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CONNECTIONS BETWEEN VARIOUS SUBCLASSES OF PLANAR HARMONIC MAPPINGS INVOLVING GENERALIZED BESSEL FUNCTIONS

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The purpose of the present paper is to establish connections between various subclasses of harmonic univalent functions by applying certain convolution operator involving generalized Bessel functions of first kind. Precisely, we investigate such connections with Goodman-Rønning-type harmonic univalent functions in the open unit disc $\mathcal{U}$.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Now, we recall that the generalized Bessel function of the first kind $w = w_{p,b,c}$ is defined as the particular solution of the second-order linear homogenous differential equation

$$z^2\omega''(z) + bz\omega'(z) + \left[cz^2 - p^2 + (1-b)p\right]\omega(z) = 0, \tag{2}$$

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where $b,p,c \in \mathcal{C}$, which is a natural generalization of Bessel's equation. This function has the familiar representation

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma(p+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p}, \quad (z \in \mathcal{C}). \tag{3}$$

The differential equation (2) permits the study of Bessel, modified Bessel, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (2) are referred to as the generalized Bessel function of order p. The particular solution given by (3) is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathcal{U}$. It is worth mentioning that, in particular, when $b = c = 1$, we reobtain the Bessel function $\omega_{p,1,1} = J_p$, and for $c = -1, b = 1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2} \omega_{p,b,c}(z^{1/2}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, \ldots$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\ldots(a+n-1), & \text{if } n = 1,2,3,\ldots, \end{cases}$$

we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p+\frac{(b+1)}{2}\right)_n} \frac{z^n}{n!}, \tag{4}$$

where $p + (b+1)/2 \neq 0, -1, -2, \ldots$. This function is analytic on $\mathcal{C}$ and satisfies the second-order linear differential equation

$$4z^2 u''(z) + 2(2p+b+1) z u'(z) + czu(z) = 0.$$

For convenience throughout in the sequel, we use the following notations:

$$u_{p,b,c} = u_p, \quad k = p + \frac{b+1}{2}.$$

Let $\mathcal{H}$ be the family of all harmonic functions of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1, \quad (z \in \mathcal{U}), \tag{5}$$

are in the class A and then $f(z)$ is given by,

$$f(z)=z+\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}|b_n|z^n},\ (0\le|b_1|<1). \tag{6}$$

For complex parameters $c_1,k_1,c_2,k_2\ \ (k_1,k_2\neq 0,-1,-2,\ldots)$, we define the functions $\phi_1(z)=zu_{p_1}(z)$ and $\phi_2(z)=zu_{p_2}(z)$.

Corresponding to these functions, we introduce the following convolution operator

$$\Omega\equiv\Omega\begin{pmatrix}k_1, & c_1\\ k_2, & c_2\end{pmatrix}:\mathcal{H}\to\mathcal{H}$$

defined by

$$\Omega\begin{pmatrix}k_1, & c_1\\ k_2, & c_2\end{pmatrix}f=f*(\phi_1+\overline{\phi_2})=h(z)*\phi_1(z)+\overline{g(z)*\phi_2(z)}$$

for any function $f=h+\overline{g}$ in $\mathcal{H}$.

Letting

$$\Omega\begin{pmatrix}k_1, & c_1\\ k_2, & c_2\end{pmatrix}f(z)=H(z)+\overline{G(z)},$$

where

$$H(z)=z+\sum_{n=2}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}A_nz^n,\quad G(z)=\sum_{n=1}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}B_nz^n. \tag{7}$$

Denote by $S_{\mathcal{H}}$ the subclass of $\mathcal{H}$ that are univalent and sense-preserving in $\mathcal{U}$. Note that $\frac{f-\overline{B_1f}}{1-|B_1|^2}\in S_{\mathcal{H}}$ whenever $f\in S_{\mathcal{H}}$. We also let the subclass $S_{\mathcal{H}}^0$ of $S_{\mathcal{H}}$, $S_{\mathcal{H}}^0=\{f=h+\overline{g}\in S_{\mathcal{H}}:g'(0)=B_1=0\}$. The classes $S_{\mathcal{H}}^0$ and $S_{\mathcal{H}}$ were first studied in [10]. Also, we let $K_{\mathcal{H}}^0$, $S_{\mathcal{H}}^{*,0}$ and $C_{\mathcal{H}}^0$ denote the subclasses of $S_{\mathcal{H}}^0$ of harmonic functions which are, respectively, convex, starlike and close-to-convex in $\mathcal{U}$. Also, let $T_{\mathcal{H}}^0$ be the class of sense-preserving, typically real harmonic functions $f=h+\overline{g}$ in $\mathcal{H}$. For definitions and properties of these classes, one may refer to ([1], [10] or [11]).

Motivated by the earlier works on the subject of harmonic functions, in this paper a new subclass of harmonic univalent functions we obtain a sufficient coefficient condition for functions $f\in\mathcal{H}$ given by (6) and also shown that this coefficient condition is necessary for functions $f\in\mathcal{T}_{\mathcal{H}}$. Further, an attempt has been made to study inclusion relations making use of Bessels functions.

2. The class $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$

For $0 \leq \lambda \leq 1, 0 < \gamma \leq 1$ we let, $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ a new subclass of $\mathcal{H}$, consist of all functions of the form (6) satisfying the condition

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-e^{i\alpha}\right)>\gamma \tag{8}$$

where

$$z'=\frac{\partial}{\partial\theta}\left(z=re^{i\theta}\right),\ f'(z)=\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right).$$

Equivalently, we have,

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))'-\overline{z(g(z))'}}{(1-\lambda)z+\lambda[h(z)+\overline{g(z)}]}-e^{i\alpha}\right)>\gamma,(z\in\mathcal{U}). \tag{9}$$

Further, for $\lambda=0$, we define a new class $\mathcal{G}_{\mathcal{H}}(0,\alpha,\gamma)\equiv\mathcal{N}_{\mathcal{H}}(\alpha,\gamma)$ satisfying the analytic criteria

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{z'}-e^{i\alpha}\right)>\gamma,\quad 0<\gamma\leq 1. \tag{10}$$

Equivalently,

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))'-\overline{z(g(z))'}}{z}-e^{i\alpha}\right)>\gamma,\ (z\in\mathcal{U}). \tag{11}$$

Also let, $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)=\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)\cap\mathcal{T}_{\mathcal{H}}$ and $\mathcal{NV}_{\mathcal{H}}(\alpha,\gamma)=\mathcal{N}_{\mathcal{H}}(\alpha,\gamma)\cap\mathcal{T}_{\mathcal{H}}$ where $\mathcal{T}_{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f=h+\overline{g}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}|a_n|z^n-\overline{\sum_{n=1}^{\infty}|b_n|z^n},\ (0\leq b_1<1). \tag{12}$$

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Theorem 2.1. *Let $f=h+\overline{g}$ be given by (6). If*

$$\sum_{n=2}^{\infty}\frac{2n-\lambda(1+\gamma)}{1-\gamma}|a_n|+\sum_{n=1}^{\infty}\frac{2n+\lambda(1+\gamma)}{1-\gamma}|b_n|\leq 1, \tag{13}$$

where $a_1=1$ and $0<\gamma\leq 1$, then $f\in\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Proof. We first show that if the inequality (13) holds for the coefficients of $f = h + \overline{g}$, then the required condition (8) is satisfied. Now, we can write

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)} - e^{i\alpha}\right) = \Re\left(\frac{A(z)}{B(z)}\right),$$

where

$$\begin{aligned} A(z) &= (1+e^{i\alpha})zf'(z) - e^{i\alpha}[(1-\lambda)z+\lambda f(z)] \\ &= z + \sum_{n=2}^{\infty}\left[n+(n-\lambda)e^{i\alpha}\right]a_n z^n - \sum_{n=1}^{\infty}\left[n+(n+\lambda)e^{i\alpha}\right]b_n z^n \end{aligned}$$

and

$$B(z) = (1-\lambda)z + \lambda f(z) = z + \lambda\sum_{n=2}^{\infty} a_n z^n - \lambda\sum_{n=1}^{\infty} b_n z^n.$$

In view of the simple assertion that $\Re\ (w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{14}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (14), we get

$$\begin{aligned} &|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \\ &\geq (2-\gamma)|z| - \sum_{n=2}^{\infty}(2n-\lambda\gamma)|a_n|\ |z|^n - \sum_{n=1}^{\infty}(2n+\lambda\gamma)|b_n|\ |z|^n \\ &-\gamma|z| - \sum_{n=2}^{\infty}(2n-2\lambda-\lambda\gamma)|a_n|\ |z|^n - \sum_{n=1}^{\infty}(2n+2\lambda+\lambda\gamma)|b_n|\ |z|^n. \\ &\geq 2(1-\gamma)|z|\left(1-\left(\sum_{n=2}^{\infty}\left[\frac{2n-\lambda(1+\gamma)}{1-\gamma}|a_n| + \frac{2n+\lambda(1+\gamma)}{1-\gamma}|b_n|\right]\right)\right) \\ &\geq 0 \end{aligned}$$

by virtue of the inequality (14). This implies that $f \in \mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$. □

Theorem 2.2. *For $a_1 = 1$ and $0 \leq \gamma < 1$, $f = h + \overline{g} \in \mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{2n-\lambda(1+\gamma)}{1-\gamma}|a_n| + \sum_{n=1}^{\infty}\frac{2n+\lambda(1+\gamma)}{1-\gamma}|b_n| \leq 1. \tag{15}$$

Proof. Since $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma) \subset \mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (12), we notice that the condition

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))' - \overline{z(g(z))'}}{(1-\lambda)z+\lambda[h(z)+\overline{g(z)}]} - e^{i\alpha}\right) > \gamma, (z \in \mathcal{U}). \tag{16}$$

Equivalently,

$$\Re\left(\frac{(1-\gamma)z-\sum\limits_{n=2}^{\infty}[2n-\lambda(1+\gamma)]|a_n|z^n-\sum\limits_{n=1}^{\infty}[2n+\lambda(1+\gamma)]\overline{b}_n\overline{z}^n}{z+\lambda\sum\limits_{n=2}^{\infty}|a_n|z^n-\lambda\sum\limits_{n=1}^{\infty}\overline{b}_n\overline{z}^n}\right)\geq 0.$$

The above required condition must hold for all values of z in $\mathcal{U}$. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}[2n-\lambda(1+\gamma)]|a_n|r^{n-1}-\sum\limits_{n=1}^{\infty}[2n+\lambda(1+\gamma)]|b_n|r^{n-1}}{1+\lambda\sum\limits_{n=2}^{\infty}|a_n|r^{n-1}-\lambda\sum\limits_{n=1}^{\infty}|b_n|r^{n-1}}\geq 0. \quad (17)$$

If the condition (15) does not hold, then the numerator in (17) is negative for r sufficiently close to 1. Hence, there exist $z_0=r_0$ in (0,1) for which the quotient of (17) is negative. This contradicts the required condition for $f\in\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$. This completes the proof of the theorem. □

Proceeding as in Theorem 2.1 and Theorem 2.2 we state the following necessary and sufficient conditions for functions $f\in\mathcal{NV}_{\mathcal{H}}(\alpha,\gamma)$ without proof.

Theorem 2.3. *Let $f=h+\overline{g}$ be given by (6). If*

$$\sum_{n=2}^{\infty}\frac{2n}{1-\gamma}|a_n|+\sum_{n=1}^{\infty}\frac{2n}{1-\gamma}|b_n|\leq 1, \quad (18)$$

where $a_1=1$ and $0<\gamma\leq 1$, then $f\in\mathcal{NV}_{\mathcal{H}}(\alpha,\gamma)$.

Further we state the following Remarks:

Remark 2.4. In [24], it is also shown that $f=h+\overline{g}$ given by (6) is in the family $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$, if and only if the coefficient condition given in Theorem 2.2 holds. Moreover, if $f\in\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$, then

$$|A_n|\leq\frac{1-\gamma}{2n-\lambda(1+\gamma)}, \qquad n\geq 2,$$

$$|B_n|\leq\frac{1-\gamma}{2n+\lambda(1+\gamma)}, \qquad n\geq 1.$$

Remark 2.5. If $f\in\mathcal{NV}_{\mathcal{H}}(\gamma)$, then $|A_n|\leq\frac{1-\gamma}{2n}$ and $|B_n|\leq\frac{1-\gamma}{2n}$, $n\geq 2$.

3. Applications to Bessel function

The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [5]- [8] and [13]). Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [2], [3], [9], [12], [14]-[23] and [25]-[26] and by work of Baricz [5]-[8]), we establish a number of connections between the classes $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$, $K^0_{\mathcal{H}}$, $S^{*,0}_{\mathcal{H}}$, $C^0_{\mathcal{H}}$ and $\mathcal{N}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ by applying the convolution operator Ω.

Throughout this paper, we will frequently use the notation

$$\Omega(f)=\Omega\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f.$$

Let

$$\phi(z)=zu_p(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!}z^n$$

$$\phi(1)=u_p(1)=1+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!} \tag{19}$$

and

$$\phi'(z)=zu_p'(z)+u_p(z)=1+\sum_{n=2}^{\infty}n\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!}z^{n-1}$$

$$\phi'(1)=u_p'(1)+u_p(1)-1=\sum_{n=2}^{\infty}n\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!} \tag{20}$$

$$\phi''(z)=zu_p''(z)+2u_p'(z)=\sum_{n=2}^{\infty}n(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!}z^{n-2}$$

$$\phi''(1)=u_p''(1)+2u_p'(1)=\sum_{n=2}^{\infty}n(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!} \tag{21}$$

$$\phi'''(z)=zu_p'''(z)+3u_p''(z)=\sum_{n=2}^{\infty}n(n-1)(n-2)\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!}z^{n-3}$$

$$\phi'''(1)=u_p'''(1)+3u_p''(1)=\sum_{n=2}^{\infty}n(n-1)(n-2)\frac{(-c/4)^{n-1}}{(k)_{n-1}\,(n-1)!} \tag{22}$$

In order to establish connections between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, we need the following results :

Lemma 3.1 ([10], [11]). *If $f=h+\overline{g}\in K^0_{\mathcal{H}}$ where h and g are given by* (5) *with $B_1=0$, then*

$$|A_n|\le\frac{n+1}{2},\ \ |B_n|\le\frac{n-1}{2}.$$

Lemma 3.2 ([8]). *If $b,p,c \in \mathcal{C}$ and $k \neq 0,-1,-2,\dots$ then the function u_p satisfies the recursive relation $4ku'_p(z) = -cu_{p+1}(z)$ for all $z \in \mathcal{C}$.*

Theorem 3.3. *Let $c_1,c_2 < 0$, $k_1,k_2 > 0$, $(k_1,k_2 \neq 0,-1,-2,\dots)$. If for some γ $(0 \leq \gamma < 1)$ the inequality*

$$2u''_{p_1}(1)+[8-\lambda(1+\gamma)]u'_{p_1}(1)+[4-2\lambda(1+\gamma)]u_{p_1}(1)+2u''_{p_2}(1)$$
$$+[4+\lambda(1+\gamma)]u'_{p_2}(1) \leq 2[3-(\lambda+\gamma+\lambda\gamma)]$$

is satisfied then $\Omega\left(K^0_{\mathcal{H}}\right) \subset \mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Proof. Let $f = h+\overline{g} \in K^0_{\mathcal{H}}$ where h and g are of the form (5) with $B_1 = 0$. We need to show that $\Omega(f) = H+\overline{G} \in \mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$, where H and G defined by (7) with $B_1 = 0$ are analytic functions in $\mathcal{U}$.

In view of Theorem 2.1, we need to prove that $P_1 \leq 1-\gamma$, where

$$P_1 = \sum_{n=2}^{\infty}[2n-\lambda(1+\gamma)]\left|\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}A_n\right|$$
$$+\sum_{n=2}^{\infty}[2n+\lambda(1+\gamma)]\left|\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}B_n\right|.$$

In view of Lemma 3.1, we have

$$\begin{aligned}P_1 &\leq \frac{1}{2}\left[\sum_{n=2}^{\infty}[(n+1)(2n-\lambda(1+\gamma))]\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.\\
&\quad\left.+\sum_{n=2}^{\infty}[(n-1)(2n+\lambda(1+\gamma))]\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right]\\
&= \frac{1}{2}\left[\sum_{n=2}^{\infty}\{2n^2-n[\lambda(1+\gamma)-2]-\lambda(1+\gamma)\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.\\
&\quad\left.+\sum_{n=2}^{\infty}\left\{2n^2+n[\lambda(1+\gamma)-2]-\lambda(1+\gamma)\right\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right]\end{aligned}$$

Writing $n^2 = n(n-1)+n$, we get

$$\begin{aligned}
&= \frac{1}{2}\left[\sum_{n=2}^{\infty}\{2n(n-1)+n[4-\lambda(1+\gamma)]-\lambda(1+\gamma)\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.\\
&\quad\left.+\sum_{n=2}^{\infty}\{2n(n-1)+n[\lambda(1+\gamma)]-\lambda(1+\gamma)\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!}\right]\\
&= \frac{1}{2}\left[2\sum_{n=2}^{\infty}\frac{n(n-1)(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}+[4-\lambda(1+\gamma)]\sum_{n=2}^{\infty}\frac{n(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.\\
&\quad-\lambda(1+\gamma)\sum_{n=2}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}+2\sum_{n=2}^{\infty}\frac{n(n-1)(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\\
&\quad\left.+\lambda(1+\gamma)\sum_{n=2}^{\infty}\frac{n(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}-\lambda(1+\gamma)\sum_{n=2}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right]\\
&= \frac{1}{2}\left[2\phi_1''(1)+[4-\lambda(1+\gamma)]\phi_1'(1)-[\lambda(1+\gamma)]\phi_1(1)+2\phi_2''(1)\right.\\
&\quad\left.+[\lambda(1+\gamma)]\phi_2'(1)-[\lambda(1+\gamma)]\phi_2(1)\right]\\
&= \frac{1}{2}\left[2(u_{p_1}''(1)+2u_{p_1}'(1))+[4-\lambda(1+\gamma)](u_{p_1}'+u_{p_1}(1)-1)\right.\\
&\quad-[\lambda(1+\gamma)](u_{p_1}(1)-1)+2(u_{p_2}''(1)+2u_{p_2}'(1))\\
&\quad\left.+[\lambda(1+\gamma)](u_{p_2}'+u_{p_2}(1)-1)-[\lambda(1+\gamma)](u_{p_2}(1)-1)\right]\\
&= \frac{1}{2}\left[2u_{p_1}''(1)+[8-\lambda(1+\gamma)]u_{p_1}'(1)+[4-2\lambda(1+\gamma)]u_{p_1}(1)\right.\\
&\quad\left.+2u_{p_2}''(1)+[4+\lambda(1+\gamma)]u_{p_2}'-[4-2\lambda(1+\gamma)]\right].
\end{aligned}$$

Now $P_1 \leq 1-\gamma$ follows from the given condition, which completes the proof. □

Analogous to Theorem 3.3, we next find conditions of the classes $S_{\mathcal{H}}^{*,0}$, $C_{\mathcal{H}}^{0}$ with $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$. However we first need the following result which may be found in [10], [11].

Lemma 3.4. *If $f = h+\overline{g} \in S_{\mathcal{H}}^{*,0}$ or $C_{\mathcal{H}}^{0}$ where h and g are given by* (5) *with $B_1 = 0$, then*

$$|A_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |B_n| \leq \frac{(2n-1)(n-1)}{6}.$$

Theorem 3.5. *Let* $c_1, c_2 < 0,\ k_1, k_2 > 0,\ (k_1, k_2 \neq 0, -1, -2, \dots)$. *If for some* $\gamma\,(0 \leq \gamma < 1)$ *the inequality*

$$\begin{aligned} &4u'''_{p_1}(1) + 2[15 - \lambda(1+\gamma)]u''_{p_1}(1) + [48 - 9\lambda(1+\gamma)]u'_{p_1}(1) \\ &+ 6[2 - \lambda(1+\gamma)]u_{p_1}(1) + 4u'''_{p_2}(1) + 2[9 + \lambda(1+\gamma)]u''_{p_2}(1) \\ &+ 3[4 + \lambda(1+\gamma)]u'_{p_2}(1) \leq 6[3 - (\lambda + \gamma + \lambda\gamma)] \end{aligned} \tag{23}$$

is satisfied, then

$$\Omega(S^{*,0}_{\mathcal{H}}) \subset \mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma) \text{ and } \Omega(C^0_{\mathcal{H}}) \subset \mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma).$$

Proof. Let $f = h + \overline{g} \in S^{*,0}_{\mathcal{H}}\ (C^0_{\mathcal{H}})$ where h and g are given by (5) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{G} \in \mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma)$, where H and G defined by (7) with $B_1 = 0$ are analytic functions in $\mathcal{U}$. In view of Theorem 2.1, it is enough to show that $P_1 \leq 1 - \gamma$, where

$$\begin{aligned} P_1 = \sum_{n=2}^{\infty} [2n - \lambda(1+\gamma)] \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} |A_n| \\ + \sum_{n=2}^{\infty} [2n + \lambda(1+\gamma)] \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} |B_n|. \end{aligned}$$

In view of Lemma 3.4, we have

$$\begin{aligned} P_1 &\leq \frac{1}{6}\left[\sum_{n=2}^{\infty}(2n+1)(n+1)(2n - \lambda(1+\gamma))\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right. \\ &\qquad \left. + \sum_{n=2}^{\infty}(2n-1)(n-1)(2n + \lambda(1+\gamma))\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right] \\ &= \frac{1}{6}\sum_{n=2}^{\infty}\left[4n^3 + 2[3 - \lambda(1+\gamma)]n^2\right. \\ &+ [2 - 3\lambda(1+\gamma)]n - \lambda(1+\gamma)\Big] \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\ &+ \frac{1}{6}\sum_{n=2}^{\infty}\left[4n^3 - 2[3 - \lambda(1+\gamma)]n^2\right. \\ &+ [2 - 3\lambda(1+\gamma)]n + \lambda(1+\gamma)\Big] \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!} \end{aligned}$$

Writing $n^3 = n(n-1)(n-2)+3n(n-1)+n$ and $n^2 = n(n-1)+n$, we have

$$
\begin{aligned}
P_1 = \frac{1}{6}\sum_{n=2}^{\infty} 4n(n-1)(n-2) + 2[9-\lambda(1+\gamma)]n(n-1)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\
+\frac{1}{6}\sum_{n=2}^{\infty}[12-5\lambda(1+\gamma)]n - \lambda(1+\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\
+\frac{1}{6}\sum_{n=2}^{\infty} 4n(n-1)(n-2)+2n(n-1)[3+\lambda(1+\gamma)]\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!} \\
-\frac{1}{6}\sum_{n=2}^{\infty}[\lambda(1+\gamma)]n + \lambda(1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!} = \\
\frac{1}{6}\Bigg[4\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} + 2[9-\lambda(1+\gamma)]\sum_{n=0}^{\infty}\frac{n(n-1)(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\
+[12-5\lambda(1+\gamma)]\sum_{n=0}^{\infty}\frac{n(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} - \lambda(1+\gamma)\sum_{n=0}^{\infty}\frac{n(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\
+4\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} + 2[3+\lambda(1+\gamma)]\sum_{n=0}^{\infty}\frac{n(n-1)(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} \\
-\lambda(1+\gamma)\sum_{n=0}^{\infty}\frac{n(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} + \lambda(1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\Bigg] \\
= \frac{1}{6}\left[4\phi_1'''(1) + 2[9-\lambda(1+\gamma)]\phi_1''(1) + [12-5\lambda(1+\gamma)]\phi_1'(1)\right. \\
-[\lambda(1+\gamma)]\phi_1(1) + 4\phi_2'''(1) + 2[3+\lambda(1+\gamma)]\phi_2''(1) \\
\left. -[\lambda(1+\gamma)]\phi_2'(1) + [\lambda(1+\gamma)]\phi_2(1)\right]. \\
= \frac{1}{6}\left[4u_{p_1}'''(1) + 2[15-\lambda(1+\gamma)]u_{p_1}''(1) + [48-9\lambda(1+\gamma)]u_{p_1}'(1)\right. \\
+6[2-\lambda(1+\gamma)]u_{p_1}(1) + 4u_{p_2}'''(1) + 2[9+\lambda(1+\gamma)]u_{p_2}''(1) \\
\left. +3[4+\lambda(1+\gamma)]u_{p_2}'(1) - 6[2-\lambda(1+\gamma)]\right].
\end{aligned}
$$

Now $P_1 \le 1-\gamma$ follows from the given condition. □

In order to determine connection between $\mathcal{NV}_{\mathcal{H}}(\gamma)$ and $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$, we need the following result:

Lemma 3.6. *If $c<0$ and $k>1$, then*

$$\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(k)_n(n+1)!} = \frac{-4(k-1)}{c}\left[u_{p-1}(1)-1\right].$$

Proof. We can write

$$\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(k)_n(n+1)!}=\frac{(k-1)}{(-c/4)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k-1)_{n+1}(n+1)!}$$
$$=\frac{-4(k-1)}{c}\left[u_{p-1}(1)-1\right]. \qquad \square$$

Theorem 3.7. *If* $c_1,c_2<0$, $k_1,k_2>1$. *If for some* β $(0\le\beta<1)$ *and* γ $(0\le\gamma<1)$ *the inequality*

$$(1-\beta)\left[\{u_{p_1}(1)-1\}+\lambda(1+\gamma)\frac{2(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1-\frac{(-c_1/4)}{k_1-1}\right]\right.$$
$$\left.+u_{p_2}(1)-\lambda(1+\gamma)\frac{2(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\right]\le 1-\gamma$$

is satisfied then

$$\Omega(\mathcal{NV}_{\mathcal{H}}(\beta))\subset\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma).$$

Proof. Let $f=h+\overline{g}\in\mathcal{NV}_{\mathcal{H}}(\beta)$ where h and g are given by (5). In view of Theorem 2.1, it is enough to show that $P_2\le 1-\gamma$, where

$$P_2=\sum_{n=2}^{\infty}\left[2n-\lambda(1+\gamma)\right]\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}|A_n|$$
$$+\sum_{n=1}^{\infty}\left[2n+\lambda(1+\gamma)\right]\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}|B_n|.$$

Using Remark 2.5, we have

$$P_2\le(1-\beta)\left[\sum_{n=2}^{\infty}\left(1-\frac{\lambda(1+\gamma)}{2n}\right)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.$$
$$\left.+\sum_{n=1}^{\infty}\left(1+\frac{\lambda(1+\gamma)}{2n}\right)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right]$$
$$=(1-\beta)\left[\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}-\frac{\lambda(1+\gamma)}{2}\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}\right.$$
$$\left.+\sum_{n=0}^{\infty}\frac{(-c_2/4)^n}{(k_2)_n n!}+\frac{(1+\gamma)}{2}\sum_{n=0}^{\infty}\frac{(-c_2/4)^n}{(k_2)_n(n+1)!}\right]=$$

$$= (1-\beta)\left[\{u_{p_1}(1)-1\} - \frac{(1+\gamma)}{2}\frac{(k_1-1)}{(-c_1/4)}\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+2}}{(k_1-1)_{n+2}(n+2)!}\right.$$
$$\left. + u_{p_2}(1) + \frac{(1+\gamma)}{2}\frac{(k_2-1)}{(-c_2/4)}\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2-1)_{n+1}(n+1)!}\right]$$
$$= (1-\beta)\left[\{u_{p_1}(1)-1\} + \frac{(1+\gamma)}{2}\frac{2(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1-\frac{(-c_1/4)}{k_1-1}\right]\right.$$
$$\left. + u_{p_2}(1) - \frac{(1+\gamma)}{2}\frac{2(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\right] \leq 1-\gamma,$$

by the given hypothesis. □

In the next theorem, we establish connections between $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ and $\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Theorem 3.8. *Let $c_1,c_2<0$, $k_1,k_2>0$. If for some γ $(0\leq\gamma<1)$ the inequality*

$$u_{p_1}+u_{p_2}\leq 2 \tag{24}$$

is satisfied, then $\Omega(\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma))\subset\mathcal{G}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Proof. Making use of Theorem 2.1, we only need to prove that $P_2\leq 1-\gamma$. Using Remark 2.4, it follows that

$$P_2 = \sum_{n=2}^{\infty}[2n-\lambda(1+\gamma)]\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}|A_n|$$
$$+\sum_{n=1}^{\infty}[2n+\lambda(1+\gamma)]\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}|B_n|$$
$$\leq (1-\gamma)\left[\sum_{n=2}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}+\sum_{n=1}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\right]$$
$$= (1-\gamma)\left[\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}+\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n}}{(k_2)_{n}n!}\right]$$
$$= (1-\gamma)\left[u_{p_1}(1)-1+u_{p_2}(1)\right]$$

by using the given condition (24), we have $P_2\leq 1-\gamma$, which completes the proof. □

In next theorem, we present conditions on the parameters k_1,k_2,c_1,c_2 and obtain a characterization for operator Ω which maps $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ on to itself.

Theorem 3.9. *Let* $c_1, c_2 < 0$*,* $k_1, k_2 > 0$ $(k_1, k_2 \neq 0, -1, -2, \ldots)$ *and* γ $(0 \leq \gamma < 1)$. *Then*

$$\Omega\left(\mathcal{GV}_{\mathcal{H}}(\lambda, \alpha, \gamma)\right) \subset \mathcal{GV}_{\mathcal{H}}(\lambda, \alpha, \gamma),$$

if and only if,

$$u_{p_1}(1) + u_{p_2}(1) \leq 2$$

Proof. The proof of the above theorem is similar to that of Theorem 3.8. Therefore we omit the details involved. □

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