vertices and $n-5$ pendent vertices. Then*

$$\begin{array}{lll} (a) & RM_2(G) = 3r^2+2r+2 & for \quad n = 3r, r \geq 2. \\ (b) & RM_2(G) = 3r^2+4r+3 & for \quad n = 3r+1, r \geq 2. \\ (c) & RM_2(G) = 3r^2+6r+1 & for \quad n = 3r+2, r \geq 2. \end{array}$$

Proof. We only prove the case (b), the other cases are similar. From Proposition 2.1 and Figure 4, we can directly write

$$\begin{aligned} RM_2(G) &= (r+2)(r+r-1+2) + r(r-1) + 1 \\ &= (r+2)(2r+1) + r^2 - r + 1 = 3r^2 + 4r + 3 \end{aligned}$$ □

Lemma 2.13. *Let a,b,c,d,e be non-negative integers and $a+b+c+d=e$. Then $ab+cd$ takes its maximum value when $a=\left\lceil \frac{e}{2} \right\rceil$, $b=\left\lfloor \frac{e}{2} \right\rfloor$ and $c=d=0$. Or by symmetry $c=\left\lceil \frac{e}{2} \right\rceil$, $d=\left\lfloor \frac{e}{2} \right\rfloor$ and $a=b=0$.*

Lemma 2.14. *Let a,b,c,d,e,f be non-negative integers and $a+b+c+d+e=f$. Then $ab+ac+de$ takes its maximum value when $e=\left\lceil \frac{f}{2} \right\rceil$, $d=\left\lfloor \frac{f}{2} \right\rfloor$ and $a=b=c=0$. Or by symmetry $a=\left\lceil \frac{f}{2} \right\rceil$, $b+c=\left\lfloor \frac{f}{2} \right\rfloor$ and $d=e=0$.*

Proof. Let $b+c=x$. Then $ab+ac+ed=a(b+c)+ed=ax+ed$. From Lemma 2.13, the desired is result acquired. □

Proposition 2.15. *Let $G \in DC_{a,b}(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b)$ with $a,b \geq 4$ and $k_2+k_a+s_2+s_b=\Omega$. Then, the maximum RM_2 index of G is $DC_{a,b}(k_1,k_2,\ldots,k_a,s_2,s_3,\ldots,s_b)$ such that $k_3=k_4=\cdots=k_{a-1}=0$, $s_3=s_4=\cdots=s_{b-1}=0$ and $k_1=\Omega$ or $|k_1-\Omega|=1$.*

Proof. By Proposition 2.1,

$$\begin{aligned}
RM_2(G) &= (k_1+3)(k_2+k_a+s_2+s_b+4)+(k_2+1)(k_3+1)+\ldots \\
&\quad +(k_{a-1}+1)(k_a+1)+(s_2+1)(s_3+1)+\cdots+(s_{b-1}+1)(s_b+1) \\
&= k_1k_2+k_1k_a+k_1s_2+k_1s_b+4k_1+3k_2+3k_a+3s_2+3s_b+12 \\
&\quad +k_2k_3+k_2+k_3+1+\cdots+k_{a-1}k_a+k_{a-1}+k_a+1+s_2s_3+s_2+s_3+1+\ldots \\
&\quad +s_{b-1}s_b+s_{b-1}+s_b+1 \\
&= k_1k_2+k_1k_a+k_1s_2+k_1s_b+k_2k_3+k_3k_4+\cdots+k_{a-2}k_{a-1}+k_{a-1}k_a \\
&\quad +s_2s_3+s_3s_4+\cdots+s_{b-2}s_{b-1}+s_{b-1}s_b+4k_1+4k_2+2k_3+\cdots+2k_{a-1} \\
&\quad +4k_a+4s_2+2s_3+\cdots+2s_{a-1}+4s_b+a+b+8= \\
&= k_1k_2+k_1k_a+k_1s_2+k_1s_b+k_2k_3+k_3k_4+\cdots+k_{a-2}k_{a-1}+k_{a-1}k_a \\
&\quad +s_2s_3+s_3s_4+\cdots+s_{b-2}s_{b-1}+s_{b-1}s_b \\
&\quad +2(k_1+k_2+\cdots+k_a+s_2+s_3+\cdots+s_b) \\
&\quad +2(k_1+k_2+k_a+s_2+s_b)+a+b+8
\end{aligned}$$

Since $k_1+k_2+\cdots+k_a+s_2+s_3+\cdots+s_b=n-a-b+1$, then

$$\begin{aligned}
RM_2(G) &= k_1k_2 + k_1k_a + k_1s_2 + k_1s_b + k_2k_3 + k_3k_4 + \cdots + k_{a-2}k_{a-1} + k_{a-1}k_a \\
&\quad + s_2s_3 + s_3s_4 + \cdots + s_{b-2}s_{b-1} + s_{b-1}s_b + 2n - 2a - 2b + 2 \\
&\quad + 2(k_1 + k_2 + k_a + s_2 + s_b) + a + b + 8 \\
&= k_1(k_2 + k_a + s_2 + s_b) + k_2k_3 + k_3k_4 + \cdots + k_{a-2}k_{a-1} + k_{a-1}k_a \\
&\quad + s_2s_3 + s_3s_4 + \cdots + s_{b-2}s_{b-1} + s_{b-1}s_b \\
&\quad + 2(k_1 + k_2 + k_a + s_2 + s_b) + 2n - a - b + 10 \\
&= k_1\Omega + k_2k_3 + k_3k_4 + \cdots + k_{a-2}k_{a-1} + k_{a-1}k_a \\
&\quad + s_2s_3 + s_3s_4 + \cdots + s_{b-2}s_{b-1} + s_{b-1}s_b + 2(k_1 + \Omega) + 2n - a - b + 10.
\end{aligned}$$

Clearly from the last equality by using Lemma 2.13, RM_2 takes its maximum value when $k_3 = k_4 = \cdots = k_{a-1} = 0$, $s_3 = s_4 = \cdots = s_{b-1} = 0$ and $k_1 = \Omega$ or $|k_1 - \Omega| = 1$. □

Theorem 2.16. *Let $G \in \Xi$ with n vertices and $n-5$ pendent vertices. Then G has maximum RM_2 value among the all graphs belong to $DC_{a,b}$ with n vertices.*

Proof. We only consider $n = 3r+1$ for $r \geq 2$. The other cases are similar. Firstly, we show that G has maximum RM_2 value among all the graphs belonging to $DC_{3,3}(k_1, k_2, k_3, s_2, s_3)$. From the definition of RM_2 index,

$$\begin{aligned}
RM_2(G) &= (k_1+3)(k_2+k_3+s_2+s_3+4) + (k_2+1)(k_3+1) + (s_2+1)(s_3+1) \\
&= (k_1+3)(k_2+k_3+s_2+s_3+4) + (k_2+k_3+s_2+s_3) + k_2k_3 + s_2s_3 + 2
\end{aligned}$$

Since $k_2 + k_3 + s_2 + s_3 = n - k_1 - 5 = 3r - k_1 - 4$ then

$$RM_2(G) = (k_1+3)(3r-k_1) + 3r - k_1 - 4 + k_2k_3 + s_2s_3 + 2.$$

By Lemma 2.13, $k_2k_3 + s_2s_3$ takes its maximum value when $k_2 = k_3$ or $k_2 = k_3 + 1$ and $s_2 = s_3 = 0$. Or by symmetry $s_2 = s_3$ or $s_2 = s_3 + 1$ and $k_2 = k_3 = 0$. We only consider the first part of the Lemma 2.13. The second part can be handled similarly. Then

$$RM_2(G) = f(k_1, k_2, k_3) = (k_1+3)(3r-k_1) + 3r - k_1 - 4 + k_2k_3 + 2.$$

Since $k_1 + k_2 + k_3 = 3r - 4$, then

$$g(k_1, k_2, k_3) = 3r - 4 - k_1 - k_2 - k_3 = 0 \tag{5}$$

can be written. By using the Lagrange multipliers method, we obtain; $3r - 2k_1 - 4 = k_2$ and $3r - 2k_1 - 4 = k_3$. Thus, $k_2 + k_3 = 6r - 4k_1 - 8$. From Equation 5,

$k_2+k_3=3r-k_1-4$. And from these last two equalities $k_1=r-1$. By Definition 2.11, $k_2=r-1$ and $k_3=r-2$.
Secondly, we show that G has maximum RM_2 value among all the graphs belonging to $DC_{a,b}(k_1,\ldots,k_a,s_2,\ldots,s_b)$ for $a+b\geq 7$ with n vertices. There are two cases in this situation.
Case 1: Let $a=4$ and $b=3$. See Figure 5. From the definition of RM_2 index,

$$\begin{aligned} RM_2(G) &= (k_1+3)(k_2+k_4+s_2+s_3+4)+(k_2+1)(k_3+1) \\ &\quad +(k_3+1)(k_4+1)+(s_2+1)(s_3+1). \end{aligned}$$

Since $k_2+k_4+s_2+s_3=n-6-k_1=3r-5-k_1$, then

$$\begin{aligned} RM_2(G) &= (k_1+3)(3r-k_1-1)+k_2+k_4+s_2+s_3 \\ &\quad +k_3+k_2k_3+k_3k_4+s_2s_3+3 \\ &= (k_1+3)(3r-k_1-1)+3r-5-k_1+k_3+k_2k_3+k_3k_4+s_2s_3+3. \end{aligned}$$

By Lemma 2.14, $k_2k_3+k_3k_4+s_2s_3$ takes its maximum value when $k_2=k_3=k_4=0$ and $s_2=s_3$ or $s_2=s_3+1$. Therefore

$$RM_2(G)=f(k_1,s_2,s_3)=(k_1+3)(3r-k_1-1)+3r-5-k_1+s_2s_3+3.$$

Since $s_2+s_3=3r-5-k_1$, then

$$g(k_1,s_2,s_3)=3r-5-k_1-s_2-s_3=0 \tag{6}$$

can be written. By using the Lagrange multipliers method, we get $s_2=3r-k_1-5$ and $s_3=3r-k_1-5$. Thus, $s_2+s_3=6r-4k_1-10$. From Equation 6, $s_2+s_3=3r-k_1-5$. And from these last two equalities $k_1=r-1$, $s_2=s_3=r-2$ can be found. Thus,

$$\begin{aligned} RM_2(G) &= (r+2)(r-2+r-2+4)+(r-1)^2+2 \\ &= r^2+2r+r^2-2r+1+2=2r^2+3. \end{aligned}$$

This last value is smaller than that of Proposition 2.12 (b). For $a\geq 5$ and $b=3$ the proof is similar.
Case 2: Let $a\geq 4$ and $b\geq 4$. By Proposition 2.15 the proof is clear. □

Now, we begin to investigate the maximum RM_2-index of the second class of bicyclic graphs with k pendent vertices.

Proposition 2.17. *Let $G\in DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$ with n vertices and $n-6$ pendent vertices. Then $G=\Psi=DC^0_{3,3}(k_1,0,0,s_1,0,0)$, with $k_1=s_1$ or $|k_1-s_1|=1$, has maximum RM_2 index among all the graphs belonging to $DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$.*

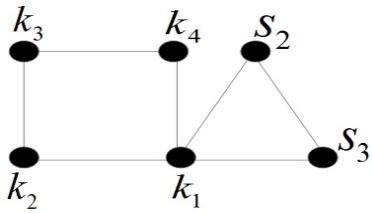


Figure 5: $G = DC_{4,3}(k_1,k_2,k_3,s_2,s_3)$ for the Case 1 of Theorem 2.16

Proof. From Proposition 2.2 (a),

$$\begin{aligned} RM_2(G) &= (k_1+2)(s_1+2)+(k_1+2)(k_2+k_3+2)+(k_2+1)(k_3+1) \\ &\quad +(s_1+2)(s_2+s_3+2)+(s_2+1)(s_3+1) \\ &= k_1s_1+k_1(k_2+k_3)+s_1(s_2+s_3)+k_2k_3+s_2s_3+14. \end{aligned}$$

$$f(k_1,k_2,k_3,s_1,s_2,s_3) = k_1s_1+k_1(k_2+k_3)+s_1(s_2+s_3)+k_2k_3+s_2s_3+14.$$

Since $k_1+k_2+k_3+s_1+s_2+s_3 = n-6$ then,

$$g(k_1,k_2,k_3,s_1,s_2,s_3) = n-k_1-k_2-k_3-s_1-s_2-s_3-6 = 0.$$

And by using the Lagrange multipliers method $k_2 = k_3 = 0$, $s_2 = s_3 = 0$, $k_1 = s_1$ or $|k_1 - s_1| = 1$. □

Corollary 2.18. *Let* $\Psi \in DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$. *Then*

$$RM_2(\Psi) = \left\lceil \frac{n-2}{2} \right\rceil \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Proof. Without loss of generality, from Proposition 2.17, let $k_1 = \left\lceil \frac{n-6}{2} \right\rceil$ and $s_1 = \left\lfloor \frac{n-6}{2} \right\rfloor$. Then with direct calculations the desired result is acquired. □

Theorem 2.19. *Let* $\Psi \in DC^0_{3,3}(k_1,k_2,k_3,s_1,s_2,s_3)$ *with n vertices and* $n-6$ *pendent vertices. Then* Ψ *has maximum* RM_2 *value among all the graphs belonging to* $DC^l_{a,b}$ *with n vertices and k pendent vertices.*

Proof. From Proposition 2.2, Lemma 2.14, Proposition 2.17 and Corollary 2.18 the desired result is acquired. □

Now, we begin to investigate the maximum RM_2-index of the third class of bicyclic graphs with k pendent vertices.

Proposition 2.20. *Let $G \in DC^0_{3+3}(k_1,k_2,k_3,s_2)$ with n vertices and $n-4$ pendent vertices. Then $G = Z = DC^0_{3+3}(k_1,k_2,k_3,s_2)$, with $|k_1-k_3| \leq 1$ and $\left|k_j-(k_2+s_2)\right| \leq 1$ ($j=1$ or $j=3$), has maximum RM_2 index among all the graphs belonging to $DC^0_{3+3}(k_1,k_2,k_3,s_2)$.*

Proof. From Proposition 2.3 (b),

$$RM_2(G) = (k_1+k_3+4)(k_2+s_2+2)+(k_1+2)(k_3+2).$$

If we put $k_2+s_2 = x$ then

$$RM_2(G) = f(k_1,k_3,x) = k_1x+k_3x+4x+k_1k_3+2k_1+2k_3+12.$$

Since $k_1+k_3+x = n-4$ then $g(k_1,k_3,x) = n-k_1-k_3-x-4 = 0$ can be written. And by using the Lagrange multipliers method we get $k_1 = k_3 = x = \frac{n-4}{3}$. Thus, $|k_1-k_3| \leq 1$ and $|k_{1,3}-(k_2+s_2)| \leq 1$. □

Proposition 2.21. *Let $G \in Z$ with n vertices and $n-4$ pendent vertices. Then*

$$\begin{array}{lll} (a) & RM_2(G) = 3r^2+4r+1 & for \quad n=3r, r\geq 2. \\ (b) & RM_2(G) = 3r^2+6r+3 & for \quad n=3r+1, r\geq 2. \\ (c) & RM_2(G) = 3r^2+8r+5 & for \quad n=3r+2, r\geq 2. \end{array}$$

Proof. By Proposition 2.3 and Proposition 2.20 we get the desired result. □

Theorem 2.22. *Let $Z \in DC^0_{3+3}(k_1,k_2,k_3,s_2)$ with n vertices and $n-4$ pendent vertices. Then Z has maximum RM_2 value among all the graphs belonging to DC^l_{a+b} with n vertices and k pendent vertices.*

Proof. From Proposition 2.3, Lemma 2.14, Proposition 2.20 and Proposition 2.21 the desired result is obtained. □

And now, from Theorem 2.16 , Theorem 2.19 and Theorem 2.22 we can state the following corollary.

Corollary 2.23. *Among all the bicyclic graphs with n vertices and k pendent vertices $Z = DC^0_{3+3}(k_1,k_2,k_3,s_2)$, with $|k_1-k_3| \leq 1$ and $\left|k_j-(k_2+s_2)\right| \leq 1$ ($j=1$ or $j=3$), has maximum RM_2 index.*

Acknowledgements

The author would like to thank the referee for his/her careful reading and making some valuable comments which have essentially improved the presentation of this paper.

REFERENCES

[1] A. Chang - F. Tian - A. Yu, *On the index of bicyclic graphs with perfect matchings*, Discrete Math. 283 (1-3) (2004), 51–59.

[2] H. Deng, *A unified approach to the extremal Zagreb indices for trees, unicyclic graphs and bicyclic graphs*, MATCH Commun. Math. Comput. Chem. 57 (3) (2007), 597–616.

[3] S. Li, *Comparing the Zagreb indices for connected bicyclic graphs*, MATCH Commun. Math. Comput. Chem. 62 (2009), 699–714.

[4] Q. Zhao - S. Li, *Sharp bounds for the Zagreb indices of bicyclic graphs with k-pendant vertices*, Discrete Appl. Math. 158 (17) (2010), 1953–1962.

[5] S. Chen - W. Liu, *Extremal modified Schultz index of bicyclic graphs*, MATCH Commun. Math. Comput. Chem. 64 (3) (2010), 767–782.

[6] L. Feng - W. Liu - K. Xu, *The hyper-Wiener index of bicyclic graphs*, Util. Math. 84 (2011), 97–104.

[7] X. F. Pan - N. N. Lv, *Sharp bounds of the zeroth-order general Randić index of bicyclic graphs with given pendent vertices*, Discrete Appl. Math. 159 (4) (2011), 240–245.

[8] L. Feng - W. Liu, *The maximal Gutman index of bicyclic graphs*, MATCH Commun. Math. Comput. Chem. 66 (2) (2011), 699–708.

[9] S. Li - O. Zhao, *Sharp upper bounds on Zagreb indices of bicyclic graphs with a given matching number*, Math. Comput. Modelling 54 (11-12) (2011), 2869–2879.

[10] K. Xu - I. Gutman, *The greatest Hosoya index of bicyclic graphs with given maximum degree*, MATCH Commun. Math. Comput. Chem. 66 (3) (2011), 795–824.

[11] K. Xu - K. C. Das, *Trees, unicyclic, and bicyclic graphs extremal with respect to multiplicative sum zagreb index*, MATCH Commun. Math. Comput. Chem. 68 (1) (2012), 257–272.

[12] Z. Du - B. Zhou, *On sum-connectivity index of bicyclic graphs*, Bull. Malays. Math. Sci. Soc. 35 (2012), 101–117.

[13] R.Sun - Z. Zhu - L.Tan, *On the Merrifield-Simmons index and Hosoya index of bicyclic graphs with a given girth*, Ars Combin. 103 (2012), 465–478.

[14] C. Du, *Minimum detour index of bicyclic graphs*, MATCH Commun. Math. Comput. Chem. 68 (1) (2012), 357–370.

[15] X. Qi, *Detour index of bicyclic graphs*, Util. Math. 90 (2013), 101–113.

[16] L.Feng - G. Yu - K. Xu - Z. Jiang, *A note on the Kirchhoff index of bicyclic graphs*, Ars Combin. 114 (2014), 33–40.

[17] B. Furtula - I. Gutman - S. Ediz, *On difference of Zagreb indices*, Discrete Appl. Math. 178 (2014), 83–88.

[18] S. Ediz, *Reduced second Zagreb index of unicyclic graphs*, (submitted).

[19] I. Gutman - N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1971), 535–538.

[20] T. Došlić - B. Furtula - A. Graovac - I. Gutman - S. Moradi - Z. Yarahmadi, *On vertex degree based molecular structure descriptors*, MATCH Commun. Math. Comput. Chem. 66 (2011), 613–626.

SÜLEYMAN EDIZ
Faculty of Education
Yüzüncü Yıl University
Van, Turkey
e-mail: `suleymanediz@yyu.edu.tr`