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HERMITE-HADAMARD TYPE INEQUALITIES FOR r -CONVEX FUNCTIONS IN q -CALCULUS

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The aim of this work is to establish the q -analogue of the Hermite-Hadamard's inequalities for convex functions and r -convex functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2},$$

is known as the Hermite-Hadamard's inequality (see [6] for more details). It was generalized in [11] to an r -convex positive function which is defined on an interval $[a, b]$. A positive function is called r -convex on $[a, b]$, if for all $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq (t(f(x))^r + (1-t)(f(y))^r)^{\frac{1}{r}}, \quad r > 0.$$

It is obvious 1-convex functions are classical convex functions. It should be noted that if f is r -convex function, then f^r is convex function.

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For $0 < q < 1$, the q -Jackson integral from 0 to b is defined by [8]

$$\int_0^b f(x) d_q x = (1-q)b \sum_{n=0}^{\infty} f(bq^n) q^n \quad (1)$$

provided the sum converge absolutely.

The q -Jackson integral in a generic interval $[a, b]$ is given by [8]

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \quad (2)$$

In [12], the authors presented a Riemann-type q -integral by

$$\begin{aligned} \int_a^b f(x) d_q^R x &= (b-a)(1-q) \sum_{k=0}^{\infty} f(a+(b-a)q^k) q^k \\ &= (b-a) \int_0^1 f((1-t)a+tb) d_q t. \end{aligned} \quad (3)$$

For $x \in \mathbb{C}$, we note $[x]_q = \frac{1-q^x}{1-q}$ and $(x, q)_n = \prod_{k=0}^{n-1} (1-xq^k)$, $n = 1, 2, \dots, \infty$.

The q -Binomial theorem is given by

$$\sum_{n=0}^{\infty} \frac{(a, q)_n}{(q, q)_n} z^n = \frac{(az, q)_{\infty}}{(z, q)_{\infty}}, \quad |z| < 1. \quad (4)$$

In [10], Pachpatte established new Hadamard-type inequalities for products of convex functions:

Let $f, g : [a, b] \rightarrow [0, \infty)$ be convex functions on $[a, b] \subset \mathbb{R}$, $a < b$. Then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3} \left(f(a)g(a) + f(b)g(b) \right) + \frac{1}{6} \left(f(a)g(b) + f(b)g(a) \right). \quad (5)$$

In [13], the authors showed the following integral inequalities of Hadamard type for r -convex functions:

Let $f : [a, b] \rightarrow [0, \infty)$ be r -convex function on $[a, b]$. Then for $0 < r \leq 1$

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{r}{r+1} ([f(a)]^r + [f(b)]^r)^{\frac{1}{r}}.$$

Moreover, if $f, g : [a, b] \rightarrow [0, \infty)$ be r -convex and s -convex functions respectively on $[a, b]$. Then for $0 \leq r, s \leq 2$:

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)dx &\leq \frac{1}{2} \frac{r}{r+2} ([f(a)]^r + [f(b)]^r)^{\frac{2}{r}} \\ &\quad + \frac{1}{2} \frac{s}{s+2} ([g(a)]^s + [g(b)]^s)^{\frac{2}{s}}, \end{aligned}$$

another, if f is r -convex and g is s -convex functions. Then the following inequality holds if $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \left(\frac{[f(a)]^r + [f(b)]^r}{2} \right)^{\frac{1}{r}} \left(\frac{[g(a)]^s + [g(b)]^s}{2} \right)^{\frac{1}{s}}.$$

For other Hermite-Hadamard type inequalities, see ([1],[2],[3],[4],[7],[13]) and the references cited therein.

In this paper, we use the Riemann-type q -integral to develop some new results of Hermite-Hadamard inequality for convex functions and r -convex functions.

2. Main Results

Theorem 2.1. *Let $f, g : [a, b] \rightarrow [0, \infty)$ be two convex functions on $[a, b] \subset \mathbb{R}$. Then*

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)d_q^R x &\leq \left(1 + \frac{1}{[3]_q} - \frac{2}{[2]_q} \right) f(a)g(a) + \frac{1}{[3]_q} f(b)g(b) \\ &\quad + \left(\frac{1}{[2]_q} - \frac{1}{[3]_q} \right) (f(a)g(b) + f(b)g(a)). \end{aligned}$$

Proof. Since f and g are convex functions on $[a, b]$ and for all $t \in [0, 1]$, we have

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$$

and

$$g((1-t)a + tb) \leq (1-t)g(a) + tg(b).$$

Then

$$\begin{aligned} f((1-t)a + tb)g((1-t)a + tb) &\leq (1-t)^2 f(a)g(a) + t^2 f(b)g(b) \\ &\quad + t(1-t) (f(a)g(b) + f(b)g(a)). \end{aligned}$$

By integrating the inequality on $[0, 1]$, we obtain

$$\begin{aligned} \int_0^1 f((1-t)a + tb)g((1-t)a + tb)d_q t &\leq f(a)g(a) \int_0^1 (1-t)^2 d_q t \\ &\quad + f(b)g(b) \int_0^1 t^2 d_q t + (f(a)g(b) + f(b)g(a)) \int_0^1 t(1-t)d_q t \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)d_q^R x &\leq \left(1 + \frac{1}{[3]_q} - \frac{2}{[2]_q} \right) f(a)g(a) + \frac{1}{[3]_q} f(b)g(b) \\ &\quad + \left(\frac{1}{[2]_q} - \frac{1}{[3]_q} \right) (f(a)g(b) + f(b)g(a)). \end{aligned}$$

Theorem 2.1 is thus proved. □

Remark 2.2. If $q \rightarrow 1$ in Theorem 2.1, we obtain the inequality (5).

We need the following Lemma for Theorem 2.4:

Lemma 2.3. For $p \geq 1$, the following inequality is valid

$$\int_0^1 (1-t)^p d_q t \leq \frac{q}{[p+1]_q}.$$

Proof. Let

$$f(t) = (1-t)_q^{(p)} := \frac{(t, q)_\infty}{(tq^p, q)_\infty}; \quad p \in \mathbb{R}, p \geq 1. \tag{6}$$

The q -derivative of f is given by

$$D_q f(t) = D_q (1-t)_q^{(p)} = -[p]_q (1-tq)^{(p-1)}. \tag{7}$$

On the other hand, we have

$$(1-t)^{-p} = 1 + \sum_{k=1}^{\infty} \frac{p(p+1) \cdots (p+k-1)}{k!} t^k, \quad t \in [0, 1], \tag{8}$$

and from the q -Binomial theorem (4)

$$\frac{1}{(1-t)_q^{(p)}} = \frac{(tq^p, q)_\infty}{(t, q)_\infty} = 1 + \sum_{k=1}^{\infty} \frac{(q^p, q)_k}{(q, q)_k} t^k. \tag{9}$$

We consider now the function

$$g_k(u) = \frac{1-u^{p+k-1}}{1-u^k}; \quad 0 < u < 1.$$

The function g_k is increasing on $]0, 1[$ and $\lim_{u \rightarrow 1^-} g_k(u) = \frac{p+k-1}{k}$, so

$$\frac{1-q^{p+k-1}}{1-q^k} \leq \frac{p+k-1}{k}, \quad 0 < q < 1. \tag{10}$$

From the relations (8), (9) and (10), we deduce that

$$(1-t)^p \leq \frac{(t, q)_\infty}{(tq^p, q)_\infty} = (1-t)_q^{(p)}. \tag{11}$$

We integrate both sides of (11) on $[0, 1]$, we obtain

$$\int_0^1 (1-t)^p d_q t \leq \int_0^1 (1-t)_q^{(p)} d_q t = \frac{q}{[p+1]_q}.$$

Lemma 2.1 is thus proved. □

Theorem 2.4. *Let $f : [a, b] \rightarrow [0, \infty)$ be r -convex function on $[a, b]$. Then the following inequality holds for $0 < r \leq 1$:*

$$\frac{1}{b-a} \int_a^b f(x) d_q^R x \leq \frac{1}{[\frac{1}{r} + 1]_q} \left([qf(a)]^r + [f(b)]^r \right)^{\frac{1}{r}}.$$

Proof. According to the definition of r -convex, for all $t \in [0, 1]$, we have

$$f((1-t)a + tb) \leq \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{1}{r}}.$$

By integrating the inequality on $[0, 1]$, we obtain

$$\int_0^1 f((1-t)a + tb) d_q t \leq \int_0^1 \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{1}{r}} d_q t.$$

Using Minkowski's inequality, we have

$$\begin{aligned} \int_0^1 \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{1}{r}} d_q t \\ \leq \left(\left(\int_0^1 (1-t)^{\frac{1}{r}} d_q t \right)^r [f(a)]^r + \left(\int_0^1 t^{\frac{1}{r}} d_q t \right)^r [f(b)]^r \right)^{\frac{1}{r}}, \end{aligned}$$

and by Lemma 2.3, we have

$$\left(\int_0^1 (1-t)^{\frac{1}{r}} d_q t \right)^r \leq \left(\frac{q}{[\frac{1}{r} + 1]_q} \right)^r.$$

Thus

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) d_q^R x &\leq \left(\left(\frac{q}{[\frac{1}{r} + 1]_q} \right)^r [f(a)]^r + \frac{1}{[\frac{1}{r} + 1]_q} [f(b)]^r \right)^{\frac{1}{r}} \\ &\leq \frac{1}{[\frac{1}{r} + 1]_q} (q^r [f(a)]^r + [f(b)]^r)^{\frac{1}{r}}. \end{aligned}$$

Theorem 2.4 is thus proved. □

Theorem 2.5. *Let $f, g : [a, b] \rightarrow [0, \infty)$ be r -convex and s -convex functions respectively on $[a, b]$. Then the following inequality holds for $0 < r, s \leq 2$:*

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x)g(x) d_q^R x &\leq \left(\frac{1}{[\frac{2}{r} + 1]_q} \right) \left([q^{\frac{1}{2}} f(a)]^r + [f(b)]^r \right)^{\frac{2}{r}} \\ &\quad + \left(\frac{1}{[\frac{2}{s} + 1]_q} \right) \left([q^{\frac{1}{2}} g(a)]^s + [g(b)]^s \right)^{\frac{2}{s}}. \end{aligned}$$

Proof. From the definition of r -convex function we can write:

$$f((1-t)a+tb) \leq ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}}$$

and

$$g((1-t)a+tb) \leq ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}}$$

for all $t \in [0, 1]$ and $r, s > 0$.

Then

$$\begin{aligned} f((1-t)a+tb)g((1-t)a+tb) \\ \leq \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{1}{r}} \left((1-t)[g(a)]^s + t[g(b)]^s \right)^{\frac{1}{s}}. \end{aligned}$$

Integrating both sides with respect to t on $[0, 1]$, we obtain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)d_q^R x \\ \leq \int_0^1 \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{1}{r}} \left((1-t)[g(a)]^s + t[g(b)]^s \right)^{\frac{1}{s}} d_q t. \end{aligned}$$

Using Cauchy's inequality, we have

$$\begin{aligned} \int_0^1 \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{1}{r}} \left((1-t)[g(a)]^s + t[g(b)]^s \right)^{\frac{1}{s}} d_q t \\ \leq \frac{1}{2} \int_0^1 \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{2}{r}} d_q t + \frac{1}{2} \int_0^1 \left((1-t)[g(a)]^s + t[g(b)]^s \right)^{\frac{2}{s}} d_q t. \end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned} \int_0^1 \left((1-t)[f(a)]^r + t[f(b)]^r \right)^{\frac{2}{r}} d_q t \\ \leq \left(\left(\int_0^1 (1-t)^{\frac{2}{r}} d_q t \right)^{\frac{r}{2}} [f(a)]^r + \left(\int_0^1 t^{\frac{2}{r}} d_q t \right)^{\frac{r}{2}} [f(b)]^r \right)^{\frac{2}{r}} \\ = \left(\left(\frac{q}{[\frac{2}{r} + 1]_q} \right)^{\frac{r}{2}} [f(a)]^r + \frac{1}{[\frac{2}{r} + 1]_q} [f(b)]^r \right)^{\frac{2}{r}}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \int_0^1 \left((1-t)[g(a)]^s + t[g(b)]^s \right)^{\frac{2}{s}} d_q t \\ \leq \left(\left(\int_0^1 (1-t)^{\frac{2}{s}} d_q t \right)^{\frac{s}{2}} [g(a)]^s + \left(\int_0^1 t^{\frac{2}{s}} d_q t \right)^{\frac{s}{2}} [g(b)]^s \right)^{\frac{2}{s}} \\ = \left(\left(\frac{q}{[\frac{2}{s} + 1]_q} \right)^{\frac{s}{2}} [g(a)]^s + \frac{1}{[\frac{2}{s} + 1]_q} [g(b)]^s \right)^{\frac{2}{s}}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x)g(x)d_q^R x &\leq \left(\left(\frac{q}{[\frac{2}{r} + 1]_q} \right)^{\frac{r}{2}} [f(a)]^r + \frac{1}{[\frac{2}{r} + 1]_q} [f(b)]^r \right)^{\frac{2}{r}} \\ &\quad + \left(\left(\frac{q}{[\frac{2}{s} + 1]_q} \right)^{\frac{s}{2}} [g(a)]^s + \frac{1}{[\frac{2}{s} + 1]_q} [g(b)]^s \right)^{\frac{2}{s}}. \end{aligned}$$

Theorem 2.5 is thus proved. □

Theorem 2.6. *Let $f, g : [a, b] \rightarrow [0, \infty)$ be r -convex and s -convex functions respectively on $[a, b]$. Then the following inequality holds if $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$:*

$$\frac{1}{b-a} \int_a^b f(x)g(x)d_q^R x \leq \left(\frac{q[f(a)]^r + [f(b)]^r}{[2]_q} \right)^{\frac{1}{r}} \left(\frac{q[g(a)]^s + [g(b)]^s}{[2]_q} \right)^{\frac{1}{s}}.$$

Proof. Since f is r -convex function and g is s -convex function, we have

$$f((1-t)a + tb) \leq ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}}$$

and

$$g((1-t)a + tb) \leq ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}}$$

for all $t \in [0, 1]$ and $r, s > 0$.

Integrate t over $[0, 1]$, we obtain

$$\begin{aligned} &\int_0^1 f((1-t)a + tb)g((1-t)a + tb)d_q t \\ &\leq \int_0^1 ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}} ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}} d_q t. \end{aligned}$$

Using q -analogue of Hölder type inequality (see[9]), we have

$$\begin{aligned} &\int_0^1 ((1-t)[f(a)]^r + t[f(b)]^r)^{\frac{1}{r}} ((1-t)[g(a)]^s + t[g(b)]^s)^{\frac{1}{s}} d_q t \\ &\leq \left(\int_0^1 (1-t)[f(a)]^r + t[f(b)]^r d_q t \right)^{\frac{1}{r}} \left(\int_0^1 (1-t)[g(a)]^s + t[g(b)]^s d_q t \right)^{\frac{1}{s}} \\ &= \left(\frac{q[f(a)]^r + [f(b)]^r}{[2]_q} \right)^{\frac{1}{r}} \left(\frac{q[g(a)]^s + [g(b)]^s}{[2]_q} \right)^{\frac{1}{s}}. \end{aligned}$$

Thus

$$\frac{1}{b-a} \int_a^b f(t)g(t)d_q^R t \leq \left(\frac{q[f(a)]^r + [f(b)]^r}{[2]_q} \right)^{\frac{1}{r}} \left(\frac{q[g(a)]^s + [g(b)]^s}{[2]_q} \right)^{\frac{1}{s}}.$$

Theorem 2.6 is thus proved. □

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