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HADAMARD PRODUCT CONCERNING CERTAIN MEROMORPHIC FUNCTIONS

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In this paper the authors introduced a new generalized differintegral operator for meromorphic univalent functions in $U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\}$. The objective of this paper is to establish certain results concerning the Hadamard product of functions in the classes $\Sigma^{*,m}_{\mu,\lambda}(\alpha,\beta,\gamma,k)$ and $\Sigma^{h}_{\mu,\lambda}(\alpha,\beta,\gamma,k)$.

1. Introduction

Throughout this paper, let the functions f of the form :

$$\varphi(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0;\, c_n \geq 0), \tag{1}$$

and

$$\psi(z) = d_1 z + \sum_{n=2}^{\infty} d_n z^n \qquad (d_1 > 0;\, d_n \geq 0) \tag{2}$$

be regular and univalent in the unit disc $U = \{z : |z| < 1\}$ and let

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_0 > 0;\, a_n \geq 0), \tag{3}$$

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$$f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \qquad (a_{0,i} > 0;\ a_{n,i} \geq 0), \tag{4}$$

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \qquad (b_0 > 0; b_n \geq 0), \tag{5}$$

and

$$g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^n \qquad (b_{0,j} > 0;\ b_{n,j} \geq 0), \tag{6}$$

be regular and univalent in the punctured unit disc $U^* = \{z : 0 < |z| < 1\} = U \backslash \{0\}$.

For a function $f(z)$ defined by (3) (with $a_0 = 1$), El-Ashwah defined the integral operator $\Im_{\lambda}^{\beta,m}(\alpha,\mu)$ (see [10, with $p = 1$]) as follows:

$$\Im_{\lambda}^{\beta,m}(\alpha,\mu) f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{\alpha+\beta}{\alpha+\beta+(\mu+\lambda)(n+1)} \right]^m a_n z^n$$

$$(\alpha,\beta,\mu,\lambda \geq 0,\ \alpha+\beta \neq 0,\ \mu+\lambda \neq 0;\ m \in \mathbb{N}_0;\ z \in U^*). \tag{7}$$

We note that
(i) $\Im_{\lambda}^{0,m}(\alpha,0) f(z) = I^m(\lambda,\alpha) f(z) (\alpha > 0, \lambda \geq 0, m \in \mathbb{N}_0)$ (see El-Ashwah [11, with $p = 1$]);
(ii) $\Im_{0}^{0,\gamma}(1,1) f(z) = P^{\gamma} f(z) (\gamma > 0)$ (see Aqlan et al. [4, with $p = 1$]);
(iii) $\Im_{0}^{0,m}(\alpha,\mu) f(z) = \mathcal{L}^m(\alpha,\mu) f(z) (\alpha > 0, \mu \geq 0, m \in \mathbb{N}_0)$ (see Bulboaca et al. [6]);
(iv) $\Im_{0}^{0,\gamma}(1,\mu) f(z) = P_{\mu}^{\gamma} f(z) (\gamma > 0, \mu > 0)$ (see Lashin [18]).

Also we note that
(i) $\Im_{1}^{0,m}(\alpha,1) f(z) = \Im^m(\alpha) f(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} \left(\frac{\alpha}{\alpha+2(k+1)} \right)^m a_n z^n (\alpha > 0, m \in \mathbb{N}_0)$:
(ii) $\Im_{p,\lambda}^{0,m}(1,0) f(z) = \Im_{p,\lambda}^{m} f(z) = \frac{1}{z^p} + \sum\limits_{n=1}^{\infty} \left(\frac{1}{1+\lambda(k+1)} \right)^m a_n z^{n-p} (\lambda \geq 0, m \in \mathbb{N}_0)$.

Now we will extend the definition of the operator $\Im_{\lambda}^{\beta,m}(\alpha,\mu) f(z)$ for $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $z \in U^*$ as follows:

$$\Im_{\lambda}^{\beta,m-1}(\alpha,\mu) f(z) = \left(\tfrac{\mu+\lambda}{\alpha+\beta} \right) z^{-\frac{\alpha+\beta}{\mu+\lambda}} \left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda} \right)+1} \Im_{\lambda}^{\beta,m}(\alpha,\mu) f(z) \right)',$$

$$\Im_{\lambda}^{\beta,m-2}(\alpha,\mu) f(z) = \left(\tfrac{\mu+\lambda}{\alpha+\beta} \right) z^{-\frac{\alpha+\beta}{\mu+\lambda}} \left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda} \right)+1} \Im_{\lambda}^{\beta,m-1}(\alpha,\mu) f(z) \right)',$$

$$\Im_{\lambda}^{\beta,1}(\alpha,\mu)f(z)=\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)z^{-\frac{\alpha+\beta}{\mu+\lambda}}\left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1}\Im_{\lambda}^{\beta,2}(\alpha,\mu)f(z)\right)',$$

$$\Im_{\lambda}^{\beta,0}(\alpha,\mu)f(z)=\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)z^{-\frac{\alpha+\beta}{\mu+\lambda}}\left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1}\Im_{\lambda}^{\beta,1}(\alpha,\mu)f(z)\right)'=f(z)$$

$$\Im_{\lambda}^{\beta,-1}(\alpha,\mu)f(z)=\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)z^{-\frac{\alpha+\beta}{\mu+\lambda}}\left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1}\Im_{\lambda}^{\beta,0}(\alpha,\mu)f(z)\right)',$$

$$\Im_{\lambda}^{\beta,-2}(\alpha,\mu)f(z)=\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)z^{-\frac{\alpha+\beta}{\mu+\lambda}}\left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1}\Im_{\lambda}^{\beta,-1}(\alpha,\mu)f(z)\right)',$$

$$\Im_{\lambda}^{\beta,-m+1}(\alpha,\mu)f(z)=\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)z^{-\frac{\alpha+\beta}{\mu+\lambda}}\left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1}\Im_{\lambda}^{\beta,-m+2}(\alpha,\mu)f(z)\right)',$$

$$\Im_{\lambda}^{\beta,-m}(\alpha,\mu)f(z)=\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)z^{-\frac{\alpha+\beta}{\mu+\lambda}}\left(z^{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1}\Im_{\lambda}^{\beta,-m+1}(\alpha,\mu)f(z)\right)'.$$

We see that for $f\in\Sigma$, we have

$$\Im_{\lambda}^{\beta,-m}(\alpha,\mu)f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{\alpha+\beta}{\alpha+\beta+(\mu+\lambda)(n+1)}\right]^{-m}a_nz^n$$

$$(\alpha,\beta,\mu,\lambda\geq 0,\ \alpha+\beta\neq 0,\ \mu+\lambda\neq 0;\ m\in\mathbb{N}_0;\ z\in U^*).$$

We note that
(i) $\Im_{0}^{0,-m}(\alpha,\mu)f(z)=J^m(\alpha,\mu)f(z)(\alpha>0,\mu\geq 0,m\in\mathbb{N}_0)$ (see El-Ashwah [13, with $p=1$]);
(ii) $\Im_{1}^{0,-m}(1,\mu)f(z)=I(m,\mu)f(z)(\mu\geq 0,m\in\mathbb{N}_0)$ (see Cho et al. [7, 8]);
(iii) $\Im_{1}^{0,-m}(\alpha,1)f(z))=D_{\alpha}^{m}f(z)(\alpha>0,m\in\mathbb{N}_0)$ (see Al-Oboudi and Al-Zkeri [1]);
(iv) $\Im_{1}^{0,-m}(1,1)f(z)=I^mf(z)(m\in\mathbb{N}_0)$ (see Uralegaddi and Somanatha [24]).

On the other hand for $\alpha+\beta=\mu+\lambda=m=1$, we have

$$\Im_{\lambda}^{\beta,-1}(\alpha,\mu)f(z)=\frac{\left(z^2f(z)\right)'}{z}=\frac{1}{z}+\sum_{n=1}^{\infty}(n+2)a_nz^n.$$

In general we will write

$$\Im_{\lambda}^{\beta,m}(\alpha,\mu)f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{m}a_nz^n$$

$$(\alpha,\beta,\mu,\lambda\geq 0,\ \alpha+\beta\neq 0,\ \mu+\lambda\neq 0;\ m\in\mathbb{Z};\ z\in U^*). \tag{8}$$

With the help of the differintegral operator $\Im_{\lambda}^{\beta,m}(\alpha,\mu)f(z)$, we define the class $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ as follows.

Denote by $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ the class of functions $f(z)$ given by (3) which satisfy the condition

$$-\text{Re}\left(\frac{z\left(\Im_{\lambda}^{\beta,m}(\alpha,\mu)f(z)\right)'}{\Im_{\lambda}^{\beta,m}(\alpha,\mu)f(z)}+\gamma\right)>k\left|\frac{z\left(\Im_{\lambda}^{\beta,m}(\alpha,\mu)f(z)\right)'}{\Im_{\lambda}^{\beta,m}(\alpha,\mu)f(z)}+1\right|$$

$$(0\leq\gamma<1;\ k\geq 0;\ \alpha,\beta,\mu,\lambda\geq 0,\ \alpha+\beta\neq 0,\ \mu+\lambda\neq 0;\ m\in\mathbb{Z};\ z\in U^*). \quad (9)$$

Taking $m=k=0$, the class $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ with $a_0=1$ will reduce to the class $\sum S^*(\gamma)$ (see Mogra et al. [21, with $\beta=1$]), and El-Ashwah and Aouf ([12, with $\beta=1$ and $k=0$]).

Also, we observe that the class $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ reduces to several interesting many other classes for different choices of $\alpha,\beta,\mu,\lambda,k$ and m.

Using similar arguments as given in [5] (see also Aouf et al. [3, with $\lambda=0$]), we can easily prove the following results for functions in the class $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$.

A function $f(z)\in\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)(\alpha,\beta,\mu,\lambda\geq 0,\alpha+\beta\neq 0,\mu+\lambda\neq 0;m\in\mathbb{Z})$ if and only if

$$\sum_{n=1}^{\infty}\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{m}[n(k+1)+(k+\gamma)]a_n\leq(1-\gamma)a_0, \quad (10)$$

where $0\leq\gamma<1,k\geq 0,\alpha,\beta,\mu,\lambda\geq 0,\alpha+\beta\neq 0,\mu+\lambda\neq 0$ and $m\in\mathbb{N}_0$.

In this paper we introduce the following class of meromorphic univalent functions in U^*.

A function $f(z)\in\sum_{\mu,\lambda}^{h}(\alpha,\beta,\gamma,k)$ if and only if

$$\sum_{n=1}^{\infty}\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{h}[n(k+1)+(k+\gamma)]a_n\leq(1-\gamma)a_0, \quad (11)$$

where $0\leq\gamma<1,k\geq 0,\alpha,\beta,\mu,\lambda\geq 0,\alpha+\beta\neq 0,\mu+\lambda\neq 0$ and $h\in\mathbb{N}_0$.

Further, $\sum_{\mu,\lambda}^{h}(\alpha,\beta,\gamma,k)\subset\sum_{\mu,\lambda}^{\varphi}(\alpha,\beta,\gamma,k)$ if $h>\varphi$, the containment being proper. Moreover, for any positive integer $h>h-1>\cdots>m+1>m$, we have the following inclusion relation
$\sum_{\mu,\lambda}^{h}(\alpha,\beta,\gamma,k)\subset\sum_{\mu,\lambda}^{h-1}(\alpha,\beta,\gamma,k)\subset\cdots\subset\sum_{\mu,\lambda}^{m+1}(\alpha,\beta,\gamma,k)\subset\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$.

We also note that, for every real number h, the class $\sum_{\mu,\lambda}^{h}(\alpha,\beta,\gamma,k)$ is nonempty as the functions of the form

$$f(z)=\frac{a_0}{z}+\sum_{n=1}^{\infty}\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-h}\left\{\frac{(1-\gamma)}{n(k+1)+(k+\gamma)}\right\}a_0\lambda_n z^n, \quad (12)$$

where $a_0 > 0$, $\lambda_n \geq 0$ and $\sum\limits_{n=1}^{\infty} \lambda_n \leq 1$, satisfy the inequality (11).

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa [22], Kumar [15–17], Mogra [19, 20], Aouf and Darwish [2], Darwish [9], Hossen [14] and Sekine [23]. Accordingly, the quasi-Hadamard product of two functions $\varphi(z)$ and $\psi(z)$ given by (1) and (2) is defined by

$$(\varphi * \psi)(z) = c_1 d_1 z + \sum_{k=2}^{\infty} c_n d_n z^n.$$

Let us define the Hadamard product of two meromorphic univalent functions $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{a_0 b_0}{z} + \sum_{n=1}^{\infty} a_n b_n z^n. \tag{13}$$

Objective of this paper is to establish certain results concerning the Hadamard product of functions for the classes $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ and $\sum_{\mu,\lambda}^{m+1}(\alpha,\beta,\gamma,k)$ analogous to the results due to Mogra [19, 20].

2. The Main Theorems

Unless otherwise mentioned we shall assume throughout the paper that $0 \leq \gamma < 1, k \geq 0, \alpha, \beta, \mu, \lambda \geq 0, \alpha+\beta \neq 0, \mu+\lambda \neq 0$ and $m \in \mathbb{N}_0$.

Theorem 2.1. *Let the functions $f_i(z)$ belong to the class $\sum_{\mu,\lambda}^{m+1}(\alpha,\beta,\gamma,k)$ for every $i = 1,2,\dots,r$. Then the Hadamard product $f_1 * f_2 * \cdots * f_r(z)$ belongs to the class $\sum_{\mu,\lambda}^{r(m+2)-1}(\alpha,\beta,\gamma,k)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty} \left\{ \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{r(m+2)-1} [n(k+1)+(k+\gamma)] \prod_{i=1}^{r} a_{n,i} \right\} \leq (1-\gamma) \left[\prod_{i=1}^{r} a_{0,i} \right]. \tag{14}$$

Since $f_i(z) \in \sum_{\mu,\lambda}^{m+1}(\alpha,\beta,\gamma,k)$, we have

$$\sum_{n=1}^{\infty} \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{m+1} [n(k+1)+(k+\gamma)] a_{n,i} \leq (1-\gamma) a_{0,i} \tag{15}$$

for every $i = 1,2,\dots,r$. Therefore,

$$\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{m+1} [n(k+1)+(k+\gamma)] a_{n,i} \leq (1-\gamma) a_{0,i}$$

or

$$a_{n,i} \leq \left\{ \left[\frac{\alpha+\beta}{\alpha+\beta+(\mu+\lambda)(n+1)} \right]^{m+1} \frac{(1-\gamma)}{n(k+1)+(k+\gamma)} \right\} a_{0,i},$$

for every $i = 1,2,\ldots,r$ implies

$$a_{n,i} \leq \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{-(m+2)} a_{0,i}, \tag{16}$$

for every $i = 1,2,\ldots,r$.

Using (16) for $i = 1,2,\ldots,r-1$, and (15) for $i = r$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\{ \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{r(m+2)-1} [n(k+1)+(k+\gamma)] \prod_{i=1}^{r} a_{n,i} \right\} \\
&\quad \leq \sum_{n=1}^{\infty} \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{r(m+2)-1} [n(k+1)+(k+\gamma)] a_{n,r} \\
&\qquad \cdot \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{-(m+2)(r-1)} \prod_{i=1}^{r-1} a_{0,i} \\
&= \left[\prod_{i=1}^{r-1} a_{0,i} \right] \sum_{n=1}^{\infty} \left\{ \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{m+1} [n(k+1)+(k+\gamma)] a_{n,r} \right\} \\
&\qquad \leq (1-\gamma) \left[\prod_{i=1}^{r} a_{0,i} \right].
\end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_r(z) \in \sum_{\mu,\lambda}^{r(m+2)-1}(\alpha,\beta,\gamma,k)$. This completes the proof of the theorem. □

Theorem 2.2. *Let the functions $f_i(z)$ belong to the class $\sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ for every $i = 1,2,\ldots,r$. Then the Hadamard product $f_1 * f_2 * \cdots * f_r(z)$ belongs to the class $\sum_{\mu,\lambda}^{r(m+1)-1}(\alpha,\beta,\gamma,k)$.*

Proof. Since $f_i(z) \in \sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$, we have

$$\sum_{n=1}^{\infty} \left\{ \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta} \right]^{m} [n(k+1)+(k+\gamma)] a_{n,i} \right\} \leq (1-\gamma) a_{0,i} \tag{17}$$

for every $i = 1,2,\ldots,r$. Therefore

$$a_{n,i} \leq \left\{ \left[\frac{\alpha+\beta}{\alpha+\beta+(\mu+\lambda)(n+1)} \right]^{m} \frac{(1-\gamma)}{n(k+1)+(k+\gamma)} \right\} a_{0,i},$$

and hence

$$a_{n,i} \leq \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-(m+1)} a_{0,i}, \tag{18}$$

for every $i = 1,2,\ldots,r$.

Using (18) for $i = 1,2,\ldots,r-1$, and (17) for $i = r$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{r(m+1)-1}[n(k+1)+(k+\gamma)]\prod_{i=1}^{r}a_{n,i}\right\}\\
&\quad\leq \sum_{n=1}^{\infty}\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{r(m+1)-1}[n(k+1)+(k+\gamma)]a_{n,r}\\
&\qquad\cdot\left[\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-(m+1)(r-1)}\prod_{i=1}^{r-1}a_{0,i}\right]\\
&\quad=\left[\prod_{i=1}^{r-1}a_{0,i}\right]\sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{m}[n(k+1)+(k+\gamma)]a_{n,r}\right\}\\
&\qquad\qquad\leq(1-\gamma)\left[\prod_{i=1}^{r}a_{0,i}\right].
\end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_r(z) \in \Sigma_{\mu,\lambda}^{r(m+1)-1}(\alpha,\beta,\gamma,k)$. This completes the proof of the theorem. □

Theorem 2.3. *Let the functions $f_i(z)$ belong to the class $\Sigma_{\mu,\lambda}^{m+1}(\alpha,\beta,\gamma,k)$ for every $i = 1,2,\ldots,r$ and let the functions $g_j(z)$ belong to the class $\Sigma_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$ for every $j = 1,2,\ldots,q$. Then the Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $\Sigma_{\mu,\lambda}^{r(m+2)+q(m+1)-1}(\alpha,\beta,\gamma,k)$.*

Proof. We denote the Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_q(z)$ by the function $H(z)$, for the sake of convenience. Clearly,

$$H(z) = \left[\prod_{i=1}^{r}a_{0,i}\cdot\prod_{j=1}^{q}b_{0,j}\right]z^{-1} + \sum_{n=1}^{\infty}\left[\prod_{i=1}^{r}a_{n,i}\cdot\prod_{j=1}^{q}b_{n,j}\right]z^{n}.$$

To prove the theorem, we need to show that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{r(m+2)+q(m+1)-1}\right.\\
&\qquad\left.\cdot[n(k+1)+(k+\gamma)]\left[\prod_{i=1}^{r}a_{n,i}\cdot\prod_{j=1}^{q}b_{n,j}\right]\right\}\leq(1-\gamma)\left[\prod_{i=1}^{r}a_{0,i}\cdot\prod_{j=1}^{q}b_{0,j}\right].
\end{aligned}$$

Since $f_i(z) \in \sum_{\mu,\lambda}^{m+1}(\alpha,\beta,\gamma,k)$, the inequalities (15) and (16) hold for every $i = 1,2,\dots,r$. Further, since $g_j(z) \in \sum_{\mu,\lambda}^{*,m}(\alpha,\beta,\gamma,k)$, we have

$$\sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{m}[n(k+1)+(k+\gamma)]b_{n,j}\right\} \le (1-\gamma)b_{0,j}, \tag{19}$$

for every $j = 1,2,\dots,q$. Whence we obtain

$$b_{n,j} \le \left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-(m+1)} b_{0,j}, \tag{20}$$

for every $j = 1,2,\dots,q$.

Using (16) for $i = 1,2,\dots,r$, (20) for $j = 1,2,\dots q-1$ and (19) for $j = q$, we get

$$\sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{r(m+2)+q(m+1)-1}[n(k+1)+(k+\gamma)]\left[\prod_{i=1}^{r}a_{n,i}\cdot\prod_{j=1}^{q}b_{n,j}\right]\right\}$$

$$\le \sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{r(m+2)+q(m+1)-1}[n(k+1)+(k+\gamma)]\right.$$

$$\left.\left[\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-r(m+2)}\prod_{i=1}^{r}a_{0,i}\cdot\prod_{j=1}^{q}b_{n,j}\right]\right\}$$

$$\le \sum_{n=1}^{\infty}\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{r(m+2)+q(m+1)-1}[n(k+1)+(k+\gamma)]$$

$$\cdot\left[\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-r(m+2)}\cdot\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{-(m+1)(q-1)}\cdot\prod_{i=1}^{r}a_{0,i}\cdot\prod_{j=1}^{q-1}b_{0,j}\right]b_{n,q}$$

$$= \left[\prod_{i=1}^{r}a_{0,i}\cdot\prod_{j=1}^{q-1}b_{0,j}\right]\sum_{n=1}^{\infty}\left\{\left[\frac{\alpha+\beta+(\mu+\lambda)(n+1)}{\alpha+\beta}\right]^{m}[n(k+1)+(k+\gamma)]b_{n,q}\right\}$$

$$\le (1-\gamma)\left[\prod_{i=1}^{r}a_{0,i}\cdot\prod_{j=1}^{q}b_{0,j}\right].$$

Hence $H(z) \in \sum_{\mu,\lambda}^{r(m+2)+q(m+1)-1}(\alpha,\beta,\gamma,k)$.

We note that the required estimate can also be obtained by using (16) for $i = 1,2,\dots,r-1$; (20) for $j = 1,2,\dots,q$and (15) for $i = r$. This completes the proof of the theorem. □

Remark 2.4. By specializing the parameters $\alpha,\beta,\mu,\lambda,k$ and m we can obtain corresponding results for various subclasses associated with various operators.

Note that other work related to differential operators can be seen in the following references (e.g. see [25, 30]).

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