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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH VARYING ARGUMENTS

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In this paper, we introduce new classes $VM(\beta)$ and $VN(\beta)$ of analytic functions with varying arguments in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Some properties such as coefficient estimates, extreme points, distortion theorems for functions $f(z)$ belonging to the classes are obtained.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Let $M(\beta)$ denote the subclass of A consisting of functions $f(z)$ which satisfy the inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \tag{2}$$

or, equivalently,

$$\left|\frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (2\beta - 1)}\right| < 1, \tag{3}$$

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for some $\beta(\beta > 1)$. Also let $N(\beta)$ denote the subclass of A consisting of functions $f(z)$ which satisfy the inequality:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \tag{4}$$

or, equivalently,

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{1 + \frac{zf''(z)}{f'(z)} - (2\beta - 1)} \right| < 1. \tag{5}$$

The classes $M(\beta)$ and $N(\beta)$ were introduced by Nishiwaki and Owa [3] (see also [1], [2], [4], [5] and [7]) and for $1 < \beta \leq \frac{4}{3}$ the classes $M(\beta)$ and $N(\beta)$ were introduced and studied by Uralegaddi et al. (see [8]). It follows from (2) and (4) we can see that (see [7])

$$f(z) \in N(\beta) \Longleftrightarrow zf^{'}(z) \in M(\beta).$$

Definition 1.1 ([6]). A function $f(z)$ defined by (1) is said to be in the class $V(\theta_n)$ if $f(z) \in A$ and $\arg(a_n) = \theta_n$ for all $n \geq 2$. If furthermore, there exists a real number α such that

$$\theta_n + (n-1)\alpha \equiv \pi \mod 2\pi,$$

then $f(z)$ is said to be in the class $V(\theta_n;\alpha)$. The union of $V(\theta_n;\alpha)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers α is denoted by V.

Denote by $VM(\beta)$ the subclass of V consisting of functions $f(z)$ in $M(\beta)$ and denote by $VN(\beta)$ the subclass of V consisting of functions $f(z)$ in $N(\beta)$.

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that $1 < \beta \leq \frac{4}{3}$.

To prove our main results we shall need the following lemmas.

Lemma 2.1 ([8]). *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\beta)\,|a_n| \leq \beta - 1, \tag{6}$$

then $f(z) \in M(\beta)$.

Lemma 2.2 ([8])**.** *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\beta)|a_n| \leq \beta - 1, \tag{7}$$

then $f(z) \in N(\beta)$.

Theorem 2.3. *Let the function $f(z)$ be of the form (1), then $f(z)$ is in the class $VM(\beta)$if and only if*

$$\sum_{n=2}^{\infty} (n-\beta)|a_n| \leq \beta - 1. \tag{8}$$

Proof. In view of Lemma 2.1, we need only to show the function $f(z)$ from the class $VM(\beta)$ satisfies the coefficient inequality (8). Let $f(z) \in VM(\beta)$. Then, from (1) and (3), we have

$$\left| \frac{\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{2(\beta-1) + \sum_{n=2}^{\infty} (n-2\beta+1)a_n z^{n-1}} \right| < 1.$$

Since $f(z) \in V$, $f(z)$ lies in the class $V(\theta_n, \alpha)$ for some sequence $\{\theta_n\}$ and a real number α such that $\theta_n + (n-1)\alpha \equiv \pi \mod 2\pi \, (n \geq 2)$, then setting $z = re^{i\alpha}$ in the above inequality, we get

$$\left| \frac{-\sum_{n=2}^{\infty} (n-1)|a_n| r^{n-1}}{2(\beta-1) - \sum_{n=2}^{\infty} (n-2\beta+1)|a_n| r^{n-1}} \right| < 1.$$

Since $Re\{w(z)\} < |w(z)| < 1$, we have

$$\Re \left\{ \frac{\sum_{n=2}^{\infty} (n-1)|a_n| r^{n-1}}{2(\beta-1) - \sum_{n=2}^{\infty} (n-2\beta+1)|a_n| r^{n-1}} \right\} < 1. \tag{9}$$

Hence

$$\sum_{n=2}^{\infty} (n-\beta)|a_n| r^{n-1} \leq (\beta-1),$$

which, upon letting $r \to 1^{-}$, readily yields the assertion (8). This completes the proof of Theorem 2.3. $\square$

Corollary 2.4. *Let the function $f(z)$ defined by (1) be in the class $VM(\beta)$, then*

$$|a_n| \leq \frac{(\beta-1)}{(n-\beta)} \ (n \geq 2). \tag{10}$$

The result is sharp for the function

$$f(z) = z + \frac{(\beta-1)}{(n-\beta)} e^{i\theta_n} z^n \ (n \geq 2). \tag{11}$$

Similarly, we can prove the following theorem for the class $VN(\beta)$.

Theorem 2.5. *Let the function $f(z)$ be of the form (1), then $f(z)$ is in the class $VN(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n\,(n-\beta)\,|a_n| \leq (\beta-1). \tag{12}$$

Corollary 2.6. *Let the function $f(z)$ defined by (1) be in the class $VN(\beta)$, then*

$$|a_n| \leq \frac{(\beta-1)}{n\,(n-\beta)} \ (n \geq 2). \tag{13}$$

The result is sharp for the function

$$f(z) = z + \frac{(\beta-1)}{n\,(n-\beta)} e^{i\theta_n} z^n \ (n \geq 2). \tag{14}$$

3. Distortion theorem

Theorem 3.1. *Let the function $f(z)$ defined by (1) be in the class $VM(\beta)$. Then*

$$|z| - \frac{(\beta-1)}{(2-\beta)}|z|^2 \leq |f(z)| \leq |z| + \frac{(\beta-1)}{(2-\beta)}|z|^2, \tag{15}$$

The result is sharp.

Proof. Since

$$\Psi(n) = (n-\beta), \tag{16}$$

is an increasing function of $n\,(n \geq 2)$, from Theorem 2.3, we have

$$(2-\beta)\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}(n-\beta)\,|a_n| \leq (\beta-1),$$

that is

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(\beta-1)}{(2-\beta)},$$

Thus

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \le |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$
$$\le |z| + \frac{(\beta-1)}{(2-\beta)} |z|^2 .$$

Similarly, we get

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} |a_n| \, |z|^n \ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$
$$\ge |z| - \frac{(\beta-1)}{(2-\beta)} |z|^2 .$$

This completes the proof of Theorem 3.1. Finally the result is sharp for the function

$$f(z) = z + \frac{(\beta-1)}{(2-\beta)} e^{i\theta_2} z^2 \tag{17}$$

$z = \pm |z| \, e^{-i\theta_2}$. □

Corollary 3.2. *Under the hypotheses of Theorem 3.1, $f(z)$ is included in a disc with center at the origin and radius r_1 given by*

$$r_1 = 1 + \frac{(\beta-1)}{(2-\beta)}.$$

Theorem 3.3. *Let the function $f(z)$ defined by (1) belong to the class $VM(\beta)$. Then*

$$1 - \frac{2(\beta-1)}{(2-\beta)} |z| \le \left| f^{'}(z) \right| \le 1 + \frac{2(\beta-1)}{(2-\beta)} |z| . \tag{18}$$

The result is sharp for the function $f(z)$ given by (17) at $z = \pm |z| \, e^{-i\theta_2}$.

Proof. Since $\{n\Psi(n)\}$, where $\Psi(n)$ given by (16) is increasing function of $n\,(n \ge 2)$, then in view of Theorem 2.5, we have

$$\frac{(2-\beta)}{2} \sum_{n=2}^{\infty} n \, |a_n| \le \sum_{n=2}^{\infty} (n-\beta) \, |a_n| \le (\beta-1) ,$$

that is

$$\sum_{n=2}^{\infty} n \, |a_n| \le \frac{2(\beta-1)}{(2-\beta)}.$$

Thus

$$\begin{aligned}\left|f^{'}(z)\right| &= \left|1+\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq 1+|z|\sum_{n=2}^{\infty} n|a_n| \\ &\leq 1+\frac{2(\beta-1)}{(2-\beta)}|z|.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\left|f^{'}(z)\right| &\geq 1-\sum_{n=2}^{\infty} n|a_n|\left|z^{n-1}\right| \geq 1-|z|\sum_{n=2}^{\infty} n|a_n| \\ &\geq 1-\frac{2(\beta-1)}{(2-\beta)}|z|.\end{aligned}$$

Finally the result is sharp for the function $f(z)$ given by (17). This completes the proof of Theorem 3.3. □

Corollary 3.4. *Let the function $f(z)$ defined by (1) be in the class $VM(\beta)$. Then $f^{'}(z)$ is included in a disc with center at the origin and radius r_2 given by*

$$r_2 = 1+\frac{2(\beta-1)}{(2-\beta)}.$$

Using the same technique as used in Theorems 3.1 and 3.3, we have the following theorems for functions in the class $VN(\beta)$:

Theorem 3.5. *Let the function $f(z)$ defined by (1) be in the class $VN(\beta)$. Then*

$$|z|-\frac{(\beta-1)}{2(2-\beta)}|z|^2 \leq |f(z)| \leq |z|+\frac{(\beta-1)}{2(2-\beta)}|z|^2,$$

The result is sharp for the function

$$f(z) = z+\frac{(\beta-1)}{2(2-\beta)}e^{i\theta_2}z^2 \tag{19}$$

at $z=\pm|z|e^{-i\theta_2}$.

Corollary 3.6. *Under the hypotheses of Theorem 3.5, $f(z)$ is included in a disc with center at the origin and radius r_3 given by*

$$r_3 = 1+\frac{(\beta-1)}{2(2-\beta)}.$$

Theorem 3.7. *Let the function $f(z)$ defined by (1) belong to the class $VN(\beta)$. Then*

$$1-\frac{(\beta-1)}{(2-\beta)}|z| \leq \left|f^{'}(z)\right| \leq 1+\frac{(\beta-1)}{(2-\beta)}|z|.$$

The result is sharp for the function $f(z)$ given by (19) at $z=\pm|z|e^{-i\theta_2}$.

Corollary 3.8. *Let the function $f(z)$ defined by (1) be in the class $VN(\beta)$. Then $f^{'}(z)$ is included in a disc with center at the origin and radius r_4 given by*

$$r_4 = 1+\frac{(\beta-1)}{(2-\beta)}.$$

4. Extreme points

Theorem 4.1. *Let the function $f(z)$ defined by (1) be in the class $VM(\beta)$, with $\arg(a_n)=\theta_n$ where $[\theta_n+(n-1)\alpha]\equiv\pi \mod 2\pi$. Define $f_1(z)=z$ and*

$$f_n(z)=z+\frac{(\beta-1)}{(n-\beta)}e^{i\theta_n}z^n. \tag{20}$$

Then $f(z)$ is in the class $VM(\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z), \tag{21}$$

where $\mu_n \geq 0$ $(n\geq 1)$ and $\sum\limits_{n=1}^{\infty}\mu_n=1$.

Proof. If $f(z)=\sum\limits_{n=1}^{\infty}\mu_n f_n(z)$ with $\sum\limits_{n=1}^{\infty}\mu_n=1$ and $\mu_n\geq 0$, then

$$\begin{aligned}&\sum_{n=2}^{\infty}(n-\beta)\frac{(\beta-1)}{(n-\beta)}\mu_n\\ &=\sum_{n=2}^{\infty}(\beta-1)\mu_n=(\beta-1)(1-\mu_1)\leq(\beta-1).\end{aligned}$$

Hence $f(z)\in VM(\beta)$.
Conversely, let the function $f(z)$ defined by (1) belongs to the class $VM(\beta)$, define

$$\mu_n=\frac{(n-\beta)}{(\beta-1)}|a_n|, \tag{22}$$

and

$$\mu_1=1-\sum_{n=2}^{\infty}\mu_n.$$

From Theorem 2.3, $\sum_{n=1}^{\infty} \mu_n \leq 1$ and so $\mu_n \geq 0$. Since $\mu_n f_n(z) = \mu_n z + a_n z^n$, then

$$\sum_{n=1}^{\infty} \mu_n f_n(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z).$$

This completes the proof of Theorem 4.1. □

Similarly, we can prove the following theorem for the class $VN(\beta)$.

Theorem 4.2. *Let the function $f(z)$ defined by (1) be in the class $VN(\beta)$, with* $\arg(a_n) = \theta_n$ *where* $[\theta_n + (n-1)\alpha] \equiv \pi \mod 2\pi$. *Define* $f_1(z) = z$ *and*

$$f_n(z) = z + \frac{(\beta - 1)}{n(n-\beta)} e^{i\theta_n} z^n. \tag{23}$$

Then $f(z)$ is in the class $VN(\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{24}$$

where $\mu_n \geq 0 \ (n \geq 1)$ *and* $\sum_{n=1}^{\infty} \mu_n = 1$.

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