LE MATEMATICHE
Vol. ??? (????) – Fasc. ??, pp. ??–??

EXPLICIT HIGHER REGULARITY ON A CAUCHY PROBLEM WITH MIXED NEUMANN-POWER TYPE BOUNDARY CONDITIONS

LUISA CONSIGLIERI

We investigate the regularity in L^p $(p > 2)$ of the gradient of any weak solution of a Cauchy problem with mixed Neumann-power type boundary conditions. Under suitable assumptions we prove the existence of weak solutions that satisfy explicit estimates. Some considerations on the steady-state regularity are discussed.

1. Introduction

In the mathematical literature, the dependence on the data is commonly hidden on the universal constants. These constants that are involved in the estimates are systematically assumed abstract, *i.e.* they may change their numerical value from line to line throughout the whole study in concern. Our objective is to find explicit estimates (also known as quantitative estimates [7]) such that allow its real and true application to other fields of science.

In the study of the regularity on the initial-boundary value problem for the second order differential equation in divergence form, at least three shortcomings appear from the real world applications. They are namely discontinuous leading coefficient, nonlinear monotone boundary conditions, and nonsmooth

Entrato in redazione: 10 giugno 2014
AMS 2010 Subject Classification: 35K20, 49N60.
Keywords: higher regularity, Caccioppoli estimate, Gehring-Giaquinta-Modica theory, L^p maximal regularity.

Lipschitz domain. One of the approaches in the investigation of regularity is based on the difference quotient technique. We refer to [14, 15, 27] where there are no boundary terms. The elliptic regularity in the halfspace can be found in [25]. For Neumann-type boundary conditions, an arbitrary bounded domain is not globally invariant with respect to translations. The difference quotient technique is only allowed by a suitable localization procedure [32]. Even the interior regularity requires the differentiability of coefficient, which is not fulfilled by our coefficient. The realization of the Laplace operator with generalized nonlinear Robin boundary conditions can be found in [6].

Also by the localization method, the higher regularity of the gradient is obtained via the reverse Hölder inequality with increasing supports (known as Gehring-Giaquinta-Modica theory, cf. [3, 4, 22, 28, 30] and the references therein). Here, we adopt this approach to determine explicit estimates for the Cauchy problem inspired in the nonlinear heat equation with the Neumann condition on one part of the boundary of the domain, and the power law condition on the remaining part of the boundary that includes the radiative effects [9, 13]. Also the constants involved in $L^{p,\infty}$-estimate are determined.

Some considerations on the steady-state case are discussed in Section 7.

2. Maximal parabolic regularity on X

Let $[0,T]\subset\mathbb{R}$ be the time interval with $T>0$, and $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a (bounded) domain of class C^1. The boundary $\partial\Omega$ is decomposed into two disjoint open subsets, namely Γ and $\partial\Omega\setminus\bar{\Gamma}$. Moreover we set $Q_T=\Omega\times]0,T[$, and $\Sigma_T=\Gamma\times]0,T[$.

In the presence of Lebesgue, Sobolev, and Bochner spaces, the functional framework is

$$\begin{aligned} L^{p,\infty}(Q_T) &= L^\infty(0,T;L^p(\Omega)); \\ V_{p,\ell}(\Omega) &= \{v\in W^{1,p}(\Omega):\ v|_\Gamma\in L^\ell(\Gamma)\}; \\ V_{p,\ell}(Q_T) &= \{v\in L^p(0,T;W^{1,p}(\Omega)):\ v|_{\Sigma_T}\in L^\ell(\Sigma_T)\}, \end{aligned}$$

for $p,\ell>1$. For $\ell\leq p_*$, with $p_*=p(n-1)/(n-p)$ if $n>p$, and any $p_*>p$ if $n=p$, observe that $V_{p,\ell}(Q_T)=L^p(0,T;W^{1,p}(\Omega))$ due to the trace embedding $W^{1,p}(\Omega)\hookrightarrow L^{p_*}(\Gamma)$.

Let us introduce the definition of a closed operator that admits maximal parabolic regularity on a Banach space X [2, 24].

Definition 2.1. We say that B *admits maximal parabolic regularity on* X if B is a closed (not necessarily linear) operator in X with dense domain $D(B)$, and for any $F\in L^p(0,T;X)$ $(1<p<\infty)$ there exists a unique function $u\in$

$L^p(0,T;D(B))$, such that $\partial_t u \in L^p(0,T;X)$, solving the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \frac{d}{dt}u(t) + Bu(t) = F(t), & \text{a.e. } t \in]0,T[\\ u(0) = u_0 \in (D(B),X)_{1/p,p} = (X,D(B))_{1/p',p} \end{cases}$$

where $(D(B),X)_{1/p,p} = \{v(0) : \ v \in L^p(0,T;D(B)), \ \partial_t v \in L^p(0,T;X)\}$ represents the interpolation space [19, Theorem 5.12], and $D(B)$ is endowed with the graph norm.

Recall that a densely defined closed operator B, such that there exists a unique solution of (ACP) for all initial values in $D(B)$, may be not a generator [1]. For every $u_0 \in D(B)$, if B is linear then this abstract Cauchy problem (ACP) has the mild solution $u \in C([0,T[;H)$ that verifies the variation of constants formula

$$u(t) = \exp[-tB]u_0 + \int_0^t \exp[-(t-\tau)B]F(\tau)\mathrm{d}\tau, \qquad t \in [0,T[.$$

Moreover, the fractional powers $B^{1/2}$ and $B^{-1/2}$ exist and global strong solutions can be obtained [31].

Here we consider the nonlinear operator $B : V_{2,\ell}(\Omega) \to [V_{2,\ell}(\Omega)]'$ defined by

$$\langle Bu, v\rangle := \int_\Omega (\mathsf{A}\nabla u)\cdot\nabla v \mathrm{dx} + \int_\Gamma b(u)uv\mathrm{ds}, \quad \forall v \in V_{2,\ell}(\Omega),$$

with the assumptions on the coefficients A and b being

- $\mathsf{A} = [A_{ij}]_{i,j=1,\cdots,n}$ is a bounded measurable $(n\times n)$ matrix-valued function such that

$$\exists a_\# > 0, \quad A_{ij}(x)\xi_i\xi_j \geq a_\#|\xi|^2, \qquad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n, \tag{1}$$

  under the summation convention over repeated indices: $\mathsf{A}\mathbf{a}\cdot\mathbf{b} = A_{ij}a_j b_i = \mathbf{b}^\top\mathsf{A}\mathbf{a}$.

- $b : \Gamma\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, that is, it is measurable in Γ and continuous in $\mathbb{R}$. Moreover, b is monotone with respect with the second variable, and it verifies for some $\ell \geq 2$

$$\exists b_\# > 0 : \qquad b(\cdot,s) \geq b_\#|s|^{\ell-2}; \tag{2}$$
$$\exists \gamma_1 \in L^\infty(\Gamma) : \qquad |b(x,s)| \leq \gamma_1(x)|s|^{\ell-2}, \tag{3}$$

  for all $s,t \in \mathbb{R}$, and a.e. in Γ.

Set

$$a^{\#} = \|\mathsf{A}\|_{\infty,\Omega}, \quad b^{\#} = \|\gamma_1\|_{\infty,\Gamma}. \tag{4}$$

Under the assumptions (1)-(3), B is monotone, hemicontinuous, bounded, and coercive. The existence and uniqueness of u of (ACP) is consequence of [33, Theorem 4.1, p. 120] if provided by $u_0 \in L^2(\Omega)$. In particular, B admits maximal parabolic regularity on $[V_{2,\ell}(\Omega)]'$, and its negative $-B$ generates a C_0-semigroup on $[V_{2,\ell}(\Omega)]'$.

We seek for the L^p-integrability of the gradient of $u \in V_{2,\ell}(Q_T)$ that verifies the variational formulation

$$\int_0^T \langle \partial_t u, v\rangle \mathrm{dt} + \int_{Q_T} (\mathsf{A}\nabla u)\cdot\nabla v \mathrm{dxdt} + \int_{\Sigma_T} b(u)uv \mathrm{dsdt} = \\ = \int_{Q_T} \mathbf{f}\cdot\nabla v \mathrm{dxdt} + \int_{Q_T} fv \mathrm{dxdt} + \int_{\Sigma_T} hv \mathrm{dsdt} \tag{5}$$

for every $v \in V_{2,\ell}(Q_T)$. The symbol $\langle\cdot,\cdot\rangle$ stands for the duality pairing in which is meaningful.

Let $p,\ell \geq 2$. We denote by $\mathcal{W}_{p,\ell}$ the set of functionals $F \in [V_{p',\ell}(Q_T)]'$ that are the form defined by

$$F(v) := \int_{Q_T} \mathbf{f}\cdot\nabla v \mathrm{dxdt} + \int_{Q_T} fv \mathrm{dxdt} + \int_{\Sigma_T} hv \mathrm{dsdt}, \quad \forall v \in V_{p',\ell}(Q_T),$$

with $\mathbf{f} \in L^p(0,T;\mathbf{L}^p(\Omega))$, $f \in L^p(Q_T)$, and $h \in L^{\ell/(\ell-1)}(\Sigma_T)$. The identification $L^p(0,T;[V_{p',\ell}(\Omega)]') \equiv [L^{p'}(0,T;V_{p',\ell}(\Omega))]'$ is due to the Phillips Theorem if provided that $V_{p,\ell}(\Omega)$ is reflexive and $1 < p < \infty$ [33, p. 104]. We simply write $\mathcal{M}_p = \mathcal{M}_{p,2}$, and $V_p(Q_T) = V_{p,2}(Q_T)$.

We state our main result in the following theorem.

Theorem 2.2. *Let Ω be a C^1 domain, $T > 0$, and the assumptions (1)-(3) be fulfilled. There exists $\delta > 0$ such that for any $p \in [2, 2+\delta]$ if $\mathbf{f} \in \mathbf{L}^{2+\delta}(Q_T)$, $f \in L^{2+\delta}(Q_T)$, $h \in L^{2+\delta}(\Sigma_T)$ and $u_0 \in L^{2+\delta}(\Omega)$, then there exists a function u in $L^{p,\infty}(Q_T) \cap V_{p,\ell+p-2}(Q_T)$ which is solution of (5) such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u\|^p_{p,\Omega}(t) \leq \mathcal{G}(a_\#, b_\#, p)\exp\left[(p-2+(p-1)v_0^{1/(p-1)})T\right]; \tag{6}$$

$$\|u\|^{\ell+p-2}_{\ell+p-2,\Sigma_T} \leq (b_\#)^{-1}\mathcal{E}(a_\#, b_\#, p); \tag{7}$$

$$\|\nabla u\|_{p,Q_T} \leq \mathcal{M}(a_\#, b_\#), \tag{8}$$

with

$$\begin{aligned}
\mathcal{G}(a_\#,b_\#,p) &= \|u_0\|^p_{p,\Omega}+\frac{1}{\nu_0}\|f\|^p_{p,Q_T}+\left(\frac{p-1}{a_\#}\right)^{p/2}\|\mathbf{f}\|^p_{p,Q_T}+\\
&+\frac{p(\ell-1)}{(\ell+p-2)b_\#^{(p-1)/(\ell-1)}}\int_{\Sigma_T}|h|^{\frac{\ell+p-2}{\ell-1}}\mathrm{dsdt};\\
\mathcal{E}(a_\#,b_\#,p) &= \mathcal{G}(a_\#,b_\#,p)\Big(1+(p-2+(p-1)\nu_0^{1/(p-1)})T\times\\
&\times\exp\left[(p-2+(p-1)\nu_0^{1/(p-1)})T\right]\Big); \qquad (9)\\
\mathcal{M}(a_\#,b_\#) &= C(n)\left[\sqrt{\frac{\mathcal{E}(a_\#,b_\#,2)}{a_\#}}+\frac{(1+\upsilon)^{1/p}}{a_\#}\Big(\sqrt{1+a_\#}\|\mathbf{f}\|_{p,Q_T}+\right.\\
&\left.+\frac{1}{\sqrt{\nu_0}}\|f\|_{p,Q_T}+\sqrt{1+a_\#}K_{2n/(n+1)}\|h\|_{p,\Sigma_T}\Big)\right]. \qquad (10)
\end{aligned}$$

Here, $\nu_0=\nu_0(f)$ is a positive constant if $f\neq 0$, and $\nu_0(0)=0$ otherwise, $C(n)$ is according to (50), $K_{2n/(n+1)}$ stands for the continuity constant of the trace embedding $W^{1,2n/(n+1)}(\Omega)\hookrightarrow L^2(\Gamma)$, and

$$\upsilon=(4^n+1)\left(\frac{6}{n+2}+2^{\frac{n(n+1)}{n+2}}+2^{\frac{n^2}{n+2}}+2^{2n+3}+2^{2(n+1)}\times\right.$$
$$\left.\times\left(\frac{2^{9n-2}}{\pi^n n^{2n}}(1-\frac{2}{n})^{n-2}\right)^{\frac{1}{n+2}}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{\frac{2}{n+2}}\left(2\left(\frac{a^\#}{a_\#}\right)^2+\frac{19}{8a_\#}\right)^{\frac{n}{n+2}}\right)^{\frac{n+2}{n}}. \qquad (11)$$

In particular, $\partial_t u\in\mathcal{W}_{p,\ell}$.

If $b_\#=0$ in (2), B is $L^2(\Omega)$-elliptic: $\langle Bw,w\rangle+a_\#\|w\|_{2,\Omega}\geq a_\#\|w\|_{1,2,\Omega}$, but is not coercive on $H^1(\Omega)$. However, it is possible to reformulate the above theorem such that similar estimates may be obtained by the Gehring-Giaquinta-Modica theory if $\ell\leq 2_*$ is provided, *i.e.* $W^{1,2}(\Omega)\hookrightarrow L^\ell(\Gamma)$ for any $\ell\geq 2$ if $n=2$, and $\ell\leq 2(n-1)/(n-2)$ if $n>2$ (cf. Remark 3.3).

3. $L^{p,\infty}(Q_T)$ and $L^{\ell+p-2}(\Sigma_T)$ estimates

Local $L^{p,\infty}(Q_T)$-estimates can be obtained under the Gehring-Giaquinta-Modica technique as can be found in [3]. Under the Moser technique as already developed in [11], $L^{p,\infty}(Q_T)$ and $L^{\ell+p-2}(\Sigma_T)$ estimates can appear as consequence of $L^\infty(Q_T)$ and $L^\infty(\Sigma_T)$ estimates, respectively. Here we provide the explicit estimates under the direct "apriori" technique in the following proposition.

Proposition 3.1. *Any function u solving (5) satisfies, for all $p \geq 2$, (6) and*

$$a_\#\||u|^{(p-2)/2}\nabla u\|^2_{2,Q_T} + b_\# \int_0^T \int_\Gamma |u|^{\ell+p-2}\mathrm{dsdt} \leq \mathcal{E}(a_\#, b_\#, p), \tag{12}$$

with $\mathcal{E}(a_\#, b_\#, p)$ being given by (9).

Proof. Fix $t \in]0,T[$ arbitrary, and let $\chi_{]0,t[} \in L^\infty(]0,T[)$ be the characteristic function. Taking $v = \chi_{]0,t[}|u|^{p-2}u$ as a test function in (5), applying the Hölder and Young inequalities, and using (1) and (2), we obtain

$$\begin{aligned}
&\frac{1}{p}\|u\|^p_{p,\Omega}(t) + a_\#(p-1)\||u|^{(p-2)/2}\nabla u\|^2_{2,Q_t} + b_\# \int_0^t \int_\Gamma |u|^{\ell+p-2}\mathrm{dsd}\tau \leq \\
&\quad \leq \frac{1}{p}\|u_0\|^p_{p,\Omega} + \int_0^t \|f\|_{p,\Omega}\|u\|^{p-1}_{p,\Omega}\mathrm{d}\tau + \frac{p-1}{2a_\#}\int_0^t \|\mathbf{f}\|^2_{p,\Omega}\|u\|^{p-2}_{p,\Omega}\mathrm{d}\tau + \\
&\quad + \frac{a_\#(p-1)}{2}\||u|^{(p-2)/2}\nabla u\|^2_{2,Q_t} + \frac{b_\#(p-1)}{\ell+p-2}\int_0^t \int_\Gamma |u|^{\ell+p-2}\mathrm{dsd}\tau + \\
&\quad + \frac{\ell-1}{(\ell+p-2)b_\#^{(p-1)/(\ell-1)}}\int_0^t \int_\Gamma |h|^{\frac{\ell+p-2}{\ell-1}}\mathrm{dsd}\tau.
\end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned}
&\frac{1}{p}\|u\|^p_{p,\Omega}(t) + \frac{a_\#(p-1)}{2}\||u|^{(p-2)/2}\nabla u\|^2_{2,Q_t} + \frac{b_\#(\ell-1)}{\ell+p-2}\int_0^t \int_\Gamma |u|^{\ell+p-2}\mathrm{dsd}\tau \\
&\leq \frac{1}{p}\left(\|u_0\|^p_{p,\Omega} + \frac{1}{\nu_0}\|f\|^p_{p,Q_t}\right) + \frac{1}{p}\left(p-2+(p-1)\nu_0^{1/(p-1)}\right)\int_0^t \|u\|^p_{p,\Omega}\mathrm{d}\tau \\
&\quad + \frac{2}{p}\left(\frac{p-1}{2a_\#}\right)^{p/2}\|\mathbf{f}\|^p_{p,Q_t} + \frac{\ell-1}{(\ell+p-2)b_\#^{(p-1)/(\ell-1)}}\int_0^t \int_\Gamma |h|^{\frac{\ell+p-2}{\ell-1}}\mathrm{dsd}\tau.
\end{aligned}$$

By Gronwall inequality, we find (6), and consequently (12) holds. □

Remark 3.2. If $f \in L^p(0,T;L^{pn/(p+n)}(\Omega))$, Proposition 3.1 remains valid with alternative estimates by considering

$$\begin{aligned}
&\int_\Omega f|u|^{p-2}u\mathrm{dx} \leq \|f\|_{pn/(p+n),\Omega} S_{p'}\left(\||u|^{p-1}\|_{p',\Omega} + (p-1)\||u|^{p-2}\nabla u\|_{p',\Omega}\right) \\
&\quad \leq S_{p'}\|f\|_{pn/(p+n),\Omega}\left(\|u\|^{p-1}_{p,\Omega} + (p-1)\|u\|^{(p-2)/2}_{p,\Omega}\||u|^{(p-2)/2}\nabla u\|_{2,\Omega}\right) \leq \\
&\quad \leq \frac{1+(p-1)^p}{p\nu_0}(S_{p'})^p\|f\|^p_{pn/(p+n),\Omega} + \frac{\nu_0^{1/p}}{2}\||u|^{(p-2)/2}\nabla u\|^2_{2,\Omega} + \\
&\quad + \left(\frac{(p-1)\nu_0^{1/(p-1)}}{p} + \frac{(p-2)\nu_0^{1/(p-2)}}{2p}\right)\|u\|^p_{p,\Omega},
\end{aligned}$$

where $S_{p'}$ ($p' < 2$) stands for the continuity constant from the Sobolev embedding $W^{1,p'}(\Omega) \hookrightarrow L^{pn/(pn-n-p)}(\Omega)$.

Remark 3.3. The estimates (6) and (8) under $b_\# = 0$ read, respectively,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u\|^p_{p,\Omega}(t) \le \mathcal{G}(a_\#,p)\exp\left[(p-1)(1+\nu_0^{1/(p-1)})T\right];$$

$$\|\nabla u\|_{p,Q_T} \le C(n)\left[\sqrt{\frac{\mathcal{G}(a_\#,2)\,(1+(1+\nu_0)T\exp[(1+\nu_0)T])}{a_\#}}+\right.$$

$$\left.+\frac{(1+\upsilon)^{1/p}}{a_\#}\left(\sqrt{1+a_\#}\left(\|\mathbf{f}\|_{p,Q_T}+K_{2n/(n+1)}\|h\|_{p,\Sigma_T}\right)+\frac{1}{\sqrt{\nu_0}}\|f\|_{p,Q_T}\right)\right],$$

with

$$\mathcal{G}(a_\#,p) = \|u_0\|^p_{p,\Omega}+\frac{1}{\nu_0}\|f\|^p_{p,Q_T}+\left(\frac{p-1}{a_\#}\right)^{p/2}\|\mathbf{f}\|^p_{p,Q_T}+$$

$$+(p-1)\left(\left(\frac{p^2}{2a_\#(p-1)}\right)^{1/(p-1)}+1\right)K^{2/(p-1)}_{2n/(n+1)}|\Omega|^{[(p-1)n]^{-1}}\|h\|^{p'}_{p',\Sigma_T}.$$

To this aim, it is sufficient to consider in the proof of Proposition 3.1

$$\int_\Gamma hu\mathrm{ds} \le \|h\|_{p',\Gamma}\||u|^{p/2}\|^{2/p}_{2,\Gamma} \le$$

$$\le \|h\|_{p',\Gamma}K^{2/p}_{2n/(n+1)}|\Omega|^{(pn)^{-1}}\left(\frac{p}{2}\||u|^{(p-2)/2}\nabla u\|_{2,\Omega}+\|u\|^{p/2}_{p,\Omega}\right)^{2/p} \le$$

$$\le \frac{1}{p}\|u\|^p_{p,\Omega}+\frac{a_\#(p-1)}{2p}\||u|^{(p-2)/2}\nabla u\|^2_{2,\Omega}+$$

$$+\frac{1}{p'}\left(\left(\frac{p^2}{2a_\#(p-1)}\right)^{1/(p-1)}+1\right)K^{2/(p-1)}_{2n/(n+1)}|\Omega|^{[(p-1)n]^{-1}}\|h\|^{p'}_{p',\Gamma}.$$

4. Auxiliary results

First, let us state a Caccioppoli-type inequality, under letting $U \in L^2(\mathbb{R})$ be defined either by $U \equiv 0$, or by

$$U(t) = \left(\int_\Omega \eta^2(x)\mathrm{dx}\right)^{-1}\int_\Omega \eta^2(x)u(x,t)\mathrm{dx}, \quad \text{if } \mathrm{supp}(\eta)\cap\Gamma = \emptyset, \tag{13}$$

where $\eta \in W^{1,\infty}_0(\mathbb{R}^n)$ satisfies $0 \le \eta \le 1$ in $\mathbb{R}^n$.

Proposition 4.1. *Let Ω be a C^1 domain, and $T > 0$. If there exists $\delta > 0$ such that $\mathbf{f} \in \mathbf{L}^{2+\delta}(Q_T)$, $f \in L^{2+\delta}(Q_T)$, $h \in L^{2+\delta}(\Sigma_T)$ and $u_0 \in L^{2+\delta}(\Omega)$, then any* *function $u\in V_{2,\ell}(Q_T)\cap C([0,T];[V_{2,\ell}(\Omega)]')$ solving (5) verifies*

$$\begin{aligned}\operatorname*{ess\ sup}_{t\in]t_0-R^2,t_0+R^2[}&\|\eta(u-U)\|^2_{2,\Omega}(t)+a_\#(1-(\nu_1+\nu_2))\|\nabla u\|^2_{2,Q_r(z_0)}\leq\\ \leq&\left(\frac{2(a^\#)^2}{a_\#}+2+\nu_0+\frac{3\nu_2}{2}\right)\frac{2}{(R-r)^2}\|\eta(u-U)\|^2_{2,Q_R(z_0)}+\frac{1}{\nu_0}\|f\|^2_{2,Q_R(z_0)}\\ &+\left(\frac{1}{a_\#\nu_1}+2\right)\|\mathbf{f}\|^2_{2,Q_R(z_0)}+2R\frac{(K_{2n/(n+1)})^2}{\nu_2}\left(\frac{1}{a_\#}+2\right)\|h\|^2_{2,\Sigma_R(z_0)},\end{aligned}\tag{14}$$

for every $z_0=(x_0,t_0)\in\overline{\Omega}\times[0,T]$, and $0<R<\sqrt{T}$. Here $\nu_0=\nu_0(f)$, $\nu_1=\nu_1(\mathbf{f})$, and $\nu_2=\nu_2(h)$ are positive constants if $f\neq 0$, $\mathbf{f}\neq\mathbf{0}$, $h\neq 0$, respectively, and $\nu_0(0)=\nu_1(\mathbf{0})=\nu_2(0)=0$ otherwise; and u (analogously for each function f, $\mathbf{f}$, h, and U) should be understood as

$$\widetilde{u}(x,t)=\begin{cases}u(x,-t), & -T<t\leq 0\\ u(x,t), & 0<t<T\\ u(x,2T-t), & T\leq t<2T\end{cases}.\tag{15}$$

Proof. Fix $-T<t_\#<t_1<t_2<t^\#<2T$, and $t\in]0,T[\cap]t_\#,t^\#[$. Let $\eta\in W^{1,\infty}_0(\mathbb{R}^n)$ satisfy $0\leq\eta\leq 1$ in $\mathbb{R}^n$, and $\zeta\in W^{1,\infty}(\mathbb{R})$ be such that $0\leq\zeta\leq 1$ in $\mathbb{R}$, $\zeta\equiv 1$ in $]t_1,t_2[$ and $\zeta\equiv 0$ in $\mathbb{R}\setminus[t_\#,t]$. Since $W^{1,\infty}_0(\mathbb{R})\hookrightarrow C(\mathbb{R})$ then $\zeta(t_\#)=0$. If $t_\#,t_1<0$ and/or $t_2,t^\#>T$, since u (analogously f, $\mathbf{f}$, h, and U) is only defined on $]0,T[$, then the extension (15) should be taken into account. For the sake of simplicity, we write briefly u instead of $\widetilde{u}$ (analogously for each function f, $\mathbf{f}$, h, and U).

Taking $v(x,\tau)=\chi_{]-T,t[}(\tau)\zeta^2(\tau)\eta^2(x)(u(x,\tau)-U(\tau))\in V_{2,\ell}(Q_T)$ as a test function in (5), making use of (1)-(2) and (4), standard computations yield

$$\begin{aligned}\frac{1}{2}\zeta^2\|\eta(u-U)\|^2_{2,\Omega}\Big|_{t_\#}^{t}&+\frac{a_\#}{2}\|\zeta\eta\nabla u\|^2_{2,\Omega\times]t_\#,t[}+b_\#\int_{t_\#}^{t}\|\zeta\eta(u-U)\|^\ell_{\ell,\Gamma}\mathrm{d}\tau\leq\\ \leq\int_{t_\#}^{t}\zeta|\zeta'|\|\eta(u-U)\|^2_{2,\Omega}\mathrm{d}\tau&+\left(\frac{2(a^\#)^2}{a_\#}+1\right)\|(u-U)\nabla\eta\|^2_{2,\Omega\times]t_\#,t[}+\\ &+\frac{a_\#\nu_1}{2}\|\zeta\eta\nabla u\|^2_{2,\Omega\times]t_\#,t[}+\left(\frac{1}{2\nu_1 a_\#}+1\right)\|\eta\mathbf{f}\|^2_{2,\Omega\times]t_\#,t[}+\\ +\int_{t_\#}^{t}\|\zeta\eta f\|_{2,\Omega}\|\zeta\eta(u-U)\|_{2,\Omega}\mathrm{d}\tau&+\int_{t_\#}^{t}\|\zeta\eta h\|_{2,\Gamma}\|\zeta\eta(u-U)\|_{2,\Gamma}\mathrm{d}\tau.\end{aligned}\tag{16}$$

Making use of the trace constant $K_{2n/(n+1)}$ correspondent to the function $\eta(u-U)\in W^{1,2n/(n+1)}(\Omega)$ with $2n/(n+1)<2\leq n$, and after applying the Young inequality, the last boundary integral in (16), denoted by I, can be computed as

$$I \leq \int_{t_\#}^{t} \|\zeta\eta h\|_{2,\Gamma} K_{2n/(n+1)} |\mathrm{supp}(\eta)|^{\frac{1}{2n}} \zeta\,(\|\eta\nabla u\|_{2,\Omega}+$$

$$+\|(u-U)\nabla\eta\|_{2,\Omega}+\|(u-U)\eta\|_{2,\Omega})\,\mathrm{d}\tau \leq$$

$$\leq (K_{2n/(n+1)})^2|\mathrm{supp}(\eta)|^{\frac{1}{n}}\left(\frac{1}{2a_\#\nu_2}+\frac{1}{\nu_2}\right)\|\eta h\|^2_{2,\Gamma\times]t_\#,t[}+$$

$$+\frac{a_\#\nu_2}{2}\|\zeta\eta\nabla u\|^2_{2,\Omega\times]t_\#,t[}+\frac{\nu_2}{2}\int_{t_\#}^{t}\zeta^2\left(\|(u-U)\nabla\eta\|^2_{2,\Omega}+\|(u-U)\eta\|^2_{2,\Omega}\right)\mathrm{d}\tau.$$

Applying the Young inequality in (16), and inserting the above inequality, we deduce

$$\frac{1}{2}\operatorname*{ess\,sup}_{t\in]t_\#,t^\#[}\zeta^2\|\eta(u-U)\|^2_{2,\Omega}(t)+\frac{a_\#}{2}(1-(\nu_1+\nu_2))\|\eta\nabla u\|^2_{2,\Omega\times]t_1,t_2[}\leq$$

$$\leq\int_{t_\#}^{t^\#}\left(|\zeta'|+\frac{\nu_0}{2}+\frac{\nu_2}{2}\right)\|\eta(u-U)\|^2_{2,\Omega}\mathrm{d}\tau+\frac{1}{2\nu_0}\|\eta f\|^2_{2,\Omega\times]t_\#,t^\#[}+$$

$$+\left(\frac{2(a^\#)^2}{a_\#}+1+\frac{\nu_2}{2}\right)\|(u-U)\nabla\eta\|^2_{2,\Omega\times]t_\#,t^\#[}+\left(\frac{1}{2a_\#\nu_1}+1\right)\|\eta\mathbf{f}\|^2_{2,\Omega\times]t_\#,t^\#[}$$

$$+(K_{2n/(n+1)})^2|\mathrm{supp}(\eta)|^{\frac{1}{n}}\left(\frac{1}{2a_\#\nu_2}+\frac{1}{\nu_2}\right)\|\eta h\|^2_{2,\Gamma\times]t_\#,t^\#[}.$$

Then, we conclude (14), by taking $\eta\equiv 1$ in $Q_r(x_0)$, $\eta\equiv 0$ in $\mathbb{R}^n\setminus Q_R(x_0)$, and $|\nabla\eta|\leq(R-r)^{-1}$ a.e. in $Q_R(x_0)\setminus Q_r(x_0)$ for any $0<r<R$ such that $(R-r)^2\leq 2$; and $|\zeta'|\leq(R-r)^{-2}$ with $t_\#=t_0-R^2$, $t_1=t_0-r^2$, $t_2=t_0+r^2$, and $t^\#=t_0+R^2$. □

Let us recall a result on the Stieltjes integral in the form that we are going to use (for the general form see [4]).

Lemma 4.2. *Suppose that $q\in]0,\infty[$, and $a\in]1,\infty[$. If $h,H_i:[1,\infty[\to[0,\infty[$ are nonincreasing functions such that*

$$\lim_{\iota\to\infty}h(\iota)=\lim_{\iota\to\infty}H_i(\iota)=0,\qquad i=1,\cdots,M_0, \tag{17}$$

and that

$$-\int_{\iota}^{\infty}\tau^q\mathrm{dh}(\tau)\leq a[\iota^q h(t)+\sum_{i=1}^{M_0}H_i^{\beta_i}(\iota)],\quad\forall\iota\geq 1, \tag{18}$$

with $\beta_i\geq 1$, then, for $\gamma\in[q,aq/(a-1)[$

$$-\int_1^\infty\iota^\gamma\mathrm{dh}(\iota)\leq\frac{q}{aq-(a-1)\gamma}\left(-\int_1^\infty\iota^q\mathrm{dh}(\iota)\right)+$$

$$+\frac{a\gamma}{aq-(a-1)\gamma}\sum_{i=1}^{M_0}H_i^{\beta_i-1}(1)\left(-\int_1^\infty\iota^{\gamma-q}\mathrm{dH_i}(\iota)\right). \tag{19}$$

Throughout this section, $z=(x,t)$ stands for spatiotemporal points. Under the parabolic metric in $\mathbb{R}^{n+1}$ being given by

$$d(z^{(1)},z^{(2)})=\max_{i=1,\cdots,n}\{|x_i^{(1)}-x_i^{(2)}|,|t^{(1)}-t^{(2)}|^{1/2}\},$$

we use the following standard notation for the parabolic parallelepiped

$$\begin{aligned} Q_R(z) &:= \{(y,\tau)\in\mathbb{R}^{n+1}:\ d((y,\tau),(x,t))<R\} \\ &= Q_R^{(n)}(x)\times]t-R^2,t+R^2[, \end{aligned} \tag{20}$$

where the spatial cubic interval $Q_R^{(n)}(x)$ stands for the cube with edges parallel to coordinate planes centered at the point x with the radius $R>0$. When no confusion arises, we shall omit the space dimension and write briefly $Q_R(x)$. Furthermore, we set

$$\begin{aligned} Q_R^+(z) &:= \{(y,\tau)\in Q_R(z):\ y_n>x_n\}; \\ \Sigma_R(z) &:= \{y\in Q_R(z):\ y_n=x_n\}. \end{aligned}$$

Next, we determine an explicit constant involved in the reverse Hölder inequality with increasing supports and an additional surface integral, where the data exponents improve the ones in [4]. Observe that in [4] the assumed restriction $(n-1)/l_1+2/l_2\geq(n+2)/s$ is not true for $l_1=l_2=s$. The elliptic version of the below result is stated in [10].

Proposition 4.3. *Let $R_0>0$, and $z_0=(x_0,t_0)$ with $x_0=(x_0',0)\in\mathbb{R}^n$ and $t_0\geq 0$. For $p>1$ and $\delta>0$, suppose that the nonnegative functions $\Phi\in L^p(Q_{R_0}^+(z_0))$, $F\in L^{m_1+\delta,m_2+\delta}(Q_{R_0}^+(z_0))$, $G\in L^{l_1+\delta,l_2+\delta}(\Sigma_{R_0}(z_0))$, and $\varphi\in L^{1+\delta}(Q_{R_0}^+(x_0))$ satisfy the estimate*

$$\begin{aligned} \frac{1}{R^{n+2}}\int_{Q_{\alpha R}(z)\cap Q_{R_0}^+(z_0)}\Phi^p\mathrm{dz} &\leq B\Big(\frac{1}{R^{n+2}}\int_{Q_R(z)\cap Q_{R_0}^+(z_0)}\Phi\mathrm{dz}\Big)^p+ \\ &+\frac{1}{R^{n+2}}\left(\|F\|^r_{m_1,m_2,Q_R(z)\cap Q_{R_0}^+(z_0)}+\|\varphi\|^d_{1,Q_R(x)\cap Q_{R_0}^+(x_0)}\right)+ \\ &+\frac{1}{R^{n+1}}\|G\|^s_{l_1,l_2,Q_R(z)\cap\Sigma_{R_0}(z_0)}, \end{aligned} \tag{21}$$

for all $z\in Q_{R_0}(z_0)$, and all $R>0$ such that $Q_R(z)\cap\Sigma_{R_0}(z_0)\neq\emptyset$ and $Q_R(z)\subset\subset Q_{R_0}(z_0)$, with some constants $\alpha\in[1/2,1[$, $B>0$, and

$$\frac{n}{m_1}+\frac{2}{m_2}\geq\frac{n+2}{r},\qquad r\geq m_2\geq m_1\geq 1; \tag{22}$$

$$\frac{n-1}{l_1}+\frac{2}{l_2}\geq\frac{n+1}{s},\qquad s\geq l_2\geq l_1\geq 1; \tag{23}$$

$$nd\geq n+2. \tag{24}$$

Then, $\Phi\in L^{p+\varepsilon}(\omega\cap Q^+_{R_0}(z_0))$, for all $\varepsilon\in[0,\delta]\cap[0,(p-1)/(\upsilon-1)[$ and measurable set $\omega\subset\subset Q_{R_0}(z_0)$. In particular, if $R_0\le 3/2$, and $\mathrm{dist}(\omega,\partial Q_{R_0}(z_0))=\beta R_0$ *with $\beta\in]0,1[$, it verifies*

$$
\begin{aligned}
\|\Phi\|^{p+\varepsilon}_{p+\varepsilon,\omega\cap Q^+_{R_0}(z_0)} &\le \frac{\beta^{-(n+2)(1+\varepsilon/p)}}{p-1-(\upsilon-1)\varepsilon}\left[\frac{p-1}{R_0^{(n+2)\varepsilon/p}}\left(\|\Phi\|^{p+\varepsilon}_{p,Q^+_{R_0}(z_0)}+\right.\right.\\
&+\|\Phi\|^{p}_{p,Q^+_{R_0}(z_0)}\left.\left(\|F\|^{r\varepsilon/p}_{m_1,m_2,Q^+_{R_0}(z_0)}+\frac{2R_0}{3}\|G\|^{s\varepsilon/p}_{l_1,l_2,\Sigma_{R_0}(z_0)}+\frac{4R_0^2}{9}\|\varphi\|^{d\varepsilon/p}_{1,Q^+_{R_0}(x_0)}\right)\right)\\
&+\upsilon(p-1+\varepsilon)\left(E_1\|F\|^{m_2+r\varepsilon/p}_{m_1+r\varepsilon/p,m_2+r\varepsilon/p,Q^+_{R_0}(z_0)}\|F\|^{r-m_2}_{m_1,m_2,Q^+_{R_0}(z_0)}+\right.\\
&+E_2\|G\|^{l_2+s\varepsilon/p}_{l_1+s\varepsilon/p,l_2+s\varepsilon/p,\Sigma_{R_0}(z_0)}\|G\|^{s-l_2}_{l_1,l_2,\Sigma_{R_0}(z_0)}+\\
&\left.\left.+3^{nd-(n+2)}(2R_0)^{n+2-nd}\|\varphi\|^{1+d\varepsilon/p}_{1+d\varepsilon/p,Q^+_{R_0}(x_0)}\|\varphi\|^{d-1}_{1,Q^+_{R_0}(x_0)}\right)\right],
\end{aligned}
\tag{25}
$$

where E_1 and E_2 are given by (32)-(33), respectively, and

$$
\begin{aligned}
\upsilon &= (4^n+1)\left(2^{n+2}(2^{n+2}B)^{1/p}+\frac{3}{p'}+\right.\\
&\left.+2^{(n/m_1+1/m_2)r/p}+2^{((n-1)/l_1+1/l_2)s/p}+2^{nd/p}+2^{2n+3}\right)^p.
\end{aligned}
\tag{26}
$$

Proof. We prolong Φ (analogously F) and φ as even functions with respect to $\Sigma_{R_0}(z_0)$:

$$
\widetilde{\Phi}(x,\cdot)=\begin{cases}\Phi(x',x_n,\cdot), & x_n>0\\ \Phi(x',-x_n,\cdot), & x_n<0\end{cases}\qquad \widetilde{\varphi}(x',x_n)=\begin{cases}\varphi(x',x_n), & x_n>0\\ \varphi(x',-x_n), & x_n<0.\end{cases}
$$

Transforming $Q_{R_0}(z_0)$ into $Q=Q_{3/2}(0)\times]-9/4,9/4[$ by the passage to new coordinates system $(y,\tau)=(3(x-x_0)/(2R_0),9(t-t_0)/(4R_0^2))$, and setting

$$
\begin{aligned}
M=\frac{3^{n+2}}{(2R_0)^{n+2}}&\left(\|\widetilde{\Phi}\|^p_{p,Q_{R_0}(z_0)}+\|\widetilde{F}\|^r_{m_1,m_2,Q_{R_0}(z_0)}+\frac{2R_0}{3}\|G\|^s_{l_1,l_2,\Sigma_{R_0}(z_0)}+\right.\\
&\left.+\frac{4R_0^2}{9}\|\widetilde{\varphi}\|^d_{1,Q_{R_0}(x_0)}\right),
\end{aligned}
$$

we define $\overline{\Phi}(y,\tau)=M^{-1/p}\widetilde{\Phi}(x_0+2R_0y/3,t_0+4R_0^2t/9)$, $\overline{F}(y,\tau)=M^{-1/r}\widetilde{F}(x_0+2R_0y/3,t_0+4R_0^2t/9)$, $\overline{G}(y,\tau)=M^{-1/s}G(x_0+2R_0y/3,t_0+4R_0^2t/9)$, and $\overline{\varphi}(y)=M^{-1/d}\widetilde{\varphi}(x_0+2R_0y/3)$.

Setting $\Sigma=\Sigma_{3/2}(0)\times]-9/4,9/4[$ with $\Sigma_{3/2}(0)=Q^{(n-1)}_{3/2}(0)\times\{0\}$. we have

$$
\max\{\|\overline{\Phi}\|^p_{p,Q},\|\overline{F}\|^r_{m_1,m_2,Q},\|\overline{G}\|^s_{l_1,l_2,\Sigma},\|\overline{\varphi}\|^d_{1,Q^{(n)}_{3/2}(0)}\}\le 1.
$$

Let us define $\Phi_0(y,\tau)=\overline{\Phi}(y,\tau)[\mathrm{dist}((y,\tau),\partial Q)]^{(n+2)/p}$.

In order to apply Lemma 4.2, our objective is to prove that

$$\int_{Q[\Phi_0>\iota]}\Phi_0^p \mathrm{d}y\mathrm{d}\tau \leq \upsilon\left(\iota^{p-1}h(\iota)+H_1^{r/m_2}(\iota)+H_2^{s/l_2}(\iota)+H_3^d(\iota)\right), \tag{27}$$

for any $\iota\in[1,\infty[$, with υ being as in (26), and

$$\begin{aligned} h(\iota) &= \int_{Q[\Phi_0>\iota]}\Phi_0 \mathrm{d}y\mathrm{d}\tau;\\ H_1(\iota) &= \int_{-9/4}^{9/4}\left(\int_{Q_{3/2}(0)[\overline{F}(\cdot,\tau)>\iota^{p/r}]}\overline{F}^{m_1}\mathrm{d}y\right)^{m_2/m_1}\mathrm{d}\tau;\\ H_2(\iota) &= \int_{-9/4}^{9/4}\left(\int_{\Sigma_{3/2}(0)[\overline{G}(\cdot,\tau)>\iota^{p/s}]}\overline{G}^{l_1}\mathrm{d}s_y\right)^{l_2/l_1}\mathrm{d}\tau;\\ H_3(\iota) &= \int_{Q_{3/2}(0)[\overline{\varphi}>\iota^{p/d}]}\overline{\varphi}\mathrm{d}y. \end{aligned}$$

Fix $\iota\geq 1$. Decompose $Q=\cup_{k\in\mathbb{N}_0}C^{(k)}=\cup_{k\in\mathbb{N}_0}\cup_{i=1,\cdots,I}D_i^{(k)}$, with $C^{(0)}=Q_{1/2}(0)$, and for each $k\geq 1$, $C^{(k)}=\{(y,\tau)\in Q:\ 2^{-k}<\mathrm{dist}((y,\tau),\partial Q)\leq 2^{-k+1}\}$, and $D_i^{(k)}$ are disjoint cubic intervals of size $1/2^{k+2}$ such that finitely ($I\in\mathbb{N}$) decompose each set $C^{(k)}$, $k\in\mathbb{N}_0$. Since

$$\frac{1}{|D_i^{(k)}|}\int_{D_i^{(k)}}\Phi_0^p(y,\tau)\mathrm{d}y\mathrm{d}\tau\leq 2^{3(n+2)},$$

the parabolic version of the Calderon-Zygmund subdivision argument implies that (for details see [4, 10]) if there exists $\lambda>2^{3(n+2)}$ then there exists a disjoint sequence of cubic intervals $Q_j^{(k)}=Q_{r_j^{(k)}}(y^{(k,j)},\tau^{(k,j)})\subset C^{(k)}$ such that $r_j^{(k)}<2^{-(k+3)}$, and

$$\int_{Q[\Phi_0>\iota\sqrt[p]{\lambda}]}\Phi_0^p \mathrm{d}y\mathrm{d}\tau\leq 2^{n+2}\iota^p\lambda\sum_{k\geq 0}\sum_{j\geq 1}|Q_j^{(k)}|; \tag{28}$$

$$\iota^p\lambda<\frac{2^{-(k-1)(n+2)}}{(2r_j^{(k)})^{n+2}}\int_{Q_j^{(k)}}\overline{\Phi}^p\mathrm{d}y\mathrm{d}\tau. \tag{29}$$

Next, in order to estimate the right hand side in (28), let us prove, for all $k\geq 0$, and $j\geq 1$, there exists $R=R_{kj}\in]r_j^{(k)},2r_j^{(k)}]$ that verifies

$$\begin{aligned} \iota 2^{2n+3}R^{n+2} &< \int_{Q_R[\Phi_0>\iota]}\Phi_0\mathrm{d}y\mathrm{d}\tau+\\ &+\iota^{-p+1}\left(I_1(R,y^{(k,j)},\tau^{(k,j)})+I_2(R,y^{(k,j)},\tau^{(k,j)})+I_3(R,y^{(k,j)})\right), \end{aligned} \tag{30}$$

with the notation $Q_R = Q_R(y^{(k,j)}, \tau^{(k,j)})$, for the points $(y^{(k,j)}, \tau^{(k,j)})$ such that $Q_R \cap \Sigma$ has positive $(n-1)$-Lebesgue measure, and

$$I_1(R,x,t) = \left(\int_{t-R^2}^{t+R^2} \left(\int_{\{y\in Q_R(x):\ \overline{F}(y,\tau)>\iota^{p/r}\}} \overline{F}^{m_1} \mathrm{d}y \right)^{m_2/m_1} \mathrm{d}\tau \right)^{r/m_2};$$

$$I_2(R,x,t) = \left(\int_{t-R^2}^{t+R^2} \left(\int_{\{y\in \Sigma_R(x):\ \overline{G}(y,\tau)>\iota^{p/s}\}} \overline{G}^{l_1} \mathrm{d}s_y \right)^{l_2/l_1} \mathrm{d}\tau \right)^{s/l_2};$$

$$I_3(R,x) = \left(\int_{\{y\in Q_R(x):\ \overline{\varphi}(y)>\iota^{p/d}\}} \overline{\varphi} \mathrm{d}y \right)^{d}.$$

Since $R \le 2r_j^{(k)} < 2^{-(k+1)}$, each Q_R only intersects the sets $C^{(k-1)}$, $C^{(k)}$, and $C^{(k+1)}$. We denote by $\mathcal{T}$ that family $\{Q_R(y^{(k,j)}, \tau^{(k,j)})\}_{k\ge 0,\, j\ge 1}$.

Rewriting (21) in terms of the new coordinates system, taking $z = (x_0 + 2R_0 y^{(k,j)}/3, t_0 + 4R_0^2 \tau^{(k,j)}/9)$, and dividing the resultant inequality by M, we deduce

$$\frac{1}{R^{n+2}} \int_{Q_{\alpha R}} \overline{\Phi}^p \mathrm{d}y\mathrm{d}\tau \le B \left(\frac{1}{R^{n+2}} \int_{Q_R} \overline{\Phi} \mathrm{d}y\mathrm{d}\tau \right)^p + $$
$$+ \frac{1}{R^{n+2}} \|\overline{F}\|^r_{m_1,m_2,Q_R} + \frac{1}{R^{n+1}} \|\overline{G}\|^s_{l_1,l_2,\Sigma_R} + \frac{1}{R^n} \|\overline{\varphi}\|^d_{1,Q_R^{(n)}},$$

where $Q_{\alpha R} = Q_{\alpha R}(y^{(k,j)}, \tau^{(k,j)}) \subset C^{(k)}$, $R \in]r_j^{(k)}, 2r_j^{(k)}]$, and $\alpha = r_j^{(k)}/R \in [1/2, 1[$, and taking (22)-(24) and $R_0 \le 3/2$ into account.

Inserting the above inequality into (29), we obtain

$$(\iota R^{n+2})^p \lambda < 2^{n+2} B \left(\int_{Q_R} \Phi_0 \mathrm{d}y\mathrm{d}\tau \right)^p + R^{(n+2)(p-1)} \|\overline{F}\|^r_{m_1,m_2,Q_R} +$$
$$+ R^{(n+2)(p-1)} \|\overline{\varphi}\|^d_{1,Q_R^{(n)}} + R^{(n+2)p-(n+1)} \|\overline{G}\|^s_{l_1,l_2,\Sigma_R}. \tag{31}$$

Each term of the above right hand side is computed as follows

$$\int_{Q_R} \Phi_0 \mathrm{d}y\mathrm{d}\tau \le \int_{Q_R[\Phi_0>\iota]} \Phi_0 \mathrm{d}y\mathrm{d}\tau + \iota (2R)^{n+2};$$

$$R^{(n+2)(p-1)/p} \|\overline{F}\|^{r/p}_{m_1,m_2,Q_R} \le$$
$$\le R^{(n+2)(p-1)/p} \left(I_1^{1/p}(R, y^{(k,j)}, \tau^{(k,j)}) + \iota 2^{\left(\frac{n}{m_1}+\frac{1}{m_2}\right)\frac{r}{p}} R^{\left(\frac{n}{m_1}+\frac{2}{m_2}\right)\frac{r}{p}} \right)$$
$$\le \frac{\iota^{-(p-1)}}{p} I_1(R, y^{(k,j)}, \tau^{(k,j)}) + \iota R^{n+2} \left(\frac{1}{p'} + 2^{(n/m_1+1/m_2)r/p} \right);$$

$$R^{(n+2)(p-1)/p}\|\overline{\varphi}\|_{1,Q_R}^{d/p} \leq \frac{\iota^{-(p-1)}}{p} I_3(R,y^{(k,j)}) + \iota R^{n+2}\left(\frac{1}{p'}+2^{nd/p}\right);$$
$$R^{(n+2)-(n+1)/p}\|\overline{G}\|_{l_1,l_2,\Sigma_R}^{s/p} \leq R^{(n+2)(p-1)/p}\|\overline{G}\|_{l_1,l_2,\Sigma_R}^{s/p} \leq$$
$$\leq \frac{\iota^{-(p-1)}}{p} I_2(R,y^{(k,j)},\tau^{(k,j)}) + \iota R^{n+2}\left(\frac{1}{p'}+2^{((n-1)/l_1+1/l_2)s/p}\right).$$

Defining

$$\lambda = \left(2^{n+2}(2^{n+2}B)^{1/p} + \frac{3}{p'} + 2^{\left(\frac{n}{m_1}+\frac{1}{m_2}\right)\frac{r}{p}} + 2^{\left(\frac{n-1}{l_1}+\frac{1}{l_2}\right)\frac{s}{p}} + 2^{nd/p} + 2^{2n+3}\right)^p,$$

we gather the above inequalities with (31) obtaining (30).

According to the Vitali covering lemma, there exist $\sigma \in]3,4[$ and a sequence of disjoint cubic intervals $\{Q_{R_i}(y^{(i)},\tau^{(i)})\}_{i\geq 1}$ from the collection $\mathcal{T}$ such that

$$\cup_{k\geq 0}\cup_{j\geq 1} Q_R(y^{(k,j)},\tau^{(k,j)}) \subset \cup_{i\geq 1} Q_{\sigma R_i}(y^{(i)},\tau^{(i)}) \subset Q.$$

Hence,

$$\sum_{k\geq 0}\sum_{j\geq 1}|Q_j^{(k)}| \leq \sum_{k\geq 0}\sum_{j\geq 1}|Q_R(y^{(k,j)},\tau^{(k,j)})| \leq \sigma^n \sum_{i\geq 1}|Q_{R_i}(y^{(i)},\tau^{(i)})|.$$

Combining the above with (28), and (30), we find

$$\int_{Q[\Phi_0>\iota\sqrt[p]{\lambda}]} \Phi_0^p \mathrm{d}y\mathrm{d}\tau \leq \lambda\sigma^n\left(\iota^{p-1}h(\iota) + H_1^{r/m_2}(\iota) + H_2^{s/l_2}(\iota) + H_3^d(\iota)\right),$$

which implies (27), by taking $\upsilon \geq \lambda(\sigma^n+1)$.

We have the relations (for details see [4], if $r\geq m_2$, $s\geq l_2$, and $d\geq 1$)

$$\begin{aligned}
-\int_1^\infty \iota^{\gamma-p+1} dH_1(\iota) &\leq \frac{m_2}{m_1} 3^{n(m_2/m_1-1)/(m_1+\delta_1)} \|\overline{F}\|_{m_1+\delta_1,m_2+\delta_1,Q}^{m_2+\delta_1};\\
-\int_1^\infty \iota^{\gamma-p+1} dH_2(\iota) &\leq \frac{l_2}{l_1} 3^{(n-1)(l_2/l_1-1)/(l_1+\delta_2)} \|\overline{G}\|_{l_1+\delta_2,l_2+\delta_2,\Sigma}^{l_2+\delta_2};\\
-\int_1^\infty \iota^{\gamma-p+1} dH_3(\iota) &\leq \|\overline{\varphi}\|_{1+\delta_3,Q_{3/2}^{(n)}(0)}^{1+\delta_3},
\end{aligned}$$

with $\delta_1 = r(\gamma-p+1)/p$, $\delta_2 = s(\gamma-p+1)/p$, and $\delta_3 = d(\gamma-p+1)/p$.

Therefore, Lemma 4.2 can be applied, concluding that, for $\gamma = p+\varepsilon-1$ such that $p\leq \gamma+1 < p+(p-1)/(\upsilon-1)$, (19) implies

$$\int_{Q[\Phi_0>1]} \Phi_0^{p+\varepsilon}\mathrm{d}y\mathrm{d}\tau \leq \frac{p-1}{\upsilon(p-1)-(\upsilon-1)\gamma}\int_{Q[\Phi_0>1]}\Phi_0^p\mathrm{d}y\mathrm{d}\tau +$$
$$+\frac{\upsilon\gamma}{\upsilon(p-1)-(\upsilon-1)\gamma}\left(\frac{m_2}{m_1}3^{\frac{n(m_2/m_1-1)}{m_1+r\varepsilon/p}}\|\overline{F}\|_{m_1+r\varepsilon/p,m_2+r\varepsilon/p,Q}^{m_2+r\varepsilon/p} H_1^{r/m_2-1}(1) + \right.$$
$$\left. +\frac{l_2}{l_1}3^{\frac{(n-1)(l_2/l_1-1)}{l_1+s\varepsilon/p}}\|\overline{G}\|_{l_1+s\varepsilon/p,l_2+s\varepsilon/p,\Sigma}^{l_2+s\varepsilon/p} H_2^{s/l_2-1}(1) + \|\overline{\varphi}\|_{1+d\varepsilon/p,Q_{3/2}^{(n)}(0)}^{1+d\varepsilon/p} H_3^{d-1}(1)\right).$$

On the other hand, since $\Phi_0^{p+\varepsilon} \leq \Phi_0^p$ a.e. in $Q\setminus Q[\Phi_0>1]$, we find for any $\omega \subset\subset Q = Q_{3/2}(0)$

$$[\mathrm{dist}(\omega,\partial Q)]^{(n+2)(1+\varepsilon/p)}\int_\omega \overline{\Phi}^{p+\varepsilon}\mathrm{d}y\mathrm{d}\tau \leq \frac{(p-1)3^{n+2}}{(\gamma-\upsilon\varepsilon)2^{n+2}}\int_Q \overline{\Phi}^p \mathrm{d}y\mathrm{d}\tau + \\ +\frac{\upsilon\gamma}{\gamma-\upsilon\varepsilon}\left(\frac{m_2}{m_1}3^{\frac{n(m_2-m_1)}{m_1(m_1+r\varepsilon/p)}}\|\overline{F}\|^{m_2+r\varepsilon/p}_{m_1+r\varepsilon/p,m_2+r\varepsilon/p,Q}\|\overline{F}\|^{r-m_2}_{m_1,m_2,Q}+ \right.\\ \left.+\frac{l_2}{l_1}3^{\frac{(n-1)(l_2-l_1)}{l_1(l_1+s\varepsilon/p)}}\|\overline{G}\|^{l_2+s\varepsilon/p}_{l_1+s\varepsilon/p,l_2+s\varepsilon/p,\Sigma}\|\overline{G}\|^{s-l_2}_{l_1,l_2,\Sigma}+\|\overline{\varphi}\|^{1+d\varepsilon/p}_{1+d\varepsilon/p,Q^{(n)}_{3/2}(0)}\|\overline{\varphi}\|^{d-1}_{1,Q^{(n)}_{3/2}(0)}\right).$$

Keeping the same designation to the transformed set $\omega\subset\subset Q_{R_0}(z_0)$, we deduce

$$\left[\mathrm{dist}(\omega,\partial Q_{R_0}(z_0))\frac{3}{2R_0}\right]^{(n+2)(1+\varepsilon/p)}\int_\omega \widetilde{\Phi}^{p+\varepsilon}\mathrm{d}z \leq \\ \leq \frac{(p-1)3^{n+2}}{(p-1-(\upsilon-1)\varepsilon)2^{n+2}}M^{\varepsilon/p}\int_{Q_{R_0}(z_0)}\widetilde{\Phi}^p\mathrm{d}z + \\ +\frac{\upsilon(p-1+\varepsilon)}{p-1-(\upsilon-1)\varepsilon}\left(E_1\|\widetilde{F}\|^{m_2+r\varepsilon/p}_{m_1+r\varepsilon/p,m_2+r\varepsilon/p,Q_{R_0}(z_0)}\|\widetilde{F}\|^{r-m_2}_{m_1,m_2,Q_{R_0}(z_0)}+ \right.\\ +E_2\|G\|^{l_2+s\varepsilon/p}_{l_1+s\varepsilon/p,l_2+s\varepsilon/p,\Sigma_{R_0}(z_0)}\|G\|^{s-l_2}_{l_1,l_2,\Sigma_{R_0}(z_0)}+ \\ \left.+\|\widetilde{\varphi}\|^{1+d\varepsilon/p}_{1+d\varepsilon/p,Q_{R_0}(x_0)}\|\widetilde{\varphi}\|^{d-1}_{1,Q_{R_0}(x_0)}3^{nd-(n+2)}(2R_0)^{n+2-nd}\right),$$

with

$$E_1 = \frac{m_2}{m_1}3^{\frac{n(m_2-m_1)}{m_1(m_1+r\varepsilon/p)}}\left(\frac{3}{2R_0}\right)^{n\left(\frac{m_2+r\varepsilon/p}{m_1+r\varepsilon/p}-\frac{m_2}{m_1}\right)}\frac{(2R_0)^{n+2}3^{r(n/m_1+2/m_2)}}{3^{n+2}(2R_0)^{r(n/m_1+2/m_2)}}; \tag{32}$$

$$E_2 = \frac{l_2}{l_1}3^{\frac{(n-1)(l_2-l_1)}{l_1(l_1+s\varepsilon/p)}}\left(\frac{3}{2R_0}\right)^{(n-1)\left(\frac{l_2+s\varepsilon/p}{l_1+s\varepsilon/p}-\frac{l_2}{l_1}\right)}\frac{(2R_0)^{n+2}3^{s((n-1)/l_1+2/l_2)}}{3^{n+2}(2R_0)^{s((n-1)/l_1+2/l_2)}}. \tag{33}$$

Therefore, by applying (22)-(24) we conclude (25) which completes the proof. □

In a similar manner that we have Proposition 4.3 the following Proposition can be obtained.

Proposition 4.4. *Under the conditions of Proposition 4.3, if instead of (21),*

$$\frac{1}{R^{n+2}}\int_{Q_{\alpha R}(z)}\Phi^p\mathrm{d}z \leq B_{\mathrm{I}}\left(\frac{1}{R^{n+2}}\int_{Q_R(z)}\Phi\mathrm{d}z\right)^p + \frac{1}{R^{n+2}}\|F\|^r_{m_1,m_2,Q_R(z)} + \\ +\frac{1}{R^{n+2}}\|\varphi\|^d_{1,Q_R(x)}, \tag{34}$$

holds for $R < \min\{\sqrt{T}, \mathrm{dist}(x,\partial\Omega)/\sqrt{n}\}$*, then* $\Phi \in L^{p+\varepsilon}(Q_r(z_0))$*, for all* $\varepsilon \in [0,\delta] \cap [0,(p-1)/(\upsilon_{\mathrm{I}}-1)[$*, with*

$$\upsilon_{\mathrm{I}} = (4^n+1)\left(2^{n+2}(2^{n+2}B_{\mathrm{I}})^{1/p} + \frac{3}{p'} + 2^{\left(\frac{n}{m_1}+\frac{1}{m_2}\right)\frac{r}{p}} + 2^{nd/p} + 2^{2n+3}\right)^p, \quad (35)$$

and $r = (1-\beta)R_0$ *with* $\beta \in]0,1[$*. In particular, it verifies*

$$\begin{aligned}\|\Phi\|^{p+\varepsilon}_{p+\varepsilon,Q_r(z_0)} \leq \frac{\beta^{-(n+2)\varepsilon/p}}{p-1-(\upsilon_{\mathrm{I}}-1)\varepsilon}\Bigg[\frac{p-1}{R_0^{(n+2)\varepsilon/p}}\Big(\|\Phi\|^{p+\varepsilon}_{p,Q_{R_0}(z_0)} + \\ + \|\Phi\|^p_{p,Q_{R_0}(z_0)}\Big(\|F\|^{r\varepsilon/p}_{m_1,m_2,Q_{R_0}(z_0)} + \frac{4R_0^2}{9}\|\varphi\|^{d\varepsilon/p}_{1,Q_{R_0}(x_0)}\Big)\Big) + \\ + \upsilon_{\mathrm{I}}(p-1+\varepsilon)\Big(E_1\|F\|^{m_2+r\varepsilon/p}_{m_1+r\varepsilon/p,m_2+r\varepsilon/p,Q_{R_0}(z_0)}\|F\|^{r-m_2}_{m_1,m_2,Q_{R_0}(z_0)} + \\ + 3^{nd-(n+2)}(2R_0)^{n+2-nd}\|\varphi\|^{1+d\varepsilon/p}_{1+d\varepsilon/p,Q_{R_0}(x_0)}\|\varphi\|^{d-1}_{1,Q_{R_0}(x_0)}\Big)\Bigg], \quad (36)\end{aligned}$$

where E_1 *and* E_2 *are given by (32)-(33), respectively.*

Remark 4.5. If $\varphi = 0$, then (26)-(35) read

$$\upsilon = (4^n+1)\left(2^{(n+2)(1+\frac{1}{p})}B^{\frac{1}{p}} + \frac{3}{p'} + 2^{\left(\frac{n}{m_1}+\frac{1}{m_2}\right)\frac{r}{p}} + 2^{\left(\frac{n-1}{l_1}+\frac{1}{l_2}\right)\frac{s}{p}} + 2^{2n+3}\right)^p; \quad (37)$$

$$\upsilon_{\mathrm{I}} = (4^n+1)\left(2^{n+2}(2^{n+2}B_{\mathrm{I}})^{1/p} + \frac{3}{p'} + 2^{(n/m_1+1/m_2)r/p} + 2^{2n+3}\right)^p. \quad (38)$$

Finally, we state a local Poincaré inequality.

Lemma 4.6. *For any* $x \in \mathbb{R}^n$ *and* $0 < R < \varepsilon(2S_{2n/(n+2)})^{-1}$*, every (non constant)* $u \in W^{1,2n/(n+2)}(Q_R(x))$ *verifies*

$$\|u\|_{2,Q_R(x)} \leq \frac{S_{2n/(n+2)}}{1-\varepsilon}\|\nabla u\|_{2n/(n+2),Q_R(x)}, \quad (39)$$

where $S_{2n/(n+2)} = \pi^{-1/2}n^{(2-3n)/(2n)}(n-2)^{(n-2)/(2n)}[\Gamma(n)/\Gamma(n/2)]^{1/n}$.

Proof. Making use of the Sobolev embedding with $q = 2n/(n+2) < 2$, and the Hölder inequality, we obtain

$$\|u\|_{2,Q_R(x)} \leq S_{2n/(n+2)}\left(\|\nabla u\|_{2n/(n+2),Q_R(x)} + |Q_R(x)|^{1/n}\|u\|_{2,Q_R(x)}\right).$$

Hence, we conclude (39). □

5. Proof of Theorem 2.2

Let $u \in V_{2,\ell}(Q_T) \cap C([0,T];[V_{2,\ell}(\Omega)]')$ solve (5) for all $v \in V_{2,\ell}(Q_T)$. Let $0 < r < R < \sqrt{T}$, and $z_0 = (x_0,t_0) \in \overline{\Omega} \times [0,T]$. Proposition 4.1 can be applied.

We split the proof by beginning to show the local interior and lateral higher integrability of the gradient of u.

5.1. Local interior higher integrability of the gradient

If $x_0 \in \Omega$, we may take $R < \mathrm{dist}(x_0,\partial\Omega)/\sqrt{n}$. Considering $R < \sqrt{T}$, $r = R/2 \leq 1$, $\nu_1 = 1/2$, and $\nu_2 = 0$, (14) reads

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in]t_0-R^2,t_0+R^2[} \|\eta(u-U)\|^2_{2,Q_R(x_0)}(t) + \frac{a_\#}{2}\|\nabla u\|^2_{2,Q_r(z_0)} \leq \\ \leq \left(\frac{2(a^\#)^2}{a_\#} + 2 + \nu_0\right)\frac{2^3}{R^2}\|\eta(u-U)\|^2_{2,Q_R(z_0)} + \\ +2\left(\frac{1}{a_\#}+1\right)\|\mathbf{f}\|^2_{2,Q_R(z_0)} + \frac{1}{\nu_0}\|f\|^2_{2,Q_R(z_0)}. \end{aligned} \tag{40}$$

In the presence of Lemma 4.6 it is sufficient to take $U = 0$, and we restrict to $R < (4S_{2n/(n+2)})^{-1}$. Denoting by $Y = 2S_{2n/(n+2)}$ the constant in the inequality (39), we integrate over time to obtain

$$\begin{aligned} \int_{t_0-R^2}^{t_0+R^2} \|\eta u\|_{2,Q_R(x_0)}\|u\|_{2,Q_R(x_0)}\mathrm{dt} \leq Y \operatorname*{ess\,sup}_{t\in]t_0-R^2,t_0+R^2[} \|\eta u\|_{2,Q_R(x_0)} \times \\ \times (2R^2)^{n-2/(2n)}\|\nabla u\|_{2n/(n+2),Q_R(z_0)}. \end{aligned} \tag{41}$$

Inserting the above inequality into (40), and after applying the Young inequality, we deduce

$$\begin{aligned} \frac{a_\#}{2}\|\nabla u\|^2_{2,Q_{R/2}(z_0)} \leq \left(\frac{2(a^\#)^2}{a_\#} + 2 + \nu_0\right)^2 \frac{2^{5-2/n}Y^2}{R^{2(n+2)/n}}\|\nabla u\|^2_{2n/(n+2),Q_R(z_0)} + \\ +2\left(\frac{1}{a_\#}+1\right)\|\mathbf{f}\|^2_{2,Q_R(z_0)} + \frac{1}{\nu_0}\|f\|^2_{2,Q_R(z_0)}. \end{aligned}$$

Employing Proposition 4.4 with $\Phi = |\nabla u|^{2n/(n+2)}$, $p = (n+2)/n$, $m_1 = m_2 = r = 2$, and

$$B_{\mathrm{I}} = \frac{2^{2(2-1/n)}}{a_\#}\left(\frac{2(a^\#)^2}{a_\#} + 2 + \nu_0\right)(4S_{2n/(n+2)})^2; \tag{42}$$

$$F = \left(\frac{4(1/a_\# + 1)|\mathbf{f}|^2 + 2|f|^2/\nu_0}{a_\#}\right)^{1/2} \in L^{2+\delta}(Q_R(z_0)), \tag{43}$$

the interior estimate

$$\|\nabla u\|_{2+\varepsilon,Q_{(1-\beta)R}(z_0)} \leq \left(\frac{2n\beta^{-\varepsilon(n+2)/2}}{4-(n+2)(\upsilon_{\rm I}-1)\varepsilon}\right)^{1/(2+\varepsilon)} \times$$
$$\times \left[\left(\frac{2(4+\varepsilon)}{n(2+\varepsilon)R^{\varepsilon(n+2)/2}}\right)^{1/(2+\varepsilon)} \|\nabla u\|_{2,Q_R(z_0)} + \right.$$
$$+ \left(\frac{2^{1+\varepsilon(n+1)/2}\varepsilon}{n(2+\varepsilon)} + \upsilon_{\rm I}\left(\frac{4+\varepsilon(n+2)}{2n}\right)\right)^{1/(2+\varepsilon)} \times$$
$$\left. \times \left(\frac{2\sqrt{(1+a_\#)}}{a_\#}\|\mathbf{f}\|_{2+\varepsilon,Q_R(z_0)} + \sqrt{\frac{2}{a_\# \nu_0}}\|f\|_{2+\varepsilon,Q_R(z_0)}\right)\right] \tag{44}$$

holds, for any $R < \min\{\sqrt{T}, \mathrm{dist}(x_0,\partial\Omega)/\sqrt{n}, (4S_{2n/(n+2)})^{-1}\}$, and for all $\varepsilon \in [0,\delta]$ with $\delta < 2/[n(\upsilon_{\rm I}-1)]$, and $\upsilon_{\rm I}$ being defined by (38), *i.e.*

$$\upsilon_{\rm I} = (4^n+1)\left(2^{2(n+1)}B_{\rm I}^{n/(n+2)} + \frac{6}{n+2} + 2^{n(n+1)/(n+2)} + 2^{2n+3}\right)^{n/(n+2)}.$$

Remark 5.1. The constant $B_{\rm I}$ defined in (42) may be differently given. For instance, it may depend on the Poincaré constant, denoted by $C_{\Omega,p}$, if we use in (41) the Minkowski, Sobolev, and Poincaré inequalities to successively compute

$$\|\eta(u-U)\|_{2,Q_R(x_0)} \leq 2\|u - \fint_{Q_R(x_0)} u\mathrm{dx}\|_{2,Q_R(x_0)}$$
$$\leq 2S_{2n/(n+2)}\left(\|\nabla u\|_{2n/(n+2),Q_R(x_0)} + \|u - \fint_{Q_R(x_0)} u\mathrm{dx}\|_{2n/(n+2),Q_R(x_0)}\right)$$
$$\leq 2S_{2n/(n+2)}\left(1 + C_{Q_R(x_0),2n/(n+2)}\right)\|\nabla u\|_{2n/(n+2),Q_R(x_0)},$$

since $u \in W^{1,2n/(n+2)}(Q_R(x_0))$ with $2n/(n+2) < 2 \leq n$. Here U is defined from (13). With this approach, the restriction of $R < (4S_{2n/(n+2)})^{-1}$ can be removed.

5.2. Local higher integrability up to the spatial boundary of the gradient

For reader's convenience, we recall the definition of C^1 domain. We use the notation $y' = (y_1,\dots,y_{n-1}) \in \mathbb{R}^{n-1}$.

Definition 5.2. We say that Ω is a domain of class C^1 (or simply C^1 domain), if Ω is an open, bounded, connected, nonempty set of $\mathbb{R}^n$ and it verifies the following:

$$\exists M \in \mathbb{N} \quad \exists \rho, \nu > 0: \qquad \partial\Omega = \cup_{m=1}^{M}\Gamma_m,$$

with

1. $\Gamma_m = O_m^{-1}(\{y=(y',y_n)\in Q_\rho^{(n-1)}(0)\times\mathbb{R} : y_n = \varpi_m(y')\}$,
2. $O_m^{-1}(\{y=(y',y_n)\in Q_\rho^{(n-1)}(0)\times\mathbb{R} : \varpi_m(y') < y_n < \varpi_m(y')+\nu\})\subset\Omega$,
3. $O_m^{-1}(\{y\in Q_\rho^{(n-1)}(0)\times\mathbb{R} : \varpi_m(y')-\nu < y_n < \varpi_m(y')\})\subset\mathbb{R}^n\setminus\Omega$,

where

$$Q_\rho^{(n-1)}(0) = \{y'=(y_1,\cdots,y_{n-1})\in\mathbb{R}^{n-1} :\ |y_i|<\rho,\ i=1,\cdots,n-1\},$$

and for each $m=1,\cdots,M$, $O_m:\mathbb{R}^n\to\mathbb{R}^n$ denotes a local coordinate system:

$$y^{(m)} = O_m(x) = \mathsf{O}x+b,\quad \mathsf{O}^{-1}=\mathsf{O}^T,\ \det\mathsf{O}=1;$$

and $\varpi_m\in C^1(Q_\rho^{(n-1)}(0))$.

By Definition 5.2, there exist $M\in\mathbb{N}$ and $\rho,\nu>0$ such that for any $x_0\in\partial\Omega$ there is $m\in\{1,\cdots,M\}$ such that a local coordinate system $y^{(m)}=O_m(x)$ and a local C^1-mapping ϖ_m verify

$$x_0\in\Gamma_m = O_m^{-1}\circ\phi_m^{-1}\left(Q_\rho^{(n-1)}(0)\times\{0\}\right), \tag{45}$$

where $\phi_m : Q_\rho^{(n-1)}(0)\times\mathbb{R}\to\mathbb{R}^n$ of class C^1 is defined by

$$\phi_m(y) = \begin{pmatrix} y' \\ y_n-\varpi_m(y') \end{pmatrix}. \tag{46}$$

For each $m\in\{1,\cdots,M\}$, we consider the change of variables

$$y\in Q_\rho^{(n-1)}(0)\times]-\nu,\nu[\quad\mapsto x=O^{-1}(\phi_m^{-1}(y)). \tag{47}$$

Since the Jacobian of the transformation $O_m^{-1}\circ\phi_m^{-1}$ is equal to 1, let us denote by the same letter any function $f=f\circ O_m^{-1}\circ\phi_m^{-1}$.

Fix $x_0\in\partial\Omega$, and $m\in\{1,\cdots,M\}$ such that $x_0\in\Gamma_m$ is in accordance with (45). Set $y_0=\phi_m\circ O_m(x_0)$, and

$$\Sigma_R(y_0) = \{y\in Q_\rho^{(n-1)}(0)\times]-\nu,\nu[:\ |y'-y_0'|<R,\ y_n=0\},$$

for any $0<R\le R_0=\min\{\rho,\nu,\mathrm{dist}(y_0',\partial' Q_\rho^{(n-1)}(0))\}$. Notice that $y_0=(y_0',0)$. Inasmuch as $\Gamma_0=O_m^{-1}\circ\phi_m^{-1}(\Sigma_{R_0}(y_0))$, different cases occur, namely $\Gamma_0\cap\Gamma\neq\emptyset$ and $\Gamma_0\cap(\partial\Omega\setminus\bar\Gamma)\neq\emptyset$; $\Gamma_0\subset\Gamma$, and $\Gamma_0\subset\partial\Omega\setminus\bar\Gamma$. Throughout the sequel, we refer to $\|\cdot\|_{2,\Sigma_R(y_0)}$ including cases where the set is empty.

Reorganizing the terms in (14) with $\nu_2 = 1/4$ as in Section 5.1, we have

$$\|\nabla u\|^2_{2,Q_r^+(z_0)} \leq \frac{B}{R^{2(n+2)/n}} \|\nabla u\|^2_{2n/(n+2),Q_R^+(z_0)} + \\ + \frac{4}{a_\#}\left[2\left(\frac{1}{a_\#}+1\right)\|\mathbf{f}\|^2_{2,Q_R^+(z_0)} + \frac{1}{\nu_0}\|f\|^2_{2,Q_R^+(z_0)} + \\ + 8R\left(\frac{1}{a_\#}+2\right)(K_{2n/(n+1)})^2\|h\|^2_{2,\Sigma_R(z_0)}\right].$$

Here B is defined by (compare to (42))

$$B = \frac{2^{5-2/n}}{a_\#}\left(\frac{2(a^\#)^2}{a_\#} + \frac{19}{8} + \nu_0\right)(4S_{2n/(n+2)})^2. \tag{48}$$

Proposition 4.3 with $\Phi = |\nabla u|^{2n/(n+2)}$, $p = (n+2)/n$, $l_1 = l_2 = s = 2$, and F being defined by, instead of (43),

$$F = 2\left(\frac{2(1/a_\# + 1)|\mathbf{f}|^2 + |f|^2/\nu_0}{a_\#}\right)^{1/2} \in L^{2+\delta}(Q_R(z_0)),$$

and

$$G = 4\left[\frac{2}{a_\#}\left(\frac{1}{a_\#}+2\right)\right]^{1/2} K_{2n/(n+1)}|h| \in L^{2+\delta}(\Sigma_R(z_0)),$$

and the application of the passage to the initial coordinates system upon choosing the neighborhood $Q_0 = O_m^{-1} \circ \phi_m^{-1}(Q_{R_0}(y_0))$ of the subset Γ_0 of the boundary $\partial\Omega$, imply that

$$\|\nabla u\|_{2+\varepsilon,Q_{(1-\beta)R}(z_0)} \leq \left(\frac{2n\beta^{-\varepsilon(n+2)/2}}{4-(n+2)(\upsilon-1)\varepsilon}\right)^{1/(2+\varepsilon)} \times \\ \times \left[\left(\frac{2(4+\varepsilon)}{n(2+\varepsilon)R^{\varepsilon(n+2)/2}}\right)^{1/(2+\varepsilon)} \|\nabla u\|_{2,Q_R(z_0)} + \\ + \left(\frac{2^{1+\varepsilon(n+1)/2}\varepsilon}{n(2+\varepsilon)} + \upsilon\left(\frac{4+\varepsilon(n+2)}{2n}\right)\right)^{1/(2+\varepsilon)} \times \\ \times \left(\frac{2\sqrt{2(1+a_\#)}}{a_\#}\|\mathbf{f}\|_{2+\varepsilon,Q_R(z_0)} + \frac{2}{\sqrt{a_\#\nu_0}}\|f\|_{2+\varepsilon,Q_R(z_0)}\right) + \\ + \left(\frac{2^{1+\varepsilon n/2}\varepsilon}{n(2+\varepsilon)R^{\varepsilon/2}} + \upsilon\left(\frac{4+\varepsilon(n+2)}{2n}\right)\right)^{1/(2+\varepsilon)} \times \\ \times \frac{4}{a_\#}\sqrt{2(1+a_\#)}K_{2n/(n+1)}\|h\|_{2+\varepsilon,\Sigma_R(z_0)}\right], \tag{49}$$

for any $R < \min\{\sqrt{T}, R_0, (4S_{2n/(n+2)})^{-1}\}$, for all $\varepsilon \in [0,\delta]$ with $\delta < 2/[n(\upsilon - 1)]$, and υ being defined by (37) with (48), that is (11).

5.3. Global higher integrability

On the one hand, Section 5.1 ensures that for each point $z \in \Omega \times [0,T]$ it is associated a sequence of cubic intervals $Q_{r(z)/2}(z)$, with side lengths $r(z) > 0$ tending to zero, such that (44) is verified. On the other hand, Section 5.2 ensures that for each point $z \in \partial\Omega \times [0,T]$ it is associated a sequence of cubic intervals $Q_{r(z)/2}(z)$, with side lengths $r(z) > 0$ tending to zero, such that (49) is verified.

From the mathematical point of view, it is indifferent to continue the proof by considering thoses cubic intervals. With in mind the view point of real and numerical applications we prefer to proceed by analysing separately the spatial domain.

According to the Besicovitch covering theorem [21, Theorem 1.2], there exists a sequence of spatial cubic intervals $\{Q_{r_m/2}(x^{(m)})\}_{m\geq 1}$ from the above collection of cubic intervals such that: $\overline{\Omega} \subset \cup_{m\geq 1} Q_{r_m/2}(x^{(m)})$; and every point of $\mathbb{R}^n$ belongs to at most $2^n + 1$ cubes in $\{Q_{r_m/2}(x^{(m)})\}_{m\geq 1}$. Since Ω is bounded, this cover is finite, *i.e.* its cardinal is an integer number M. Let us define

$$r_\# = \min\{r_m : \ m = 1, \cdots, M\}.$$

Indeed, there exists N (depending on the dimension of the space) families of pairwise disjoint cubes such that (for details see [10, 21])

$$\{Q_{r_m}(x^{(m)})\}_{m=1,\cdots,M} = \cup_{m=1}^N \{Q_{r_i}(x^{(i)})\}_{i=1,\cdots,\mathcal{I}(m)\cup\mathcal{J}(m)},$$

where $\mathcal{I}(m)$ contains the indices with $x^{(i)} \in \Omega$, while $\mathcal{J}(m)$ contains the indices with $x^{(i)} \in \partial\Omega$. For each $i \in \mathcal{I}(m)$ (analogously for $i \in \mathcal{J}(m)$) there exists $d = d_i > 0$ such that $dr_i^2/4 = T$. If $d < 1$, we take $t^{(i)} = 0$ observing that $]0,T[\subset]0, r_i^2/4[$. If the integer part $\lfloor d \rfloor$ is even, *i.e.* $\lfloor d \rfloor = 2k$, $k \in \mathbb{N}$, then we may build $k+1$ parabolic interval cubes $Q_{r_i}(z^{(i,j)})$ centered at $z^{(i,j)} = (x^{(i)}, t^{(j)})$ where $t^{(j)} = 2(j-1)r_i^2/4$ for $j = 1, \cdots, k+1 := \mathcal{K}(i)$, observing that $t^{(m+1)} = 2mr_i^2/4 < T$. If $\lfloor d \rfloor$ is odd, *i.e.* $\lfloor d \rfloor = 2k-1$, $k \in \mathbb{N}$, then we may build k parabolic interval cubes $Q_{r_i}(z^{(i,j)})$ centered at $z^{(i,j)} = (x^{(i)}, t^{(j)})$ where $t^{(j)} = (2j-1)r_i^2/4$ for $j = 1, \cdots, k := \mathcal{K}(i)$.

Hence, combining (44) and (49) with

$$\|\nabla u\|_{p,Q_T} \leq \sum_{m=1}^N \left(\sum_{\substack{i\in\mathcal{I}(m)\\ j=1,\cdots,\mathcal{K}(i)}} \|\nabla u\|_{p,Q_{r_i/2}(z^{(i,j)})} + \sum_{\substack{i\in\mathcal{J}(m)\\ j=1,\cdots,\mathcal{K}(i)}} \|\nabla u\|_{p,Q_{r_i/2}(z^{(i,j)})\cap\Omega} \right),$$

we find the no optimal, but simplified, estimate

$$\|\nabla u\|_{p,Q_T} \leq C(n)\left[\|\nabla u\|_{2,Q_T} + \frac{(1+\upsilon)^{1/(2+\varepsilon)}}{a_\#}\left(\sqrt{1+a_\#}\|\mathbf{f}\|_{2+\varepsilon,Q_T} + \right.\right.$$
$$\left.\left. + \frac{1}{\sqrt{\nu_0}}\|f\|_{2+\varepsilon,Q_T} + \sqrt{1+a_\#}K_{2n/(n+1)}\|h\|_{2+\varepsilon,\Sigma_T}\right)\right],$$

where

$$C(n) = N2^{c(n)}(r_\#)^{-(n+2)/2}\left(\frac{2n\beta^{-\varepsilon(n+2)/2}}{4-(n+2)(\upsilon-1)\varepsilon}\right)^{1/(2+\varepsilon)}. \tag{50}$$

Here, $c(n)$ stands for a positive polynomial function of degree 1 on the space dimension n. Therefore, from (12) we conclude (8).

6. $W^{1,p}$ regularity ($\ell = 2$ and isotropic case)

In this section, we reformulate the explicit L^p-estimate of the gradient of a weak solution. The leading coefficient is assumed to be $\mathsf{A} = a\mathsf{I}$.

Let us state the following results whose extends to the problem under study the result obtained in [5, 29] for the Dirichlet problem. To this end, we introduce the Robin-Laplacian operator $\Delta^{\mathrm{R}} \in \mathcal{L}(V_p(Q_T); \mathcal{W}_p)$ and the perturbation $P : u \in V_p(Q_T) \mapsto Pu \in \mathcal{W}_p$ defined by

$$\langle -\Delta^{\mathrm{R}} u, v\rangle := \int_{Q_T} \nabla u \cdot \nabla v \mathrm{dxdt} + \int_{\Sigma_T} uv\mathrm{dsdt};$$
$$\langle Pu, v\rangle := \int_{Q_T} (1-a)\nabla u \cdot \nabla v \mathrm{dxdt} + \int_{\Sigma_T} (1-b(u))uv\mathrm{dsdt},$$

for all $v \in V_{p'}(Q_T)$. The term $b(u)uv$ belongs to $L^1(\Sigma_T)$ due to the embedding $L^2(0,T;W^{1,2}(\Omega)) \hookrightarrow L^2(\Sigma_T)$, and the growthness of b implies that $b(u) \in L^\infty(\Sigma_T)$.

The following first result is established.

Proposition 6.1. *If $L^{-1} : \mathcal{W}_p \to V_p(Q_T)$ is an isomorphism, then*

$$\|L^{-1}Pu\|_{V_p(Q_T)} \leq \|L^{-1}\|_{\mathrm{op}}\left((1-a_\#)\|\nabla u\|_{p,Q_T} + (1-b_\#)\|u\|_{2,\Sigma_T}\right),$$

where $\|L\|_{\mathrm{op}}$ stands for the operator norm of L.

Proof. This property is a consequence of definition of P, and the assumptions (1)-(3) with $a_\#, b_\# < 1$ and $\ell = 2$. □

The existence and uniqueness of weak solutions to the linearized variational problem (5), *i.e.* $\mathsf{A} = \mathsf{I}$ and $b \equiv 1$, guarantee that $L = \partial_t - \Delta^{\mathrm{R}}$ is an isomorphism from $\{w \in D(L) : Lw \in \mathcal{W}_p\}$ onto $\mathcal{W}_p$, for any $1 < p < \bar{p}$ and some $\bar{p} > 1$, such that $D(L) \subset \mathcal{W}_p \subset R(L)$. In particular, this restriction of L to $\mathcal{W}_p$ (called the $\mathcal{W}_p$ realization of the operator L) satisfies

$$\|L^{-1}\|_{\mathrm{op}} \leq \sup_{\substack{\mathbf{f}\in \mathbf{L}^p(Q_T)\\ \|\mathbf{f}\|_{p,Q_T}=1}} \sup_{\substack{f\in L^p(Q_T)\\ \|f\|_{p,Q_T}=1}} \sup_{\substack{h\in L^2(\Sigma_T)\\ \|h\|_{2,\Sigma_T}=1}} \left(\mathcal{M}(1,1) + \mathcal{E}(1,1,p)\right)|_{\ell=2} := \Lambda_p,$$

where $\mathcal{M}(1,1)$ and $\mathcal{E}(1,1,p)$ are according to (10) and (9), respectively.

Therefore, we state the following version of [5, Thm. 2.2, p. 272].

Proposition 6.2. *Under the assumptions (1)-(3) with $a_\#, b_\# \in]1-1/\Lambda_p, 1[$ and $\ell = 2$, then any weak solution $u \in V_2(Q_T)$ of (5) enjoys the following properties*

1. *u satisfies the variational problem $\langle Lu - Pu - F, v\rangle_{[V_{p'}(Q_T)]' \times V_{p'}(Q_T)} = 0$,*
2. *u verifies the following estimate*

$$(1-\Lambda_p(1-a_\#))\|\nabla u\|_{p,Q_T} + (1-\Lambda_p(1-b_\#))\|u\|_{2,\Sigma_T} \leq \\ \leq \Lambda_p\left(\|\mathbf{f}\|_{p,Q_T} + \|f\|_{p,Q_T} + \|h\|_{2,\Sigma_T}\right).$$

Proof. The point 1 is consequence of the definitions of the operators. We give an outline of the proof of the point 2. From the point 1, we have $u = L^{-1}(Pu + F)$. Then, the claimed estimate follows from Proposition 6.1. □

Finally, we observe that different explicit estimates are obtained via the interpolative approach, namely the Marcinkiewicz interpolation theorem [16, pp. 228-230].

Theorem 6.3. *Let T be a linear mapping from $L^q(\Omega) \cap L^r(\Omega)$ into itself, $1 \leq q < r < \infty$, and suppose there are constants T_1 and T_2 such that*

$$\mu_{Tf}(t) \leq \left(\frac{T_1\|f\|_{q,\Omega}}{t}\right)^q, \qquad \mu_{Tf}(t) \leq \left(\frac{T_2\|f\|_{r,\Omega}}{t}\right)^r,$$

for all $f \in L^q(\Omega) \cap L^r(\Omega)$, and $t > 0$. Then, T extends as a bounded linear mapping from $L^p(\Omega)$ into itself for any p such that $q < p < r$, and

$$\|Tf\|_{p,\Omega} \leq 2\left(\frac{p}{p-q} + \frac{p}{r-p}\right)^{1/p} T_1^\alpha T_2^{1-\alpha}\|f\|_{p,\Omega} \tag{51}$$

holds for all $f \in L^q(\Omega) \cap L^r(\Omega)$, where $1/p = \alpha/q + (1-\alpha)/r$.

7. Steady-state $W^{1,p}$ regularity

The higher integrability of the gradient is an useful tool in order to obtain Hölder continuity (by embedding if $p > n$). As one knows since long (see e.g. [26, Ch. 3] or [16, Ch. 8]), Hölder continuity can be achieved directly. When the domain is only Lipschitz, the coefficients are discontinuous, and the boundary conditions are mixed, it is proved in [23], for the most interesting dimensions $n = 2,3,4$. For all dimensions it is (unfortunately, rather implicitly) shown in [17, 18] by use of Sobolev-Campanato spaces (which embed for suitable indices in corresponding Hölder spaces).

An explicit estimate is established in [10]. However, in there the dependence on the data has a wordy expression. In view of this, such estimate is traced back to the celebrated paper by Gröger and Rehberg [20] in the context of elliptic regularity theory for weak solutions in the case of mixed boundary conditions. In this approach, it is assumed to be known the upper bound

$$M_q = \sup\{\|u\|_{1,q,\Omega} :\ u \in W^{1,q}_{\Gamma}(\Omega),\ \|(-\Delta + I)u\|_{[W^{1,q'}_{\Gamma}(\Omega)]'} \leq 1\}, \tag{52}$$

with $W^{1,q}_{\Gamma}(\Omega) = \{v \in W^{1,q}(\Omega) :\ v = 0 \text{ on } \Gamma\}$.

Here, we consider the following mixed Neumann-power type problem to a linear elliptic equation:
(NPP) Find u such that verifies, in the sense of distributions,

$$-\nabla\cdot(\mathsf{A}\nabla u) = f - \nabla\cdot\mathbf{f} \quad \text{in} \quad \Omega; \tag{53}$$

$$(\mathsf{A}\nabla u - \mathbf{f})\cdot\mathbf{n} = (h - b(u)u)\chi_\Gamma \quad \text{on} \quad \partial\Omega, \tag{54}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$. Even more, instead of (52) we set

$$M_p = \|(-\Delta^{\mathrm{R}}_{\Omega})^{-1}\|_{\mathrm{op}},$$

where $\Delta^{\mathrm{R}}_{\Omega}$ is the isomorphism from $W^{1,p}(\Omega)$ onto $[W^{1,p'}(\Omega)]'$ defined by

$$\langle -\Delta^{\mathrm{R}}_{\Omega} w, v\rangle = \int_\Omega \nabla w\cdot\nabla v \mathrm{dx} + \int_\Gamma wv \mathrm{ds}.$$

We endow the Sobolev space $W^{1,p}(\Omega)$ with the norm

$$\|v\|_{1,p,\Omega} = \|\nabla v\|_{p,\Omega} + \|v\|_{2,\Gamma}.$$

Although its existence and uniqueness of solutions to (NPP) are classical in appropriate subspace of $H^1(\Omega)$, namely $V_{2,\ell}(\Omega)$, the $W^{1,p}$-regularity $(p > 2)$ of the weak solution is a hardship. Even if the leading coefficient is assumed either to be in VMO [34] or to verify a minimal condition [8] or if provided by the

Laplacian operator, *i.e.* $\mathsf{A} = \mathsf{I}$ [12, 35], the use of H^2-regularity is not allowed since our right hand side does not belong to a Lebesgue space.

For reader convenience, we exhibit the explicit constant involved in the $W^{1,p}(\Omega)$ estimate established in [10].

Theorem 7.1. *Let Ω be a C^1 domain, the assumptions (1)-(3) be fulfilled, and*

$$\upsilon_{\mathrm{U}} = (8^n+1)2^{6n}\left[P_n\left(\left(\frac{4a^\#}{a_\#}\right)^2+\frac{4+\nu_0}{a_\#}\right)^{1/2}+1\right]^2,$$

where $\nu_0 = \nu_0(f)$ is a positive constant if $f \neq 0$, and $\nu_0(0) = 0$ otherwise. Here, P_n is a function on the spatial variable n, and in particular $P_2 = 3/\sqrt{\pi}$, and

$$P_3 = \frac{2}{\sqrt{\pi}}\left(\frac{2}{\Gamma(3/2)}\right)^{1/3}\left(3^{-7/6}+\frac{4}{\pi}\frac{3^{1/3}}{5^{1/6}}\sin(5\pi/6)\right).$$

If $\mathbf{f} \in \mathbf{L}^{2+\varepsilon}(\Omega)$, $f \in L^{2+\varepsilon}(\Omega)$, and $h \in L^{2+\varepsilon}(\Gamma)$ for any $\varepsilon \in]0, 4/((n+2)(\upsilon_{\mathrm{U}} - 1))[$, then there exists a weak solution $u \in V_{2,\ell}(\Omega)$ to (53)-(54), in the sense

$$\int_\Omega (\mathsf{A}\nabla u)\cdot\nabla v \mathrm{dx} + \int_\Gamma b(u)uv\mathrm{ds} = \int_\Omega \mathbf{f}\cdot\nabla v\mathrm{dx} + \\ + \int_\Omega fv\mathrm{dx} + \int_\Gamma hv\mathrm{ds}, \quad \forall v \in V_{2,\ell}(\Omega), \tag{55}$$

such that belongs to $W^{1,2+\varepsilon}(\Omega)$. In particular,

$$\|\nabla u\|_{2+\varepsilon,\Omega}^{2+\varepsilon} \le \frac{2^{n(1+\varepsilon/2)}(2^n+1)}{4-(n+2)(\upsilon_{\mathrm{U}}-1)\varepsilon}\left[\left(\frac{8}{(r_\#)^n}\right)^{\varepsilon/2} 4\|\nabla u\|_{2,\Omega}^{2+\varepsilon} + \right. \\ + \left(2^{2+(n+1)\varepsilon/2}+\upsilon_{\mathrm{U}}(4+(n+2)\varepsilon)\right)\|\mathcal{F}\|_{2+\varepsilon,\Omega}^{2+\varepsilon} + \\ \left. +(4+\upsilon_{\mathrm{U}}(4+(n+2)\varepsilon))\|\mathcal{H}\|_{2+\varepsilon,\Gamma}^{2+\varepsilon}\right], \tag{56}$$

where

$$\mathcal{F} = (a_\#)^{-1/2}\left[\left(\frac{2}{a_\#}+2\right)|\mathbf{f}|^2+\frac{1}{\nu_0}|f|^2\right]^{1/2};$$

$$\mathcal{H} = 2\frac{\sqrt{2+2^{-1/n}a_\#}}{a_\#}K_{2n/(n+1)}|h+b^\#\left(\operatorname*{ess\,sup}_\Omega|u|\right)^{\ell-1}|,$$

with $r_\# > 0$ being dependent on the space dimension.

Let us extend the existence result for the mixed Dirichlet-Neumann problem [20, Theorem 1] to the following one for the mixed Neumann-power type problem (NPP).

Proposition 7.2. *Suppose $\ell=2$. If A is symmetric, $\mathbf{f}\in \mathbf{L}^p(\Omega)$, $f\in L^{pn/(p+n)}(\Omega)$, and $h\in L^2(\Gamma)$, with $p>2$ such that*

$$a^\# /M_p > \varkappa := \max\{\sqrt{(a^\#)^2-(a_\#)^2}, |a^\# - a_\# b_\# /a^\#|\}, \tag{57}$$

then the weak solution $u\in V_{2,\ell}(\Omega)$ of (55) satisfies

$$\|\nabla u\|_{p,\Omega} + \|u\|_{2,\Gamma} \leq \frac{M_p a_\#}{a^\# - \varkappa M_p} (\|\mathbf{f}\|_{p,\Omega} + S_{p'}\|f\|_{pn/(p+n),\Omega} + \|h\|_{2,\Gamma}), \tag{58}$$

where $S_{p'}$ is according to Remark 3.2.

Proof. The monotone theory for elliptic equations (see for instance [33, Corollary 2.2, p. 39]) ensures the existence of $u\in V_{2,\ell}(\Omega)$ solving

$$\int_\Omega \nabla u\cdot\nabla v\mathrm{dx} + \int_\Gamma b(u)uv\mathrm{ds} = \int_\Omega \mathbf{F}\cdot \nabla v\mathrm{dx} + t\int_\Omega fv\mathrm{dx} + \\ + \int_\Gamma Gv\mathrm{ds}, \quad \forall v\in V_{2,\ell}(\Omega), \tag{59}$$

with

$$\begin{aligned} \mathbf{F} &= (\mathsf{I}-t\mathsf{A})\nabla u + t\mathbf{f}; \\ G &= (1-tb(u))u + th, \end{aligned}$$

for any $t>0$.

We seek for a unique fixed point of the continuous linear mapping $Q: W^{1,p}(\Omega)\to W^{1,p}(\Omega)$ defined by $Qw = (-\Delta_\Omega^{\mathrm{R}})^{-1}(L_t w)$, with

$$\langle L_t w, v\rangle = \int_\Omega [(\mathsf{I}-t\mathsf{A})\nabla w + t\mathbf{f}]\cdot\nabla v\mathrm{dx} + t\int_\Omega fv\mathrm{dx} + \int_\Gamma [(1-tb(u))w + th]v\mathrm{ds},$$

for all $v\in W^{1,p'}(\Omega)$.

The existence of a unique fixed point is guaranteed if Q is strictly contractive. Let u_1, $u_2\in W^{1,p}(\Omega)$ be arbitrary, then

$$\|Qu_1 - Qu_2\|_{1,p,\Omega} \leq M_p\left(\|(\mathsf{I}-t\mathsf{A})\nabla(u_1-u_2)\|_{p,\Omega} + \|(1-tb(u))(u_1-u_2)\|_{2,\Gamma}\right).$$

For all $\mathbf{y}\in \mathbf{L}^p(\Omega)$, we have the relation [20]

$$\|(\mathsf{I} - \frac{a_\#}{(a^\#)^2}\mathsf{A})\mathbf{y}\|_{p,\Omega} \leq \sqrt{1-(a_\#/a^\#)^2}\|\mathbf{y}\|_{p,\Omega}. \tag{60}$$

Letting $t = a_\#(a^\#)^{-2}$, gathering the two above inequalities we obtain

$$\|Qu_1 - Qu_2\|_{1,p,\Omega} \le M_p \left(\sqrt{1-(a_\#/a^\#)^2}\|\nabla(u_1-u_2)\|_{p,\Omega} + \right.$$
$$\left. + |1 - a_\# b_\#/(a^\#)^2| \|u_1 - u_2\|_{2,\Gamma} \right).$$

By (57), Q is a strict contraction, and then there exists $w \in W^{1,p}(\Omega)$ such that $w = (-\Delta_\Omega^{\mathrm{R}})^{-1}(L_t w)$. By uniqueness of solution in $V_{2,\ell}(\Omega)$, then $w \equiv u$ verifies

$$\|u\|_{1,p,\Omega} \le \frac{M_p}{a^\#}\left(\varkappa \|u\|_{1,p,\Omega} + \frac{a_\#}{a^\#}(\|\mathbf{f}\|_{p,\Omega} + S_{p'}\|f\|_{pn/(p+n),\Omega} + \|h\|_{2,\Gamma}) \right),$$

since $p' < 2$, which implies (58). □

Remark 7.3. The choice of the involved constant in (57), which comes from (60), is not optimal. In the work [29] the author shows that if there exists $\theta \in [0,1[$ such that A verifies

$$\sum_{i,j=1}^n \frac{1}{2}(A_{ij} - A_{ji})\xi_i \eta_j \le a_\# \theta \left(\sum_{i=1}^n \xi_i^2 \right)^{1/2} \left(\sum_{j=1}^n \eta_j^2 \right)^{1/2}$$

then

$$\|(\mathsf{I} - \frac{1}{a^\#}\mathsf{A})\mathbf{y}\|_{p,\Omega} \le c(1-(1-\theta)a_\#/a^\#)\|\mathbf{y}\|_{p,\Omega}, \quad \forall \mathbf{y} \in \mathbf{L}^p(\Omega),$$

where $c > 1$ is dependent on p $(p \ge 2)$ and the space dimension n as follows

$$\left(\sum_{i=1}^n |y_i|^2 \right)^{p/2} \le c^p \sum_{i=1}^n |y_i|^p.$$

In particular, $c = 2^{1/2-1/p}$ if $n = 2$. For a symmetric A, we emphasize that $\theta = 0$, and

$$c(1 - a_\#/a^\#) \le \sqrt{1-(a_\#/a^\#)^2} \quad \text{if } a_\#/a^\# \ge \frac{c^2-1}{c^2+1}.$$

Moreover, if instead (57) we suppose

$$M_p \max\{c(1-a_\#/a^\#), |1 - a_\# b_\#/(a^\#)^2|\} \le 1 - a_\#/(2a^\#),$$

then (58) reads

$$\|\nabla u\|_{p,\Omega} + \|u\|_{2,\Gamma} \le \frac{2M_p}{a_\#}(\|\mathbf{f}\|_{p,\Omega} + S_{p'}\|f\|_{pn/(p+n),\Omega} + \|h\|_{2,\Gamma}).$$

Remark 7.4. We emphasize that Proposition 7.2 does not contradicts the counterexample of the existence of a function $u \in H^1(\mathbb{R}^2)$ solving the elliptic equation $\nabla \cdot (\mathsf{A}\nabla u) = 0$ in $\mathbb{R}^2$ such that does not belong to $W^{1,p}(\Omega)$ for some $p > 2$.

REFERENCES

[1] W. Arendt, *The abstract Cauchy problem, special semigroups and perturbation*, in One-parameter semigroups of positive operators, R. Nagel Ed. Springer-Verlag, Berlin-Heidelberg 1986, 26–46.

[2] W. Arendt, *Semigroups and evolution equations: functional calculus, regularity and kernel estimates*, in Handbook of Differential Equations, Evolutionary Equation 1, C. M. Dafermos and E. Feireisl Eds. Elsevier 2004, 1–86.

[3] A. A. Arkhipova, *L_p-estimates of the gradients of solutions of initial/boundary-value problems for quasilinear parabolic systems*, J. Math. Sci. 73 (6) (1995), 609–617.

[4] A. A. Arkhipova, *Modifications of the Gehring lemma appearing in the study of parabolic initial-boundary-value problems*, J. Math. Sci. 97 (4) (1999), 4189–4205.

[5] A. Bensoussan - J. L. Lions - G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam 1978.

[6] M. Biegert - M. Warma, *The heat equation with nonlinear generalized Robin boundary conditions*, J. Differential Equations 247 (2009), 1949–1979.

[7] M. Bonforte - G. Grillo - J. L. Vazquez, *Quantitative bounds for subcritical semilinear elliptic equations*, Milan J. Math. 80 (1) (2012), 65–118.

[8] S. Byun - H. Chen - M. Kim - L. Wang, *L^p regularity theory for linear elliptic systems*, Discrete Contin. Dyn. Syst. 18 (1) (2007), 121–134.

[9] L. Consiglieri, *Mathematical analysis of selected problems from fluid thermomechanics. The $(p-q)$ coupled fluid-energy systems*, Lambert Academic Publishing, Saarbrücken 2011.

[10] L. Consiglieri, *Radiative effects on the thermoelectric problems*, arXiv:1312.3960

[11] L. Consiglieri, *Explicit estimates on a mixed Neumann-Robin-Cauchy problem*, arXiv:1406.1895

[12] H. Ding, *A regularity result for boundary value problems on Lipschitz domains*, Ann. Fac. Sci. Toulouse Math. 5^e série 10 (2) (1989), 325–333.

[13] P.-É. Druet, *Weak solutions to a time-dependent heat equation with nonlocal radiation boundary condition and arbitrary p-summable right-hand side*, Appl. Math. 55 (2) (2010), 111–149.

[14] C. Ebmeyer, *Nonlinear elliptic problems under mixed boundary value conditions in nonsmooth domains*, SIAM J. Math. Anal. 32 (1) (2000), 103–118.

[15] L. Fattorusso - M. Marino, *Differenziabilità locale per sistemi parabolici non lineari del secondo ordine con non linearità $q \geq 2$*, Ric. Mat. 41 (1992), 89–112.

[16] D. Gilbarg - N. S. Trudinger, *Elliptic partial differential equations of second order*, GMW 224, Springer-Verlag, New York 1983.

[17] J. A. Griepentrog - L. Recke, *Linear elliptic boundary value problems with nonsmooth data: Normal solvability on Sobolev-Campanato spaces*, Math. Nachr. 225 (2001), 39–74.

[18] J. A. Griepentrog, *Linear elliptic boundary value problems with non-smooth data: Campanato spaces of functionals*, Math. Nachr. 243 (2002), 19–42.

[19] P. Grisvard, *Équations différentielles abstraites*, Annales Scientifiques de l'É.N.S. 4^e série 2 (3) (1969), 311–395.

[20] K. Gröger - J. Rehberg, *Resolvent estimates in $W^{1,-p}$ for second order elliptic differential operators in case of mixed boundary conditions*, Math. Ann. 285 (1989), 105–113.

[21] M. Guzmán, *Differentiation of integrals in R^n*, Springer-Verlag, Berlin, Heidelberg, New York 1975.

[22] J. Haga - N. Kikuchi, *On the higher integrability for the gradients of the solutions to difference partial differential systems of elliptic-parabolic type*, Z. Angew. Math. Phys. 51 (2000), 290–303.

[23] R. Haller-Dintelmann - C. Meyer - J. Rehberg - A. Schiela, *Hölder continuity and optimal control for nonsmooth elliptic problems*, Appl. Math. Optim. 60 (3) (2009), 397–428.

[24] R. Haller-Dintelmann - J. Rehberg, *Maximal parabolic regularity for divergence operators including mixed boundary conditions*, J. Differential Equations 247 (5) (2009), 1354–1396.

[25] M. Kassmann - W. R. Madych, *Difference quotients and elliptic mixed boundary value problems of second order*, Indiana Univ. Math. J. 56 (3) (2007), 1047–1082.

[26] O. A. Ladyzhenskaya - N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, "Nauka", Moskow, 1964, English translation Mathematics in Science and Engineering 46, Academic Press, New York-London 1968.

[27] J. Naumann - J. Wolf, *Interior differentiability of weak solutions of parabolic systems with quadratic growth nonlinearities*, Rend. Sem. Mat. Univ. Padova 98 (1997), 253–272.

[28] J. Naumann - M. Wolff, *A global L^p-estimate on weak solutions of nonlinear parabolic systems under mixed boundary conditions*, www2.mathematik.hu-berlin.de/publ/pre/1995/p-95-2.ps

[29] J. Nečas, *Sur la regularité des solutions faibles des équations elliptiques non linéaires*, Comment. Math. Univ. Carolin. 9 (3) (1968), 365–413.

[30] M. Parviainen, *Global higher integrability for parabolic quasiminimizers in nonsmooth domains*, Calc. Var. 31 (2008), 75–98.

[31] S. M. Rankin III, *Semilinear evolution equations in Banach spaces with application to parabolic partial differential equations*, Trans. Amer. Math. Soc. 336 (2) (1993), 523–535.

[32] G. Savaré, *Regularity and perturbation results for mixed second order elliptic problems*, Comm. Partial Differential Equations 22 (5-6) (1997), 869–899.

[33] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs 49, AMS, Providence RI 1997.

[34] C. Vitanza, *$W^{1,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients*, Le Matematiche 47 (1) (1992), 177–186.

[35] I. Wood, *Maximal L^p-regularity for the Laplacian on Lipschitz domains*, Math. Z. 255 (4) (2007), 855–875.

LUISA CONSIGLIERI
Independent Researcher Professor, European Union
http://sites.google.com/site/luisaconsiglieri
e-mail: lconsiglieri@gmail.com