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ON GENERALIZED COMPOSITE FRACTIONAL q-DERIVATIVE

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In the present paper, we define a generalized composite fractional q-derivative $D_q^{\alpha,\beta;\nu}$ and obtain some results for it. These results are image of power function under $D_q^{\alpha,\beta;\nu}$, composition of Riemann-Liouville type fractional q-integral I_q^{α} with $D_q^{\alpha,\beta;\nu}$ and q-Laplace transform of $D_q^{\alpha,\beta;\nu}$.

We also obtain solutions of a q-difference equation with derivative as $D_q^{\alpha,\beta;\nu}$ and discuss some special cases. A q-difference equation of $D_q^{\alpha,\beta;\nu}$ is solved using q-Laplace transform and its inverse.

1. Introduction

The study of q-analysis is an old subject, which dates back to the end of the19^{th} century. A detailed account of the work on this subject can be seen in the books by Exton [9], Gasper and Rahman [10] and Ernst [8]. The q-analysis has found many applications in such areas as the theory of partitions, combinatorics, exactly solvable models in statistical mechanics, computer algebra etc. The subject of q-analysis concerns mainly the properties of the so-called q-special functions, which are the extensions of the classical special functions based on a parameter, or the base q. In recent years, mathematicians have reconsidered q-difference equations for their links with other branches of mathematics such as

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quantum algebras and q-combinatory. The q-difference equations involve a new kind of difference operator, the q-derivative, which can be viewed as a sort of deformation of the ordinary derivative.

Fractional calculus is the theory of integrals and derivatives to an arbitrary order, which generalizes integer-order differentiation and integration. Fractional derivatives have proved to be very efficient and adequate to describe many phenomena with memory and hereditary processes. These phenomena are abundant in science, engineering, viscoelasticity, control, porous media, mechanics, electrical engineering, electromagnetism etc. Recent books on fractional calculus ([6], [7], [16], [20], [21]) exhibit its application in various fields of science and engineering. Unlike the classical derivatives, fractional derivatives have the ability to characterize adequately, the processes involving a past history. Different from classical (or integer-order) derivatives there are several definitions for fractional derivatives given in different contexts (see [12], [13], [15], [18], [19]).

From the point of view of q-calculus several authors have introduced various fractional q-integrals and fractional q-derivatives ([2], [3], [17], see also[5]).

In the present paper, we define a new fractional q-derivative termed as generalized composite fractional q-derivative and obtain some basic useful results for it. We also obtain solution of a q-difference equation with this q-derivative, using q-Laplace transform.

2. Preliminaries

We shall use the following definitions and results in subsequent sections.

The *q-shifted factorial* (q-analogue of Pochhammer symbol) is defined as in [10]:

$$(a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right),\ n \in N \tag{1}$$

with $(a;q)_0 = 1$, $q \neq 1$.
If we consider $(a;q)_\infty$, then as the infinite product diverges when $a \neq 0$ and $|q| \geq 1$, therefore whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. Also, for any complex number α, we have

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{2}$$

where the principal value of q^α is taken.

The *q-derivative* of a function [10] is defined by:

$$(D_q f)(t) = \frac{f(z) - f(qt)}{(1-q)t}, \; (t \neq 0, q \neq 1) \tag{3}$$
$$(D_q f)(0) = \lim_{t \to 0} (D_q f)(t)$$

The Riemann-Liouville type fractional q-integral of order $\alpha > 0$, for a real valued function $f(t)$, is defined as ([17], see also [5])

$$I_q^\alpha f(t) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t \left(q\tau/t; q\right)_{\alpha-1} f(\tau)\, d_q\tau, \; 0 < |q| < 1, \tag{4}$$

with $I_q^0 f(t) = f(t)$.

The Riemann-Liouville type fractional q-derivative of order α $(m-1 < \alpha \leq m, \; \mathrm{m} \in \mathbb{N})$, for a real valued function $f(t)$, is defined as [17]:

$$D_q^\alpha f(t) = D_q^m I_q^{m-\alpha} f(t), \; 0 < |q| < 1, \tag{5}$$

where $D_q^m \equiv D_q.D_q \ldots D_q \; (m\, times)$.

The Caputo type fractional q-derivative of order α $(m-1 < \alpha \leq m, m \in \mathbb{N})$, is defined as [17]

$$_*D_q^\alpha f(t) = I_q^{m-\alpha} D_q^m f(t), \; 0 < |q| < 1. \tag{6}$$

The q-Laplace transform of a function $f(t)$ is defined by means of following q-integral [1]

$$\tilde{f}(s) = {}_qL_s\{f(t)\} = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q^{-qst} f(t)\, d_qt, s > 0, \tag{7}$$

where E_q^t is the q-exponential function given by

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q;q)_n} = (-z;q)_\infty. \tag{8}$$

Now we mention some results for I_q^α, D_q^α and $_*D_q^\alpha$ and q-Laplace transform which will be required subsequently [5].

$$\text{(i)} \quad I_q^\alpha I_q^\beta f(t) = I_q^\beta I_q^\alpha f(t) = I_q^{\alpha+\beta} f(t), \quad \alpha \geq 0, \beta \geq 0. \tag{9}$$

$$\text{(ii)} \quad I_q^\alpha t^{\lambda-1} = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)} t^{\lambda+\alpha-1}, \quad \alpha \geq 0, \; \lambda > 0. \tag{10}$$

(iii) Composition of Riemann-Liouville type fractional q-integral and Riemann-Liouville type fractional q-derivative

$$I_q^\alpha D_q^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{(t)^{\alpha-k-1}}{\Gamma_q(\alpha-k)} D_q^{\alpha-k-1} f(t)\big|_{t\to 0^+}, \quad m-1<\alpha,\beta\le m,\ m\in\mathbb{N}. \tag{11}$$

(iv) $$ {}_qL_s\left[I_q^\alpha f(t)\right] = \frac{(1-q)^\alpha}{s^\alpha}\, {}_qL_s[f(t)],\ \alpha\ge 0. \tag{12}$$

(v) $$ {}_qL_s\left[D_q^\alpha f(t)\right] = \frac{s^\alpha}{(1-q)^\alpha}\, {}_qL_s[f(t)] - \sum_{k=0}^{m-1} \frac{s^k}{(1-q)^{k+1}} D_q^{\alpha-k-1} f(t)\big|_{t\to 0^+}, \quad m-1<\alpha\le m,\ m\in\mathbb{N}. \tag{13}$$

(vi) $$ {}_qL_s\left[{}_*D_q^\alpha f(t)\right] = \frac{s^\alpha}{(1-q)^\alpha}\, {}_qL_s[f(t)] - \sum_{k=0}^{m-1} \frac{s^{\alpha-k-1}}{(1-q)^{\alpha-k}} \left[D_q^k f(t)\right]_{t\to 0^+}, \quad m-1<\alpha\le m,\ \ m\in\mathbb{N}. \tag{14}$$

(vii) $$ {}_qL_s\{t^\nu\} = \frac{(q;q)_\nu}{s^{\nu+1}} = \frac{(1-q)^\nu\,\Gamma_q(\nu+1)}{s^{\nu+1}},\ \ 0<|q|<1,\ \ Re(\nu)>-1. \tag{15}$$

(viii) $$ {}_qL_s(f *_q g)(t) = {}_qL_s\{f(t)\}\, {}_qL_s\{g(t)\}, \tag{16}$$

where $f *_q g$ is the Laplace q-convolution of two analytic functions $f(t)$ and $g(t)$, defined as follows ([11], see also [5])

$$(f *_q g)(t) = \frac{1}{(1-q)} \int_0^t f(u)\, g[t-qu]\, d_q u, \tag{17}$$

where $g[t-qu] = \sum_{n=0}^\infty a_n (t-qu)(t-q^2u)\dots(t-q^nu)$, for the function $g(t) = \sum_{n=0}^\infty a_n t^n$.

3. Generalized composite fractional q-derivative

We define the generalized composite fractional q-derivative for $m-1<\alpha,\beta\le m,\ 0\le\nu\le 1,\ m\in\mathbb{N},\ 0<|q|<1$, as follows

$$D_q^{\alpha,\beta;\nu} f(t) = I_q^{\nu(m-\beta)} D_q^m I_q^{(1-\nu)(m-\alpha)} f(t). \tag{18}$$

For $\nu = 0$, (18) reduces to the Riemann-Liouville type fractional q-derivative of order α defined by (5) and for $\nu = 1$, it reduces to the Caputo type fractional q-derivative of order β defined (6)

For $0 < \nu < 1$, it interpolates continuously between the Riemann-Liouville type fractional q-derivative of order α and the Caputo type fractional q-derivative of order β.

Remark 3.1. If we let $q \to 1$in (18), we get a new definition named as generalized composite fractional derivative

$$D^{\alpha,\beta;\nu} f(t) = I^{\nu(m-\beta)} D^m I^{(1-\nu)(m-\alpha)} f(t), \tag{19}$$

$$m-1 < \alpha, \beta \leq m,\ 0 \leq \nu \leq 1,\ m \in \mathbb{N}.$$

Remark 3.2. For $\alpha = \beta$, (18) reduces to the following composite fractional q-derivative

$$D_q^{\alpha,\alpha;\nu} f(t) = I_q^{\nu(m-\alpha)} D_q^m I_q^{(1-\nu)(m-\alpha)} f(t) = D_q^{\alpha,\nu} f(t), \tag{20}$$

$$m-1 < \alpha \leq m,\ 0 \leq \nu \leq 1,\ m \in \mathbb{N},\ 0 < |q| < 1.$$

This $D_q^{\alpha,\nu}$ is a q-extension of the generalized Riemann-Liouville fractional derivative $D^{\alpha,\nu}$ defined by Hilfer [12].

We now obtain some results for the generalized composite fractional q-derivative $D_q^{\alpha,\beta;\nu}$.

Theorem 3.3. *For* $m-1 < \alpha, \beta \leq m,\ 0 \leq \nu \leq 1,\ m \in \mathbb{N},\ 0 < |q| < 1,\ t > 0$ *and* $\lambda > \max\{0, 1 + (\nu-1)(m-\nu)\}$ *we have the* image of power function under $D_q^{\alpha,\beta;\nu}$ *as follows*

$$D_q^{\alpha,\beta;\nu} t^{\lambda-1} = \frac{\Gamma_q(\lambda)}{\Gamma_q(\nu(\alpha-\beta)+\lambda-\alpha)} t^{\nu(\alpha-\beta)+\lambda-\alpha-1}. \tag{21}$$

Proof. In view of definition (18) and the result (10), we get

$$\begin{aligned} D_q^{\alpha,\beta;\nu} t^{\lambda-1} &= I_q^{\nu(m-\beta)} D_q^m \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+(1-\nu)(m-\alpha))} t^{(1-\nu)(m-\alpha)+\lambda-1} \\ &= I_q^{\nu(m-\beta)} \frac{\Gamma(\lambda)}{\Gamma_q(\lambda+\alpha\nu-\alpha-m\nu)} t^{\alpha\nu-\alpha-m\nu+\lambda-1}. \end{aligned} \tag{22}$$

Using result (10) again, we get (21). □

Theorem 3.4. *For* $m-1<\alpha,\beta\leq m,\ 0\leq\nu\leq 1,\ m\in\mathbb{N},\ 0<|q|<1$*, the* composition of Riemann-Liouville type fractional q-integral (4) with the generalized composite fractional q-derivative (18) *is given by*

$$I_q^{\alpha+\nu(\beta-\alpha)}D_q^{\alpha,\beta;\nu}f(t)$$
$$=f(t)-\sum_{k=0}^{m-1}\frac{t^{\alpha+\nu(m-\alpha)-k-1}}{\Gamma_q(\alpha+\nu(m-\alpha)-k)}D_q^{\alpha+\nu(m-\alpha)-k-1}f(t)\Big|_{t\to 0^+}. \quad (23)$$

Proof. From (18) we can write

$$D_q^{\alpha,\beta;\nu}f(t)=I_q^{\nu(m-\beta)}D_q^mI_q^{(1-\nu)(m-\alpha)}f(t)=I_q^{\nu(m-\beta)}D_q^{\alpha+\nu(m-\alpha)}f(t). \quad (24)$$

Applying $I_q^{\alpha+\nu(\beta-\alpha)}$ on both the sides and using the semigroup property (9), we get

$$I_q^{\alpha+\nu(\beta-\alpha)}D_q^{\alpha,\beta;\nu}f(t)=I_q^{\alpha+\nu(m-\alpha)}D_q^{\alpha+\nu(m-\alpha)}f(t). \quad (25)$$

In view of the result (11), we arrive at (23). □

Theorem 3.5. *For* $m-1<\alpha,\beta\leq m,\ 0\leq\nu\leq 1,\ m\in\mathbb{N},\ 0<|q|<1$*, the* q-Laplace transform of the generalized composite fractional derivative *is given by*

$${}_qL_s\left[D_q^{\alpha,\beta;\nu}f(t)\right]=\frac{s^{\alpha+\nu(\beta-\alpha)}}{(1-q)^{\alpha+\nu(\beta-\alpha)}}\,{}_qL_s[f(t)]$$
$$-\sum_{k=0}^{m-1}\frac{s^{k-\nu(m-\beta)}}{(1-q)^{k-\nu(m-\beta)+1}}D_q^{m-k-1}I_q^{(1-\nu)(m-\alpha)}f(t)\Big|_{t\to 0^+}. \quad (26)$$

Proof. For convenience, let us write $g(t)=D_q^mI_q^{(1-\nu)(m-\alpha)}f(t)=D_q^mh(t)$. Now by definition of $D_q^{\alpha,\beta;\nu}$ and in view of the result (12), we have

$${}_qL_s\left[D_q^{\alpha,\beta;\nu}f(t)\right]={}_qL_s\left[I_q^{\nu(m-\beta)}g(t)\right]=\frac{(1-q)^{\nu(m-\beta)}}{s^{\nu(m-\beta)}}\,{}_qL_s[g(t)]. \quad (27)$$

Now writing $g(t)$ in terms of $h(t)$ and using the following formula of q-Laplace transform of m^{th} q-derivative of a function,

$${}_qL_s\left\{D_q^mf(t)\right\}=\frac{s^m}{(1-q)^m}\,{}_qL_s\{f(t)\}-\sum_{k=0}^{m-1}\frac{s^k}{(1-q)^{k+1}}\left[D_q^{(m-k-1)}f(t)\right]_{t\to 0^+}, \quad (28)$$

we can write the right side of (27) as

$$\begin{aligned}&{}_qL_s\left[D_q^{\alpha,\beta;\nu}f(t)\right]\\&=\frac{(1-q)^{\nu(m-\beta)}}{s^{\nu(m-\beta)}}\left[\frac{s^m}{(1-q)^m}\,{}_qL_s\left[h(t)\right]-\sum_{k=0}^{m-1}s^{m-k-1}\,D^{m-k-1}h(t)\Big|_{t\to 0^+}\right],\end{aligned}\tag{29}$$

where $h(t)=I_q^{(1-\nu)(m-\alpha)}f(t)$. Using the result (12) for q-Laplace transform of $h(t)$, we arrive at the result (26). □

On taking $q\to 1$ in Theorems 3.3 to 3.5, we get corresponding results for generalized composite fractional derivative $D^{\alpha,\beta;\nu}$ defined by (19). On further taking $\alpha=\beta$ in the results thus obtained, we get corresponding results for the composite fractional derivative $D^{\alpha,\nu}$, as given in the works of [18], [12] and [19] respectively.

4. Solution of q-difference equation with generalized composite fractional q-derivative

Theorem 4.1. *Consider the following q-initial value problem with $D_q^{\alpha,\beta;\nu}$, the generalized composite fractional q-derivative defined by* (18)

$$D_q^{\alpha,\beta;\nu}y(t)-\lambda y(t)=g(t),\ m-1<\alpha,\beta\le m,0\le\nu\le 1,m\in\mathbb{N},\tag{30}$$

λ is a real number and $g(t)$ is some known function, with initial conditions

$$D_q^{m-k-1}I_q^{(1-\nu)(m-\alpha)}y(t)\Big|_{t\to 0^+}=y_k;\ k=0,1,2,\ldots,m-1.\tag{31}$$

The solution of above problem is given by

$$\begin{aligned}y(t)=&\int_0^t u^{\alpha+\nu(\beta-\alpha)-1}e_{\alpha+\nu(\beta-\alpha),\alpha+\nu(\beta-\alpha)}\left(\lambda u^{\alpha+\nu(\beta-\alpha)};q\right)g\left[t-qu\right]d_qu\\&+\sum_{k=0}^{m-1}y_k\,t^{\alpha+\nu(m-\alpha)-k-1}e_{\alpha+\nu(\beta-\alpha),\alpha+\nu(m-\alpha)-k}\left(\lambda t^{\alpha+\nu(\beta-\alpha)};q\right),\end{aligned}\tag{32}$$

with $\left|\lambda\left\{t(1-q)\right\}^{\alpha+\nu(\beta-\alpha)}\right|<1.$

Proof. Taking q-Laplace transform of (30), using Theorem 3.5 and initial conditions (31), we get

$$\widetilde{y}(s) = \frac{(1-q)^{\alpha+\nu(\beta-\alpha)}}{s^{\alpha+\nu(\beta-\alpha)}} \frac{\widetilde{g}(s)}{\left\{1-\lambda\frac{(1-q)^{\alpha+\nu(\beta-\alpha)}}{s^{\alpha+\nu(\beta-\alpha)}}\right\}} + \frac{\sum_{k=0}^{m-1} y_k \frac{s^{k-\alpha-\nu(m-\alpha)}}{(1-q)^{k-\alpha-\nu(m-\alpha)+1}}}{\left\{1-\lambda\frac{(1-q)^{\alpha+\nu(\beta-\alpha)}}{s^{\alpha+\nu(\beta-\alpha)}}\right\}}, \tag{33}$$

where $\widetilde{y}(s)$ is q-Laplace transform of $y(t)$. Next, we write the binomial expression occurring in (33) in series form and rewrite (33) as

$$\widetilde{y}(s) = \sum_{r=0}^{\infty} \lambda^r \frac{(1-q)^{\alpha+\nu(\beta-\alpha)+r(\alpha+\nu(\beta-\alpha))}}{s^{\alpha+\nu(\beta-\alpha)+r(\alpha+\nu(\beta-\alpha))}} \widetilde{g}(s) + \sum_{k=0}^{m-1}\sum_{r=0}^{\infty} y_k \lambda^r \frac{(1-q)^{\alpha+\nu(m-\alpha)-k+r(\alpha+\nu(\beta-\alpha))-1}}{s^{\alpha+\nu(m-\alpha)-k+r(\alpha+\nu(\beta-\alpha))}}. \tag{34}$$

We take q-Laplace inversion of (34) and using the result (15) and (16), we arrive at the following

$$y(t) = \int_0^t u^{\alpha+\nu(\beta-\alpha)-1} \sum_{r=0}^{\infty} \frac{\left(\lambda u^{\alpha+\nu(\beta-\alpha)}\right)^r}{\Gamma_q(\alpha+\nu(\beta-\alpha)+r(\alpha+\nu(\beta-\alpha)))} g[t-qu]\, d_q u + \sum_{k=0}^{m-1} y_k t^{\alpha+\nu(m-\alpha)-k-1} \sum_{r=0}^{\infty} \frac{\left(\lambda t^{\alpha+\nu(\beta-\alpha)}\right)^r}{\Gamma_q(\alpha+\nu(m-\alpha)-k+r(\alpha+\nu(\beta-\alpha)))}. \tag{35}$$

In view of the q-Mittag-Leffler function [5] defined as

$$e_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n+\beta)}, \quad \left(\left|z(1-q)^{\alpha}\right| < 1\right), \tag{36}$$

we arrive at (32). □

Special Cases:
(i) If we take $\alpha = \beta$ in Theorem 4.1, we obtain the following result

Corollary 4.2. *Consider the q-initial value problem with* $D_q^{\alpha,\nu}$ *is the composite fractional q-derivative defined by* (20)

$$D_q^{\alpha,\nu} y(t) - \lambda y(t) = g(t), \; m-1 < \alpha \le m, \; 0 \le \nu \le 1, m \in \mathbb{N}. \tag{37}$$

λ *is a real number and* $g(t)$ *is some known function, with initial conditions*

$$D_q^{k-m-1} I_q^{(1-\nu)(m-\alpha)} y(t)\Big|_{t\to 0} = y_k; \; k = 0,1,2,\dots,m-1. \tag{38}$$

The solution of above problem is given by

$$y(t)=\int_0^t u^{\alpha-1}\,e_{\alpha,\alpha}\left(\lambda u^{\alpha};q\right)\,g\left[t-qu\right]d_qu$$
$$+\sum_{k=0}^{m-1} y_k\,t^{\alpha+\nu(m-\alpha)-k-1}e_{\alpha,\alpha+\nu(m-\alpha)-k}\left(\lambda t^{\alpha};q\right), \tag{39}$$

with $\left|\lambda\left\{t\left(1-q\right)\right\}^{\alpha}\right|<1.$

In the above result if we take the limit as $q\to 1$, we arrive at a result essentially similar to the result given in [12]. If we further take $g(t)=0$ and $m=1$, we arrive at the generalized fractional relaxation problem considered in the book by Hilfer [13].

(ii) On taking $\nu=0$ in Theorem 4.1, we get the following result

Corollary 4.3. *Consider the q-initial value problem with D_q^{α} is the Riemann-Liouville type fractional q-derivative of order α defined by* (5)

$$D_q^{\alpha}y(t)-\lambda y(t)=g(t),\; m-1<\alpha\le m, m\in\mathbb{N}. \tag{40}$$

λ is a real number and $g(t)$is some known function, with initial conditions

$$D_q^{k-m-1}I_q^{(m-\alpha)}y(t)\Big|_{t=0}=y_k;\; k=0,1,2,\ldots,m-1. \tag{41}$$

The solution of above problem is given by

$$y(t)=\int_0^t u^{\alpha-1}\,e_{\alpha,\alpha}\left(\lambda u^{\alpha};q\right)g\left[t-qu\right]d_qu+\sum_{k=0}^{m-1} y_k\,t^{\alpha-k-1}e_{\alpha,\alpha-k}\left(\lambda t^{\alpha};q\right), \tag{42}$$

with $\left|\lambda\left\{t\left(1-q\right)\right\}^{\alpha}\right|<1.$

(iii) On taking $\nu=1$ in Theorem 4.1, we get the following result

Corollary 4.4. *Consider the q-initial value problem with ${}_*D_q^{\beta}$ is the Caputo type fractional q-derivative of order β defined by* (6).

$${}_*D_q^{\beta}y(t)-\lambda y(t)=g(t),\; m-1<\beta\le m,\; m\in\mathbb{N}, \tag{43}$$

λ is a real number and $g(t)$ is some known function, with initial conditions

$${}_*D_q^{k-m-1}y(t)\big|_{t=0}=y_k;\; k=0,1,2,\ldots,m-1. \tag{44}$$

The solution of above problem is given by

$$y(t) = \int_0^t u^{\beta-1} e_{\beta,\beta}\left(\lambda u^{\beta};q\right) g[t-qu]\, d_q u + \sum_{k=0}^{m-1} y_k t^{m-k-1} e_{\beta,m-k}\left(\lambda t^{\beta};q\right), \tag{45}$$

with $\left|\lambda\{t(1-q)\}^{\beta}\right| < 1.$

(iv) If we let $q \to 1$ in Theorem 4.1, we obtain the following result

Corollary 4.5. *Consider the initial value problem with $D^{\alpha,\beta;\nu}$ is the new generalized fractional derivative defined by* (19).

$$D^{\alpha,\beta;\nu} y(t) - \lambda y(t) = g(t),\; m-1 < \alpha \le m, m \in \mathbb{N}. \tag{46}$$

λ is a real number and $g(t)$is some known function, with initial conditions

$$D^{m-k-1} I^{(1-\nu)(m-\alpha)} y(t)\Big|_{t=0} = y_k;\; k = 0,1,2,\ldots,m-1. \tag{47}$$

The solution of above problem is given by

$$\begin{aligned} y(t) = &\int_0^t u^{\alpha+\nu(\beta-\alpha)-1} E_{\alpha+\nu(\beta-\alpha),\alpha+\nu(\beta-\alpha)}\left(\lambda u^{\alpha+\nu(\beta-\alpha)}\right) g[t-u]\, du \\ &+ \sum_{k=0}^{m-1} y_k t^{\alpha+\nu(m-\alpha)-k-1} E_{\alpha+\nu(\beta-\alpha),\alpha+\nu(m-\alpha)-k}\left(\lambda t^{\alpha+\nu(\beta-\alpha)}\right). \end{aligned} \tag{48}$$

Here, $E_{\alpha,\beta}$ is the Mittag-Leffler function defined as [15]

$$E_{\alpha,\beta} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{49}$$

where $z, \beta \in \mathbb{C}$, $Re(\alpha) > 0$.

5. Conclusion

In this paper, we have defined a generalized composite fractional q-derivative which is a q-extension of generalized Riemann Liouville fractional derivative defined by Hilfer in 2009. It also provides generalization of both Riemann-Liouville and Caputo type fractional q-derivatives. We study some basic properties for this derivative which are useful in its applications to physical problems. We also solve a q-difference equation with this fractional q-derivative using q-Laplace transform.

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