

LE MATEMATICHE

Vol. LXX (2015) – Fasc. II, pp. 173–190

doi: 10.4418/2015.70.2.12

## SOME FRACTIONAL CALCULUS RESULTS ASSOCIATED WITH THE $I$ -FUNCTION

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The effect of Marichev-Saigo-Maeda (MSM) fractional operators involving third Appell function on the  $I$ -function is studied. It is shown that the order of the  $I$ -function increases on application of these operators to the power multiple of the  $I$ -function. The Caputo-type MSM fractional derivatives are introduced and studied for the  $I$ -function. As special cases, the corresponding assertions for Saigo and Erdélyi-Kober fractional operators are also presented. The results obtained in this paper generalize several known results obtained recently in the literature.

### 1. Introduction

The Fox's  $H$ -function [16] has found numerous applications in the fields of statistical distributions, fractional calculus, statistical mechanics and some areas of engineering. Inayat-Hussain [14] generalized this function to the  $\bar{H}$ -function. Generalized Riemann zeta functions, polylogarithm functions of complex order and the exact partition functions of the Gaussian model in statistical mechanics, among others, are the special cases of the  $\bar{H}$ -function, which are not particular cases of the  $H$ -function. Rathie [15] introduced the  $I$ -function, which includes

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Entrato in redazione: 2 settembre 2014

*AMS 2010 Subject Classification:* 26A33, 44A20, 33C60, 33C65.

*Keywords:*  $I$ -function, Marichev-Saigo-Maeda fractional operators, Appell function.

The research of K. K. Kataria was supported by UGC, Govt. of India.

the  $\bar{H}$ -function as a special case. The  $I$ -function is represented by the following Mellin-Barnes type contour integral

$$I_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_C \chi(s) z^{-s} ds, \quad (1)$$

where

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma^{\beta_j}(b_j + B_j s) \prod_{i=1}^n \Gamma^{\alpha_i}(1 - a_i - A_i s)}{\prod_{j=m+1}^q \Gamma^{\beta_j}(1 - b_j - B_j s) \prod_{i=n+1}^p \Gamma^{\alpha_i}(a_i + A_i s)}. \quad (2)$$

In the above definition,  $z \neq 0, m, n, p, q$  are integers satisfying  $0 \leq m \leq q$  and  $0 \leq n \leq p$  with  $\alpha_i, A_i > 0$  for  $i = 1, 2, \dots, p$  and  $\beta_j, B_j > 0$  for  $j = 1, 2, \dots, q$ . Also,  $a_i$ 's and  $b_j$ 's are complex numbers such that no singularity of  $\Gamma^{\beta_j}(b_j + B_j s)$  coincides with any singularity of  $\Gamma^{\alpha_i}(1 - a_i - A_i s)$ . Here,  $\Gamma^k(z)$  denotes the generalized  $k$ -gamma function and an empty product is to be interpreted as unity. The path  $C$  of integration, in the complex  $s$ -plane runs from  $c - i\infty$  to  $c + i\infty$  for some real number  $c$  such that the singularity of  $\Gamma^{\beta_j}(b_j + B_j s)$  lie entirely to the left of the path and the singularity of  $\Gamma^{\alpha_i}(1 - a_i - A_i s)$  lie entirely to the right of the path, that is, (1) is a Mellin-Barnes integral [4]. For other possible contours  $C$ , convergence conditions, various series representations and for some elementary properties of the  $I$ -function, one can refer [15]. The  $\bar{H}$ -function follows as a particular case when  $\alpha_i = 1$  for  $i = n + 1, \dots, p$  and  $\beta_j = 1$  for  $j = 1, 2, \dots, m$  in (1). Recently, the extension of the  $I$ -function to several complex variables (see [12], [13]) and its applications in wireless communication were studied by several authors (see [1], [2], [11]).

In this paper, we derive some fractional calculus results associated with the  $I$ -function. For  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$  and  $x > 0$  with  $\text{Re}(\gamma) > 0$ , the left- and right-hand sided Marichev-Saigo-Maeda (MSM) fractional integral operators [20] associated with third Appell function are defined by

$$\begin{aligned} & \left( \mathcal{I}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f \right) (x) \\ &= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3 \left( \alpha, \alpha', \beta, \beta', \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt \end{aligned} \quad (3)$$

and

$$\begin{aligned} & \left( \mathcal{I}_-^{\alpha, \alpha', \beta, \beta', \gamma} f \right) (x) \\ &= \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3 \left( \alpha, \alpha', \beta, \beta', \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) dt, \end{aligned} \quad (4)$$

respectively. The corresponding fractional differential operators [3] have their respective forms as

$$\left(\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(\mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m} f\right)(x) \tag{5}$$

and

$$\left(\mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^m \left(\mathcal{I}_-^{-\alpha',-\alpha,-\beta',-\beta+m,-\gamma+m} f\right)(x), \tag{6}$$

where  $m = [\text{Re}(\gamma)] + 1$  and  $[\text{Re}(\gamma)]$  denotes the integer part of  $\text{Re}(\gamma)$ . The third Appell function  $F_3$  (also known as Horn function [18]), is defined by

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x; y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n x^m y^n}{(\gamma)_{m+n} m! n!},$$

such that  $\max\{|x|, |y|\} < 1$ . Here  $(z)_n$  is the Pochhammer symbol, defined for  $z \in \mathbb{C}$  by

$$(z)_n = \begin{cases} 1 & \text{if } n = 0, \\ z(z+1)(z+2)\dots(z+(n-1)) & \text{if } n \in \mathbb{N}. \end{cases}$$

Saigo [21] introduced the fractional integral and differential operators involving Gauss hypergeometric function  ${}_2F_1$  as the kernel. For  $\alpha, \beta, \gamma \in \mathbb{C}$  and  $x > 0$  with  $\text{Re}(\alpha) > 0$ , the left- and right-hand sided Saigo fractional integral operators are defined by

$$\left(\mathcal{I}_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma, \alpha; 1-\frac{t}{x}\right) f(t) dt \tag{7}$$

and

$$\left(\mathcal{I}_-^{\alpha,\beta,\gamma} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma, \alpha; 1-\frac{x}{t}\right) f(t) dt, \tag{8}$$

respectively. The corresponding fractional differential operators are

$$\left(\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(\mathcal{I}_{0+}^{-\alpha+m,-\beta-m,\alpha+\gamma-m} f\right)(x) \tag{9}$$

and

$$\left(\mathcal{D}_-^{\alpha,\beta,\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^m \left(\mathcal{I}_-^{-\alpha+m,-\beta-m,\alpha+\gamma} f\right)(x), \tag{10}$$

where now  $m = [\text{Re}(\alpha)] + 1$ . For  $\beta = -\alpha$  and  $\beta = 0$  in (7)-(10), we get the corresponding Riemann-Liouville and Erdélyi-Kober fractional operators respectively (for definition see [19]). The Gauss hypergeometric function is related to third Appell function as

$$F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta, \gamma; x; y) = {}_2F_1(\alpha, \beta, \gamma; x + y - xy).$$

The MSM fractional operators (3)-(6) are connected to Saigo operators (7)-(10) by

$$\begin{aligned} (\mathcal{I}_{0+}^{\alpha, 0, \beta, \beta', \gamma} f)(x) &= (\mathcal{I}_{0+}^{\gamma, \alpha - \gamma, -\beta} f)(x), \\ (\mathcal{I}_{-}^{\alpha, 0, \beta, \beta', \gamma} f)(x) &= (\mathcal{I}_{-}^{\gamma, \alpha - \gamma, -\beta} f)(x), \end{aligned} \tag{11}$$

and

$$\begin{aligned} (\mathcal{D}_{0+}^{0, \alpha', \beta, \beta', \gamma} f)(x) &= (\mathcal{D}_{0+}^{\gamma, \alpha' - \gamma, \beta' - \gamma} f)(x), \\ (\mathcal{D}_{-}^{0, \alpha', \beta, \beta', \gamma} f)(x) &= (\mathcal{D}_{-}^{\gamma, \alpha' - \gamma, \beta' - \gamma} f)(x). \end{aligned} \tag{12}$$

In Section 2, some preliminary results are stated, which will be used in the proofs of subsequent theorems.

## 2. Preliminaries

The following notations are used throughout the paper:

$$\begin{aligned} \mu &= \sum_{j=1}^q \beta_j B_j - \sum_{i=1}^p \alpha_i A_i, \\ \Omega &= \sum_{i=1}^p \left(\frac{1}{2} - \text{Re}(a_i)\right) \alpha_i - \sum_{j=1}^q \left(\frac{1}{2} - \text{Re}(b_j)\right) \beta_j, \\ \Delta &= \sum_{j=1}^m \beta_j B_j - \sum_{j=m+1}^q \beta_j B_j + \sum_{i=1}^n \alpha_i A_i - \sum_{i=n+1}^p \alpha_i A_i, \end{aligned}$$

where  $m, n, a_i, A_i, \alpha_i$  and  $b_j, B_j, \beta_j$  appear in the definition of the  $I$ -function. Moreover, the  $I$ -function is analytic if  $\mu \geq 0$  and the integral in (1) converges if  $|\arg(z)| < \Delta \frac{\pi}{2}$  and  $\Delta > 0$ . Also, if  $|\arg(z)| = \Delta \frac{\pi}{2}$  with  $\Delta \geq 0$ , then it converges absolutely under the following conditions (see [15]): (i)  $\mu = 0$  and  $\Omega < -1$ , (ii)  $|\mu| \neq 0$  with  $s = \sigma + it$ , where  $\sigma, t \in \mathbb{R}$  and are such that for  $|t| \rightarrow \infty$  we have  $\Omega + \sigma\mu < -1$ .

The following are well known results for MSM integral operators of power functions (see [3]).

**Lemma 2.1.** Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$  such that  $\text{Re}(\gamma) > 0$ .

(a) If  $\text{Re}(\rho) > \max\{0, \text{Re}(\alpha' - \beta'), \text{Re}(\alpha + \alpha' + \beta - \gamma)\}$ , then

$$\begin{aligned} & \left( \mathcal{I}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma, \rho-1} \right) (x) & (13) \\ &= \frac{\Gamma(\rho)\Gamma(-\alpha' + \beta' + \rho)\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho)}{\Gamma(\beta' + \rho)\Gamma(-\alpha - \alpha' + \gamma + \rho)\Gamma(-\alpha' - \beta + \gamma + \rho)} x^{-\alpha - \alpha' + \gamma + \rho - 1}. \end{aligned}$$

(b) If  $\text{Re}(\rho) > \max\{\text{Re}(\beta), \text{Re}(-\alpha - \alpha' + \gamma), \text{Re}(-\alpha - \beta' + \gamma)\}$ , then

$$\begin{aligned} & \left( \mathcal{I}_{-}^{\alpha, \alpha', \beta, \beta', \gamma, \rho} \right) (x) & (14) \\ &= \frac{\Gamma(-\beta + \rho)\Gamma(\alpha + \alpha' - \gamma + \rho)\Gamma(\alpha + \beta' - \gamma + \rho)}{\Gamma(\rho)\Gamma(\alpha - \beta + \rho)\Gamma(\alpha + \alpha' + \beta' - \gamma + \rho)} x^{-\alpha - \alpha' + \gamma - \rho}. \end{aligned}$$

In subsequent theorems, the conditions for the absolute convergence of the integral involved in (1) is assumed. Also, the contour  $C$  of integration is assumed to be the imaginary axis, i.e.,  $\text{Re}(s) = 0$ .

### 3. The MSM fractional integration of the $I$ -function

First, we present the left-hand sided MSM fractional integration of the  $I$ -function.

**Theorem 3.1.** Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$  be such that  $\text{Re}(\gamma), \mu > 0$  and  $\text{Re}(\rho) > \max\{0, \text{Re}(\alpha' - \beta'), \text{Re}(\alpha + \alpha' + \beta - \gamma)\}$ . Then for  $x > 0$ ,

$$\begin{aligned} & \left( \mathcal{I}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{-\alpha - \alpha' + \gamma + \rho - 1} I_{p+3,q+3}^{m,n+3} \left[ ax^\mu \left| \begin{matrix} (1 - \rho, \mu, 1) & (1 + \alpha' - \beta' - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1 - \beta' - \rho, \mu, 1) \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1 + \alpha + \alpha' + \beta - \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1 + \alpha + \alpha' - \gamma - \rho, \mu, 1) & (1 + \alpha' + \beta - \gamma - \rho, \mu, 1) \end{matrix} \right. \right]. \quad (15) \end{aligned}$$

*Proof.* The left-hand side (lhs) of (15) is given by

$$\left( \mathcal{I}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{\rho-1} \frac{1}{2\pi i} \int_C \chi(s) (at^\mu)^{-s} ds \right) \right) (x), \quad (16)$$

where  $\chi(s)$  is given by (2). Interchanging the order of integration and using (13), (16) is equal to

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \chi(s) a^{-s} \left( \mathcal{I}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma, \rho - \mu s - 1} \right) (x) ds \\ &= x^{-\alpha - \alpha' + \gamma + \rho - 1} \frac{1}{2\pi i} \int_C \chi(s) \chi_1(s) (ax^\mu)^{-s} ds, \end{aligned}$$

where

$$\chi_1(s) = \frac{\Gamma(\rho - \mu s)\Gamma(-\alpha' + \beta' + \rho - \mu s)\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho - \mu s)}{\Gamma(\beta' + \rho - \mu s)\Gamma(-\alpha - \alpha' + \gamma + \rho - \mu s)\Gamma(-\alpha' - \beta + \gamma + \rho - \mu s)}.$$

The result now follows from (1).  $\square$

In view of (11), we have the following result for Saigo operators.

**Corollary 3.2.** *Let  $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$  be such that  $\operatorname{Re}(\alpha), \mu > 0$  and  $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(\beta - \gamma)\}$ . Then the left-hand sided generalized fractional integration  $\mathcal{I}_{0+}^{\alpha, \beta, \gamma}$  of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( \mathcal{I}_{0+}^{\alpha, \beta, \gamma} \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{array}{l} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{array} \right. \right] \right) \right) (x) \\ &= x^{-\beta+\rho-1} I_{p+2,q+2}^{m,n+2} \left[ ax^\mu \left| \begin{array}{l} (1-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} \\ (1+\beta-\gamma-\rho, \mu, 1) \quad (a_i, A_i, \alpha_i)_{1,p} \\ (1+\beta-\rho, \mu, 1) \quad (1-\alpha-\gamma-\rho, \mu, 1) \end{array} \right. \right]. \end{aligned}$$

The above corollary leads to Erdélyi-Kober fractional integral as follows.

**Corollary 3.3.** *Let  $\alpha, \gamma, \rho, a \in \mathbb{C}$  be such that  $\operatorname{Re}(\alpha), \mu > 0$  and  $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(-\gamma)\}$ . Then the left-hand sided Erdélyi-Kober fractional integration  $\mathcal{I}_{\gamma,\alpha}^+$  ( $= \mathcal{I}_{0+}^{\alpha,0,\gamma}$ ) of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( \mathcal{I}_{\gamma,\alpha}^+ \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{array}{l} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{array} \right. \right] \right) \right) (x) \\ &= x^{\rho-1} I_{p+1,q+1}^{m,n+1} \left[ ax^\mu \left| \begin{array}{l} (1-\gamma-\rho, \mu, 1) \quad (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \quad (1-\alpha-\gamma-\rho, \mu, 1) \end{array} \right. \right]. \end{aligned}$$

The following result corresponds to the right-hand sided MSM fractional integration of the  $I$ -function.

**Theorem 3.4.** *Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$  be such that  $\operatorname{Re}(\gamma), \mu > 0$  and*

$\text{Re}(\rho) > \max\{\text{Re}(\beta), \text{Re}(-\alpha - \alpha' + \gamma), \text{Re}(-\alpha - \beta' + \gamma)\}$ . Then

$$\begin{aligned} & \left( \mathcal{I}_-^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{-\rho} I_{p,q}^{m,n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{-\alpha - \alpha' + \gamma - \rho} I_{p+3, q+3}^{m, n+3} \left[ ax^{-\mu} \left| \begin{matrix} (1 + \beta - \rho, \mu, 1) & (1 - \alpha - \alpha' + \gamma - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1 - \rho, \mu, 1) \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1 - \alpha - \beta' + \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1 - \alpha + \beta - \rho, \mu, 1) & (1 - \alpha - \alpha' - \beta' + \gamma - \rho, \mu, 1) \end{matrix} \right. \right], \quad (17) \end{aligned}$$

for  $x > 0$ .

*Proof.* Using (1), the lhs of (17) is equal to

$$\left( \mathcal{I}_-^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{-\rho} \frac{1}{2\pi i} \int_C \chi(s) (at^{-\mu})^{-s} ds \right) \right) (x), \quad (18)$$

where  $\chi(s)$  is given by (2). Interchanging the order of integration and using (14), (18) reduces to

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \chi(s) a^{-s} \left( \mathcal{I}_-^{\alpha, \alpha', \beta, \beta', \gamma} t^{-(\rho - \mu s)} \right) (x) ds \\ &= x^{-\alpha - \alpha' + \gamma - \rho} \frac{1}{2\pi i} \int_C \chi(s) \chi_2(s) (ax^{-\mu})^{-s} ds, \end{aligned}$$

where

$$\chi_2(s) = \frac{\Gamma(-\beta + \rho - \mu s) \Gamma(\alpha + \alpha' - \gamma + \rho - \mu s) \Gamma(\alpha + \beta' - \gamma + \rho - \mu s)}{\Gamma(\rho - \mu s) \Gamma(\alpha - \beta + \rho - \mu s) \Gamma(\alpha + \alpha' + \beta' - \gamma + \rho - \mu s)}.$$

The result follows from (1). □

The Saigo and Erdélyi-Kober fractional integration of the  $I$ -function follow as corollaries.

**Corollary 3.5.** Let  $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$  be such that  $\text{Re}(\alpha), \mu > 0$  and  $\text{Re}(\rho) > \max\{\text{Re}(-\beta), \text{Re}(-\gamma)\}$ . Then the right-hand sided generalized fractional integration  $\mathcal{I}_-^{\alpha, \beta, \gamma}$  of the  $I$ -function, for  $x > 0$ , is given by

$$\begin{aligned} & \left( \mathcal{I}_-^{\alpha, \beta, \gamma} \left( t^{-\rho} I_{p,q}^{m,n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{-\beta - \rho} I_{p+2, q+2}^{m, n+2} \left[ ax^{-\mu} \left| \begin{matrix} (1 - \gamma - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1 - \beta - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1 - \rho, \mu, 1) & (1 - \alpha - \beta - \gamma - \rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**Corollary 3.6.** *Let  $\alpha, \gamma, \rho, a \in \mathbb{C}$  be such that  $\operatorname{Re}(\alpha), \mu > 0$  and  $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(-\gamma)\}$ . Then the right-hand sided Erdélyi-Kober fractional integration  $\mathcal{K}_{\gamma, \alpha}^- (= \mathcal{I}_{-}^{\alpha, 0, \gamma})$  of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( \mathcal{K}_{\gamma, \alpha}^- \left( t^{-\rho} I_{p, q}^{m, n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, q} \end{matrix} \right. \right] \right) \right) (x) \\ & = x^{-\rho} I_{p+1, q+1}^{m, n+1} \left[ ax^{-\mu} \left| \begin{matrix} (1 - \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, q} & (1 - \alpha - \gamma - \rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**4. The MSM fractional differentiation of the  $I$ -function**

In this section, we derive the MSM fractional derivative of the  $I$ -function. We first require the following result.

**Lemma 4.1.** *Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ .*

(a) *If  $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(-\alpha + \beta), \operatorname{Re}(-\alpha - \alpha' - \beta' + \gamma)\}$ , then*

$$\begin{aligned} & \left( \mathcal{D}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x) \\ & = \frac{\Gamma(\rho)\Gamma(\alpha - \beta + \rho)\Gamma(\alpha + \alpha' + \beta' - \gamma + \rho)}{\Gamma(-\beta + \rho)\Gamma(\alpha + \alpha' - \gamma + \rho)\Gamma(\alpha + \beta' - \gamma + \rho)} x^{\alpha + \alpha' - \gamma + \rho - 1}. \end{aligned} \tag{19}$$

(b) *If  $\operatorname{Re}(\rho) > \max\{\operatorname{Re}(-\beta'), \operatorname{Re}(\alpha' + \beta - \gamma), \operatorname{Re}(\alpha + \alpha' - \gamma) + [\operatorname{Re}(\gamma)] + 1\}$ , then*

$$\begin{aligned} & \left( \mathcal{D}_{-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{-\rho} \right) (x) \\ & = \frac{\Gamma(\beta' + \rho)\Gamma(-\alpha - \alpha' + \gamma + \rho)\Gamma(-\alpha' - \beta + \gamma + \rho)}{\Gamma(\rho)\Gamma(-\alpha' + \beta' + \rho)\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho)} x^{\alpha + \alpha' - \gamma - \rho}. \end{aligned} \tag{20}$$

*Proof.* (a) Let  $m = [\operatorname{Re}(\gamma)] + 1$ . Using (5) and (13), the lhs of (19) is equal to

$$\begin{aligned} & \left( \frac{d}{dx} \right)^m \left( \mathcal{I}_{0+}^{-\alpha', -\alpha, -\beta'+m, -\beta, -\gamma+m} t^{\rho-1} \right) (x) \\ & = \frac{d^m}{dx^m} \frac{\Gamma(\rho)\Gamma(\alpha - \beta + \rho)\Gamma(\alpha + \alpha' + \beta' - \gamma + \rho)x^{\alpha + \alpha' - \gamma + \rho + m - 1}}{\Gamma(-\beta + \rho)\Gamma(\alpha + \alpha' - \gamma + \rho + m)\Gamma(\alpha + \beta' - \gamma + \rho)} \\ & = \frac{\Gamma(\rho)\Gamma(\alpha - \beta + \rho)\Gamma(\alpha + \alpha' + \beta' - \gamma + \rho)}{\Gamma(-\beta + \rho)\Gamma(\alpha + \alpha' - \gamma + \rho + m)\Gamma(\alpha + \beta' - \gamma + \rho)}, \end{aligned}$$



which on differentiation yields (19).

(b) Using (6) and (14), the lhs of (20) reduces to

$$\begin{aligned} &\left(-\frac{d}{dx}\right)^m \left(\mathcal{I}_-^{-\alpha', -\alpha, -\beta', -\beta+m, -\gamma+m} t^{-\rho}\right)(x) \\ &= \frac{(-1)^m \Gamma(\beta' + \rho) \Gamma(-\alpha - \alpha' + \gamma + \rho - m)}{\Gamma(\rho) \Gamma(-\alpha' + \beta' + \rho)} \\ &\quad \times \frac{\Gamma(-\alpha' - \beta + \gamma + \rho)}{\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho)} \frac{d^m}{dx^m} x^{\alpha + \alpha' - \gamma - \rho + m}. \end{aligned} \quad (21)$$

Also note that

$$\frac{d^m}{dx^m} x^{\alpha + \alpha' - \gamma - \rho + m} = \frac{\Gamma(\alpha + \alpha' - \gamma - \rho + m + 1)}{\Gamma(\alpha + \alpha' - \gamma - \rho + 1)} x^{\alpha + \alpha' - \gamma - \rho}. \quad (22)$$

By the reflection formula for the gamma function (see [4]), we obtain

$$\begin{aligned} &\frac{\Gamma(-\alpha - \alpha' + \gamma + \rho - m) \Gamma(1 - (-\alpha - \alpha' + \gamma + \rho - m))}{\pi} \\ &= \frac{\pi}{\sin(-\alpha - \alpha' + \gamma + \rho - m) \pi} = \frac{\pi}{(-1)^m \sin(-\alpha - \alpha' + \gamma + \rho) \pi}. \end{aligned} \quad (23)$$

Similarly,

$$\begin{aligned} &\Gamma(-\alpha - \alpha' + \gamma + \rho) \Gamma(1 - (-\alpha - \alpha' + \gamma + \rho)) \\ &= \frac{\pi}{\sin(-\alpha - \alpha' + \gamma + \rho) \pi}. \end{aligned} \quad (24)$$

On substituting (22)-(24) in (21), the result follows.  $\square$

**Remark 4.2.** The following connections between Lemma 2.1 and Lemma 4.1 are clear in view of (5) and (6).

(a) If in the hypothesis of Lemma 2.1(a), we make the changes

$$\alpha \rightarrow -\alpha', \alpha' \rightarrow -\alpha, \beta \rightarrow -\beta' + [\text{Re}(\gamma)] + 1, \beta' \rightarrow -\beta, \gamma \rightarrow -\gamma + [\text{Re}(\gamma)] + 1$$

and in the rhs of (19),  $\alpha \rightarrow -\alpha', \alpha' \rightarrow -\alpha, \beta \rightarrow -\beta', \beta' \rightarrow -\beta, \gamma \rightarrow -\gamma$  respectively, then Lemma 4.1(a) follows.

(b) If in the hypothesis of Lemma 2.1(b), we make the changes

$$\alpha \rightarrow -\alpha', \alpha' \rightarrow -\alpha, \beta \rightarrow -\beta', \beta' \rightarrow -\beta + [\text{Re}(\gamma)] + 1, \gamma \rightarrow -\gamma + [\text{Re}(\gamma)] + 1$$

and in the rhs of (20),  $\alpha \rightarrow -\alpha', \alpha' \rightarrow -\alpha, \beta \rightarrow -\beta', \beta' \rightarrow -\beta, \gamma \rightarrow -\gamma$  respectively, then Lemma 4.1(b) is obtained.

The next result gives the left-hand sided MSM fractional derivative of the  $I$ -function.

**Theorem 4.3.** *Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$  be such that  $\mu > 0$  and  $\text{Re}(\rho) > \max\{0, \text{Re}(-\alpha + \beta), \text{Re}(-\alpha - \alpha' - \beta' + \gamma)\}$ . Then for  $x > 0$ ,*

$$\begin{aligned} & \left( \mathcal{D}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) & (25) \\ & = x^{\alpha+\alpha'-\gamma+\rho-1} I_{p+3,q+3}^{m,n+3} \left[ ax^\mu \left| \begin{matrix} (1-\rho, \mu, 1) & (1-\alpha+\beta-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1+\beta-\rho, \mu, 1) \\ (1-\alpha-\alpha'-\beta'+\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1-\alpha-\alpha'+\gamma-\rho, \mu, 1) & (1-\alpha-\beta'+\gamma-\rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

*Proof.* On using (1), the lhs of (25) equals

$$\left( \mathcal{D}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{\rho-1} \frac{1}{2\pi i} \int_C \chi(s) (at^\mu)^{-s} ds \right) \right) (x), \tag{26}$$

where  $\chi(s)$  is given by (2). Interchanging the order of integration and differentiation and using (19), (26) becomes

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \chi(s) a^{-s} \left( \mathcal{D}_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-\mu s-1} \right) (x) ds \\ & = x^{\alpha+\alpha'-\gamma+\rho-1} \frac{1}{2\pi i} \int_C \chi(s) \chi_3(s) (ax^\mu)^{-s} ds, \end{aligned}$$

where

$$\chi_3(s) = \frac{\Gamma(\rho - \mu s) \Gamma(\alpha - \beta + \rho - \mu s) \Gamma(\alpha + \alpha' + \beta' - \gamma + \rho - \mu s)}{\Gamma(-\beta + \rho - \mu s) \Gamma(\alpha + \alpha' - \gamma + \rho - \mu s) \Gamma(\alpha + \beta' - \gamma + \rho - \mu s)}.$$

The result follows from (1). □

Following corollaries for Saigo and Erdélyi-Kober fractional derivatives follow immediately.

**Corollary 4.4.** *Let  $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$  be such that  $\text{Re}(\rho) > \max\{0, \text{Re}(-\alpha - \beta - \gamma)\}$  and  $\mu > 0$ . Then the left-hand sided generalized fractional differentiation*

$\mathcal{D}_{0+}^{\alpha,\beta,\gamma}$  of the  $I$ -function is given for  $x > 0$  by

$$\begin{aligned} & \left( \mathcal{D}_{0+}^{\alpha,\beta,\gamma} \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{\beta+\rho-1} I_{p+2,q+2}^{m,n+2} \left[ ax^\mu \left| \begin{matrix} (1-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1-\alpha-\beta-\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1-\gamma-\rho, \mu, 1) & (1-\beta-\rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**Corollary 4.5.** Let  $\alpha, \gamma, \rho \in \mathbb{C}$  be such that  $\text{Re}(\rho) > \max\{0, \text{Re}(-\alpha - \gamma)\}$  and  $\mu > 0$ . Then the left-hand sided Erdélyi-Kober fractional differentiation  $\mathcal{D}_{\gamma,\alpha}^+$  ( $= \mathcal{D}_{0+}^{\alpha,0,\gamma}$ ) of the  $I$ -function is given for  $x > 0$  by

$$\begin{aligned} & \left( \mathcal{D}_{\gamma,\alpha}^+ \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{\rho-1} I_{p+1,q+1}^{m,n+1} \left[ ax^\mu \left| \begin{matrix} (1-\alpha-\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} & (1-\gamma-\rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

The next theorem yields the right-hand sided MSM fractional derivative of the  $I$ -function.

**Theorem 4.6.** Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$  be such that  $\mu > 0$  and  $\text{Re}(\rho) > \max\{\text{Re}(-\beta'), \text{Re}(\alpha' + \beta - \gamma), \text{Re}(\alpha + \alpha' - \gamma) + [\text{Re}(\gamma)] + 1\}$ . Then

$$\begin{aligned} & \left( \mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma} \left( t^{-\rho} I_{p,q}^{m,n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \tag{27} \\ &= x^{\alpha+\alpha'-\gamma-\rho} I_{p+3,q+3}^{m,n+3} \left[ ax^{-\mu} \left| \begin{matrix} (1-\beta'-\rho, \mu, 1) & (1+\alpha+\alpha'-\gamma-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1-\rho, \mu, 1) \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1+\alpha'+\beta-\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1+\alpha'-\beta'-\rho, \mu, 1) & (1+\alpha+\alpha'+\beta-\gamma-\rho, \mu, 1) \end{matrix} \right. \right], \end{aligned}$$

for  $x > 0$ .

*Proof.* Using (20) and the definition of the  $I$ -function in (1), the lhs of (27) is

$$\begin{aligned} & \left( \mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma} \left( t^{-\rho} \frac{1}{2\pi i} \int_C \chi(s) (at^{-\mu})^{-s} ds \right) \right) (x) \\ &= \frac{1}{2\pi i} \int_C \chi(s) a^{-s} \left( \mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma} t^{-(\rho-\mu s)} \right) (x) ds \\ &= x^{\alpha+\alpha'-\gamma-\rho} \frac{1}{2\pi i} \int_C \chi(s) \chi_4(s) (ax^{-\mu})^{-s} ds, \end{aligned}$$

where  $\chi(s)$  is given by (2) and

$$\chi_4(s) = \frac{\Gamma(\beta' + \rho - \mu s)\Gamma(-\alpha - \alpha' + \gamma + \rho - \mu s)\Gamma(-\alpha' - \beta + \gamma + \rho - \mu s)}{\Gamma(\rho - \mu s)\Gamma(-\alpha' + \beta' + \rho - \mu s)\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho - \mu s)}.$$

Thus, the theorem now follows from (1). □

**Corollary 4.7.** *Let  $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$  be such that  $\text{Re}(\rho) > \max\{\text{Re}(-\alpha - \gamma), \text{Re}(\beta) + [\text{Re}(\alpha)] + 1\}$  and  $\mu > 0$ . Then the right-hand sided generalized fractional differentiation  $\mathcal{D}_-^{\alpha, \beta, \gamma}$  of the I-function is given for  $x > 0$  by*

$$\begin{aligned} & \left( \mathcal{D}_-^{\alpha, \beta, \gamma} \left( t^{-\rho} I_{p, q}^{m, n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{\beta - \rho} I_{p+2, q+2}^{m, n+2} \left[ ax^{-\mu} \left| \begin{matrix} (1 + \beta - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1, q} \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1 - \alpha - \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1, p} \\ (1 - \rho, \mu, 1) & (1 + \beta - \gamma - \rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**Corollary 4.8.** *Let  $\alpha, \gamma, \rho, a \in \mathbb{C}$  be such that  $\mu > 0$  and  $\text{Re}(\rho) > \max\{[\text{Re}(\alpha)] + 1, \text{Re}(-\alpha - \gamma)\}$ . Then the right-hand sided Erdélyi-Kober fractional differentiation  $\mathcal{D}_{\gamma, \alpha}^-$  ( $= \mathcal{D}_-^{\alpha, 0, \gamma}$ ) of the I-function is given for  $x > 0$  by*

$$\begin{aligned} & \left( \mathcal{D}_{\gamma, \alpha}^- \left( t^{-\rho} I_{p, q}^{m, n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{-\rho} I_{p+1, q+1}^{m, n+1} \left[ ax^{-\mu} \left| \begin{matrix} (1 - \alpha - \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, q} & (1 - \gamma - \rho, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**Remark 4.9.** Theorems 4.3 and 4.6 respectively follow from Theorems 3.1 and 3.4, in view of Remark 4.2.

### 5. The Caputo-type MSM fractional differentiation of the I-function

Since the Riemann-Liouville fractional derivatives have some drawbacks while applying to real world problems, especially in the context of initial conditions, the Caputo derivative is being extensively used in applications as the initial conditions have physical significance. Rao *et al.* [17] introduced Caputo-type fractional derivative which involve Gauss hypergeometric function in the kernel. For  $\alpha, \beta, \gamma \in \mathbb{C}$  and  $x > 0$  with  $\text{Re}(\alpha) > 0$ , the left- and right-hand sided Caputo

fractional differential operators associated with Gauss hypergeometric function are defined by

$$\left({}^c\mathcal{D}_{0+}^{\alpha,\beta,\gamma}f\right)(x) = \left(\mathcal{I}_{0+}^{-\alpha+m,-\beta-m,\alpha+\gamma-m}f^{(m)}\right)(x) \tag{28}$$

and

$$\left({}^c\mathcal{D}_-^{\alpha,\beta,\gamma}f\right)(x) = (-1)^m \left(\mathcal{I}_-^{-\alpha+m,-\beta-m,\alpha+\gamma}f^{(m)}\right)(x), \tag{29}$$

where  $m = [\text{Re}(\alpha)] + 1$  and  $f^{(m)}$  denotes the  $m$ -th derivative of  $f$ . The relation between the Caputo-type MSM fractional derivative and the MSM fractional derivative is same as the relation between the Caputo fractional derivative and the Riemann-Liouville fractional derivative.

For  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$  and  $x > 0$  with  $\text{Re}(\gamma) > 0$ , we define the left- and right-hand sided Caputo-type MSM fractional differential operators associated with third Appell function as

$$\left({}^c\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(\mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}f^{(m)}\right)(x) \tag{30}$$

and

$$\left({}^c\mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = (-1)^m \left(\mathcal{I}_-^{-\alpha',-\alpha,-\beta',-\beta+m,-\gamma+m}f^{(m)}\right)(x), \tag{31}$$

respectively, where  $m = [\text{Re}(\gamma)] + 1$ . The fractional operators (30) and (31) are connected to (28) and (29) as follows:

$$\begin{aligned} \left({}^c\mathcal{D}_{0+}^{0,\alpha',\beta,\beta',\gamma}f\right)(x) &= \left({}^c\mathcal{D}_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x), \\ \left({}^c\mathcal{D}_-^{0,\alpha',\beta,\beta',\gamma}f\right)(x) &= \left({}^c\mathcal{D}_-^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x). \end{aligned} \tag{32}$$

In this section, we study the Caputo-type MSM fractional differentiation of the  $I$ -function. First we state and prove the following lemma.

**Lemma 5.1.** *Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$  and  $m = [\text{Re}(\gamma)] + 1$ .*

(a) *If  $\text{Re}(\rho) - m > \max\{0, \text{Re}(-\alpha + \beta), \text{Re}(-\alpha - \alpha' - \beta' + \gamma)\}$ , then*

$$\begin{aligned} &\left({}^c\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}I_t^{\rho-1}\right)(x) \tag{33} \\ &= \frac{\Gamma(\rho)\Gamma(\alpha - \beta + \rho - m)\Gamma(\alpha + \alpha' + \beta' - \gamma + \rho - m)}{\Gamma(-\beta + \rho - m)\Gamma(\alpha + \alpha' - \gamma + \rho)\Gamma(\alpha + \beta' - \gamma + \rho - m)}x^{\alpha+\alpha'-\gamma+\rho-1}. \end{aligned}$$

(b) *If  $\text{Re}(\rho) + m > \max\{\text{Re}(-\beta'), \text{Re}(\alpha' + \beta - \gamma), \text{Re}(\alpha + \alpha' - \gamma) + [\text{Re}(\gamma)] + 1\}$ , then*

$$\begin{aligned} &\left({}^c\mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma}I_t^{-\rho}\right)(x) \tag{34} \\ &= \frac{\Gamma(\beta' + \rho + m)\Gamma(-\alpha - \alpha' + \gamma + \rho)\Gamma(-\alpha' - \beta + \gamma + \rho + m)}{\Gamma(\rho)\Gamma(-\alpha' + \beta' + \rho + m)\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho + m)}x^{\alpha+\alpha'-\gamma-\rho}. \end{aligned}$$

*Proof.* (a) From (30), we have

$$\begin{aligned} \left({}^c\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x) &= \left(\mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}\frac{d^m}{dt^m}t^{\rho-1}\right)(x) \\ &= \frac{\Gamma(\rho)}{\Gamma(\rho-m)}\left(\mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}t^{\rho-m-1}\right)(x), \end{aligned}$$

which on using (13) gives (33).

(b) From (31), we have

$$\begin{aligned} \left({}^c\mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho}\right)(x) &= (-1)^m\left(\mathcal{I}_-^{-\alpha',-\alpha,-\beta',-\beta+m,-\gamma+m}\frac{d^m}{dt^m}t^{-\rho}\right)(x) \\ &= \frac{\Gamma(\rho+m)}{\Gamma(\rho)}\left(\mathcal{I}_-^{-\alpha',-\alpha,-\beta',-\beta+m,-\gamma+m}t^{-\rho-m}\right)(x), \end{aligned}$$

and thus (34) follows from (14). □

We next present the left-hand sided Caputo-type MSM fractional derivative of the  $I$ -function.

**Theorem 5.2.** *Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$ ,  $m = [\text{Re}(\gamma)] + 1$  be such that  $\mu > 0$  and  $\text{Re}(\rho) - m > \max\{0, \text{Re}(-\alpha + \beta), \text{Re}(-\alpha - \alpha' - \beta' + \gamma)\}$ . Then for  $x > 0$*

$$\begin{aligned} &\left({}^c\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}I_{p,q}^{m,n}\left[at^\mu\left|\begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix}\right.\right]\right)\right)(x) \\ &= x^{\alpha+\alpha'-\gamma+\rho-1}I_{p+3,q+3}^{m,n+3}\left[ax^\mu\left|\begin{matrix} (1-\rho, \mu, 1) & (1-\alpha+\beta-\rho+m, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1+\beta-\rho+m, \mu, 1) \end{matrix}\right.\right] \\ &\quad \left.\begin{matrix} (1-\alpha-\alpha'-\beta'+\gamma-\rho+m, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1-\alpha-\alpha'+\gamma-\rho, \mu, 1) & (1-\alpha-\beta'+\gamma-\rho+m, \mu, 1) \end{matrix}\right]. \end{aligned}$$

*Proof.* From (1) and (33), we get

$$\begin{aligned} &\left({}^c\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}\frac{1}{2\pi i}\int_C\chi(s)(at^\mu)^{-s}ds\right)\right)(x) \\ &= \frac{1}{2\pi i}\int_C\chi(s)a^{-s}\left({}^c\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-\mu s-1}\right)(x)ds \\ &= x^{\alpha+\alpha'-\gamma+\rho-1}\frac{1}{2\pi i}\int_C\chi(s)\chi_5(s)(ax^\mu)^{-s}ds, \end{aligned}$$

where  $\chi(s)$  is given by (2) and

$$\begin{aligned} \chi_5(s) &= \frac{\Gamma(\rho-\mu s)}{\Gamma(-\beta+\rho-\mu s-m)} \\ &\times \frac{\Gamma(\alpha-\beta+\rho-\mu s-m)\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho-\mu s-m)}{\Gamma(\alpha+\alpha'-\gamma+\rho-\mu s)\Gamma(\alpha+\beta'-\gamma+\rho-\mu s-m)}. \end{aligned}$$

The result now follows from (1). □

**Corollary 5.3.** *Let  $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$ ,  $m = [\operatorname{Re}(\alpha)] + 1$  be such that  $\mu > 0$  and  $\operatorname{Re}(\rho) - m > \max\{0, \operatorname{Re}(-\alpha - \beta - \gamma)\}$ . The left-hand sided generalized Caputo fractional differentiation  ${}^c\mathcal{D}_{0+}^{\alpha, \beta, \gamma}$  of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( {}^c\mathcal{D}_{0+}^{\alpha, \beta, \gamma} \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{\beta+\rho-1} I_{p+2,q+2}^{m,n+2} \left[ ax^\mu \left| \begin{matrix} (1-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1-\alpha-\beta-\gamma-\rho+m, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1-\beta-\rho, \mu, 1) & (1-\gamma-\rho+m, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**Corollary 5.4.** *Let  $\alpha, \gamma, \rho, a \in \mathbb{C}$ ,  $m = [\operatorname{Re}(\alpha)] + 1$  be such that  $\operatorname{Re}(\rho) - m > \max\{0, \operatorname{Re}(-\alpha - \gamma)\}$  and  $\mu > 0$ . Then the left-hand sided Caputo-type Erdelyi-Kober fractional differentiation  ${}^c\mathcal{D}_{\gamma,\alpha}^+$  ( $= {}^c\mathcal{D}_{0+}^{\alpha,0,\gamma}$ ) of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( {}^c\mathcal{D}_{\gamma,\alpha}^+ \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{\rho-1} I_{p+1,q+1}^{m,n+1} \left[ ax^\mu \left| \begin{matrix} (1-\alpha-\gamma-\rho+m, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} & (1-\gamma-\rho+m, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

Finally, we present the right-hand sided Caputo-type MSM fractional derivative of the  $I$ -function.

**Theorem 5.5.** *Let  $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$ ,  $m = [\operatorname{Re}(\gamma)] + 1$  be such that  $\mu > 0$  and  $\operatorname{Re}(\rho) + m > \max\{\operatorname{Re}(-\beta'), \operatorname{Re}(\alpha' + \beta - \gamma), \operatorname{Re}(\alpha + \alpha' - \gamma) + m\}$ . Then*

$$\begin{aligned} & \left( {}^c\mathcal{D}_-^{\alpha, \alpha', \beta, \beta', \gamma} \left( t^{-\rho} I_{p,q}^{m,n} \left[ at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) = x^{\alpha+\alpha'-\gamma-\rho} \\ & \quad \times I_{p+3,q+3}^{m,n+3} \left[ ax^{-\mu} \left| \begin{matrix} (1-\beta'-\rho-m, \mu, 1) & (1+\alpha+\alpha'-\gamma-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1-\rho, \mu, 1) \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1+\alpha'+\beta-\gamma-\rho-m, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1+\alpha'-\beta'-\rho-m, \mu, 1) & (1+\alpha+\alpha'+\beta-\gamma-\rho-m, \mu, 1) \end{matrix} \right. \right], \end{aligned}$$

for  $x > 0$ .

*Proof.* Using (34) and the definition of the  $I$ -function in (1), we have

$$\begin{aligned} & \left( {}^c\mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma} \left( t^{-\rho} \frac{1}{2\pi i} \int_C \chi(s) (at^{-\mu})^{-s} ds \right) \right) (x) \\ &= \frac{1}{2\pi i} \int_C \chi(s) a^{-s} \left( {}^c\mathcal{D}_-^{\alpha,\alpha',\beta,\beta',\gamma} t^{-(\rho-\mu s)} \right) (x) ds \\ &= x^{\alpha+\alpha'-\gamma-\rho} \frac{1}{2\pi i} \int_C \chi(s) \chi_6(s) (ax^{-\mu})^{-s} ds, \end{aligned}$$

where  $\chi_6(s)$  is given by

$$\begin{aligned} \chi_6(s) &= \frac{\Gamma(\beta' + \rho - \mu s + m)}{\Gamma(\rho - \mu s)} \\ &\times \frac{\Gamma(-\alpha - \alpha' + \gamma + \rho - \mu s) \Gamma(-\alpha' - \beta + \gamma + \rho - \mu s + m)}{\Gamma(-\alpha' + \beta' + \rho - \mu s + m) \Gamma(-\alpha - \alpha' - \beta + \gamma + \rho - \mu s + m)}. \end{aligned}$$

The theorem is now proved using (1). □

**Corollary 5.6.** *Let  $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$ ,  $m = [\text{Re}(\alpha)] + 1$  be such that  $\mu > 0$  and  $\text{Re}(\rho) + m > \max\{\text{Re}(\beta) + m, \text{Re}(-\alpha - \gamma)\}$ . Then the right-hand sided generalized Caputo fractional differentiation  ${}^c\mathcal{D}_-^{\alpha,\beta,\gamma}$  of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( {}^c\mathcal{D}_-^{\alpha,\beta,\gamma} \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{\beta-\rho} I_{p+2,q+2}^{m,n+2} \left[ ax^{-\mu} \left| \begin{matrix} (1 + \beta - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right. \\ & \quad \left. \left. \begin{matrix} (1 - \alpha - \gamma - \rho - m, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1 - \rho, \mu, 1) & (1 + \beta - \gamma - \rho - m, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$

**Corollary 5.7.** *Let  $\alpha, \gamma, \rho, a \in \mathbb{C}$ ,  $m = [\text{Re}(\alpha)] + 1$  be such that  $\mu > 0$  and  $\text{Re}(\rho) + m > \max\{m, \text{Re}(-\alpha - \gamma)\}$ . Then the right-hand sided Caputo-type Erdelyi-Kober fractional differentiation  ${}^c\mathcal{D}_{\gamma,\alpha}^-$  ( $= {}^c\mathcal{D}_-^{\alpha,0,\gamma}$ ) of the  $I$ -function is given for  $x > 0$  by*

$$\begin{aligned} & \left( {}^c\mathcal{D}_{\gamma,\alpha}^- \left( t^{\rho-1} I_{p,q}^{m,n} \left[ at^\mu \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} \end{matrix} \right. \right] \right) \right) (x) \\ &= x^{-\rho} I_{p+1,q+1}^{m,n+1} \left[ ax^{-\mu} \left| \begin{matrix} (1 - \alpha - \gamma - \rho - m, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} & (1 - \gamma - \rho - m, \mu, 1) \end{matrix} \right. \right]. \end{aligned}$$



## 6. Conclusion

The  $I$ -function is one of the most generalized function available in literature, which generalizes the  $\bar{H}$ -function,  $H$ -function, Meijer  $G$ -function, generalized Wright function, hypergeometric function, generalized Mittag-Leffler function and many other functions. On the other hand, the MSM fractional operators generalize, among others, Saigo, Riemann-Liouville, Weyl and Erdélyi-Kober fractional operators. In view of this fact, several recent results obtained by Srivastava *et al.* [10], Purohit *et al.* [5], Kilbas and Sebastian in a series of papers [9],[8],[7],[6] become particular cases of our results.

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