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NEW CLASSES OF INTEGRAL INEQUALITIES OF FRACTIONAL ORDER

ZOUBIR DAHMANI

In this paper, the Riemann-Liouville fractional operator is used to generate new classes of integral inequalities using a family of n positive functions, $(n \in \mathbb{N}^*)$. For our results, some interesting classical inequalities can be deduced as some special cases.

1. Introduction

It is a well known truth that the integral inequalities play an important role in the theory of differential and integral equations. Indeed this importance seems to have increased during the last two decades. For details, we refer to [7, 9, 10, 12–14] and the references therein. Moreover, the study of fractional type inequalities is also of great importance in the existence and uniqueness theory for fractional differential equations. We refer the reader to [1–4, 11] for further information and applications.

The aim of this paper is to generalize some classical inequalities. By using the Riemann-Liouville fractional operator, we generate new classes of integral inequalities using a family of n positive functions defined on $[a,b]$. Some interesting classical inequalities of [8] can be deduced as some special cases. Furthermore, our results can be considered as generalizations of order n for some results in [4, 5].

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2. Preliminaries

In this section, we introduce some definitions and properties related to the fractional integral operator of Riemann-Liouville [6, 11].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function $f : [a,b] \to \mathbb{R}$ is defined as

$$\begin{gathered} J^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, a < t \leq b, \\ J^0[f(t)] = f(t), \end{gathered} \tag{1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du.$

For the convenience of establishing the results, we give the following properties:

$$J^{\alpha} J^{\beta}[f(t)] = J^{\alpha+\beta}[f(t)], \alpha \geq 0, \beta \geq 0, \tag{2}$$

and

$$J^{\alpha} J^{\beta}[f(t)] = J^{\beta} J^{\alpha}[f(t)]. \tag{3}$$

3. Main Results

In this section, we prove three classes of fractional integral inequalities. These results allow us in particular to generalize some classical inequalities. The first class is given by the following two theorems:

Theorem 3.1. *Suppose that $(f_i)_{i=1,\dots,n}$ are n positive continuous and decreasing functions on $[a,b]$. Then, the following inequality*

$$\frac{J^{\alpha}\left[\prod_{i \neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right]}{J^{\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \geq \frac{J^{\alpha}\left[(t-a)^{\delta} \prod_{i \neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\right]}{J^{\alpha}\left[(t-a)^{\delta} \prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \tag{4}$$

is valid for any $a < t \leq b, \alpha > 0, \delta > 0, \beta \geq \gamma_p > 0$, where p is a fixed integer in $\{1, 2, \dots, n\}$.

Proof. Since $(f_i)_{i=1,\dots,n}$ are positive, continuous and decreasing functions on $[a,b]$, then we have

$$\left((\rho-a)^{\delta} - (\tau-a)^{\delta}\right)\left(f_p^{\beta-\gamma_p}(\tau) - f_p^{\beta-\gamma_p}(\rho)\right) \geq 0, \tag{5}$$

for any fixed $p \in \{1, \dots, n\}$ and for any $\beta \geq \gamma_p > 0, \delta > 0, \tau, \rho \in [a,t]; a < t \leq b$. Denote

$$K_p(\tau,\rho) := \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \prod_{i=1}^{n} f_i^{\gamma_i}(\tau)\left((\rho-a)^{\delta} - (\tau-a)^{\delta}\right)\left(f_p^{\beta-\gamma_p}(\tau) - f_p^{\beta-\gamma_p}(\rho)\right). \tag{6}$$

We have

$$K_p(\tau,\rho) \geq 0. \tag{7}$$

The inequality (7) implies that

$$\begin{aligned} 0 \leq \int_a^t K_p(\tau,\rho)d\tau = (\rho-a)^{\delta} J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] \\ + f_p^{\beta-\gamma_p}(\rho) J^{\alpha}\Big[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] - J^{\alpha}\Big[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] \\ -(\rho-a)^{\delta} f^{\beta-\gamma_p}(\rho) J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]. \end{aligned} \tag{8}$$

And consequently,

$$\begin{aligned} J^{\alpha}\Big[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] \\ \geq J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] J^{\alpha}\Big[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big]. \end{aligned} \tag{9}$$

Theorem 3.1 is thus proved. □

Remark 3.2. The inequality (4) is reversed if the functions $(f_i)_{i=1,...,n}$ are increasing on $[a,b]$.

Remark 3.3. Applying Theorem 3.1 for $\alpha = 1, t = b, n = 1$, we obtain Theorem 3 in [8].

The second result is the following theorem:

Theorem 3.4. *Suppose that $(f_i)_{i=1,...,n}$ are positive, continuous and decreasing functions on $[a,b]$. Then for any fixed p in $\{1,2,\ldots,n\}$ and for any $a < t \leq b, \alpha > 0, \omega > 0, \delta > 0, \beta \geq \gamma_p > 0$, we have*

$$\begin{aligned} &\Big\{ J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\omega}\Big[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] \\ &\quad + J^{\omega}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\alpha}\Big[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]\Big\} \Big/ \\ &\Big\{ J^{\alpha}\Big[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\omega}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] \\ &\quad + J^{\omega}\Big[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]\Big\} \geq 1. \end{aligned} \tag{10}$$

Proof. Multiplying both sides of (8) by $\frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)}\prod_{i=1}^{n} f_i^{\gamma_i}(\rho), \omega > 0$, then integrating the resulting inequality with respect to ρ over $(a,t), a < t \leq b$ and using Fubini's theorem, we obtain

$$
\begin{aligned}
0 \leq \int_a^t \int_a^t \frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)} \prod_{i=1}^{n} f_i^{\gamma_i}(\rho) K_p(\tau,\rho)\, d\tau d\rho
&= J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\omega}\Big[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] \\
&+ J^{\omega}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\alpha}\Big[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] \\
&- J^{\alpha}\Big[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\omega}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] \\
&- J^{\omega}\Big[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]. \qquad (11)
\end{aligned}
$$

This completes the proof of Theorem 3.4.

□

Remark 3.5. (i) Applying Theorem 3.4 for $\alpha = \omega$, we obtain Theorem 3.1.
(ii) Applying Theorem 3.4 for $\alpha = \omega = 1, t = b, n = 1$, we obtain Theorem 3 of [8].

Another class of fractional integral inequalities which generalizes the above theorems is described in the following theorems. We have

Theorem 3.6. *Let $(f_i)_{i=1,\ldots,n}$ and g be positive continuous functions on $[a,b]$, such that g is increasing and $(f_i)_{i=1,\ldots,n}$ are decreasing on $[a,b]$. Then, the following inequality*

$$
\frac{J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\alpha}\Big[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]}{J^{\alpha}\Big[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]} \geq 1 \qquad (12)
$$

holds for any $a < t \leq b, \alpha > 0, \delta > 0, \beta \geq \gamma_p > 0$, where p is a fixed integer in $\{1,2,\ldots,n\}$.

Proof. Using the conditions of Theorem 3.6, we can write

$$
\Big(g^{\delta}(\rho) - g^{\delta}(\tau)\Big)\Big(f_p^{\beta-\gamma_p}(\tau) - f_p^{\beta-\gamma_p}(\rho)\Big) \geq 0, \qquad (13)
$$

for all $p = 1,\ldots,n, \delta > 0, \beta \geq \gamma_p > 0, \tau,\rho \in [a,t]; a < t \leq b.$

Now, let us consider the quantity

$$L_p(\tau,\rho) := \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\prod_{i=1}^{n} f_i^{\gamma_i}(\tau)\Big(g^{\delta}(\rho)-g^{\delta}(\tau)\Big)\Big(f_p^{\beta-\gamma_p}(\tau)-f_p^{\beta-\gamma_p}(\rho)\Big). \quad (14)$$

It is clear that

$$L_p(\tau,\rho)\geq 0. \quad (15)$$

Therefore,

$$0 \leq \int_a^t L_p(\tau,\rho)d\tau = g^{\delta}(\rho)J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] + f_p^{\beta-\gamma_p}(\rho)J^{\alpha}\Big[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]$$
$$-J^{\alpha}\Big[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] - (g^{\delta}(\rho)f^{\beta-\gamma_p}(\rho)J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]. \quad (16)$$

Consequently,

$$J^{\alpha}\Big[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big] \geq J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]J^{\alpha}\Big[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big]. \quad (17)$$

Theorem 3.6 is thus proved. □

Remark 3.7. Applying Theorem 3.6 for $\alpha = 1, t = b, n = 1$, we obtain Theorem 4 of [8].

We give also the following result:

Theorem 3.8. *Suppose that $(f_i)_{i=1,\dots,n}$ and g are positive and continuous functions on $[a,b]$, such that g is increasing and $(f_i)_{i=1,\dots,n}$ are decreasing on $[a,b]$. Then for any fixed $p \in \{1,2,\dots,n\}$ and for all $\alpha > 0, \omega > 0, \delta > 0, \beta \geq \gamma_p > 0$, we have*

$$\left\{J^{\alpha}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big]J^{\omega}\Big[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] + J^{\omega}\Big[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big]J^{\alpha}\Big[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]\right\} \Bigg/$$
$$\left\{J^{\alpha}\Big[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big]J^{\omega}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big] + J^{\omega}\Big[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\beta}(t)\Big]J^{\alpha}\Big[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\Big]\right\} \geq 1. \quad (18)$$

Proof. Using (16), we can write

$$0 \leq \int_a^t \int_a^t \prod_{i=1}^n f_i^{\gamma_i}(\rho) \frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)} L_p(\tau,\rho) d\tau d\rho =$$
$$J^\alpha\Big[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^\beta(t)\Big] J^\omega\Big[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\Big] + J^\omega\Big[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^\beta(t)\Big] J^\alpha\Big[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\Big]$$
$$-J^\alpha\Big[g(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^\beta(t)\Big] J^\omega\Big[\prod_{i=1}^n f_i^{\gamma_i}(t)\Big] - J^\omega\Big[g(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^\beta(t)\Big] J^\alpha\Big[\prod_{i=1}^n f_i^{\gamma_i}(t)\Big]. \quad (19)$$

This ends the proof of Theorem 3.8. □

Remark 3.9. (i) Applying Theorem 3.8 for $\alpha = \omega$, we obtain Theorem 3.6.
(ii) Applying Theorem 3.8 for $\alpha = \omega = 1, t = b, n = 1$, we obtain Theorem 4 of [8].

We further have:

Theorem 3.10. *Let* $(f_i)_{i=1,\dots,n}$ *and* g *be positive continuous functions on* $[a,b]$*. Suppose that for any fixed integer* $p \in \{1,2,\dots,n\}$*,*

$$\Big(f_p^\delta(\tau)g^\delta(\rho) - f_p^\delta(\rho)g^\delta(\tau)\Big)\Big(f_p^{\beta-\gamma_p}(\tau) - f_p^{\beta-\gamma_p}(\rho)\Big) \geq 0;$$
$$\tau,\rho \in [a,t], t \in]a,b], \delta > 0, \beta \geq \gamma_p > 0. \quad (20)$$

Then for any $\alpha > 0$*, we have*

$$\frac{J^\alpha\Big[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\beta+\delta}(t)\Big] J^\alpha\Big[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\Big]}{J^\alpha\Big[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^\beta(t)\Big] J^\alpha\Big[f_p^\delta \prod_{i=1}^n f_i^{\gamma_i}(t)\Big]} \geq 1. \quad (21)$$

Proof. The proof is quite similar to that for Theorem 3.4, provided we replace the quantity $\Big(g^\delta(\rho) - g^\delta(\tau)\Big)$ (in $L_p(\tau,\rho)$) by: $\Big(f_p^\delta(\tau)g^\delta(\rho) - f_p^\delta(\rho)g^\delta(\tau)\Big)$. □

Remark 3.11. It is clear that on $[a,b]$, Theorem 6 of [8] would follow as a special case of Theorem 3.10 when $\alpha = 1, t = b, n = 1$.

Also, with the same assumptions as before, we get the following generalization of Theorem 3.10:

Theorem 3.12. *Let $(f_i)_{i=1,\dots,n}$ and g be positive continuous functions on $[a,b]$. Suppose that for any fixed integer $p\in\{1,2,\dots,n\}$,*

$$\Big(f^{\delta}(\tau)g^{\delta}(\rho)-f^{\delta}(\rho)g^{\delta}(\tau)\Big)\Big(f^{\beta-\gamma}(\tau)-f^{\beta-\gamma}(\rho)\Big)\geq 0;$$
$$\tau,\rho\in[a,t],t\in]a,b],\delta>0,\beta\geq\gamma_p>0. \quad (22)$$

Then for all $\alpha>0,\omega>0$, the inequality

$$\Bigg\{J^{\alpha}\Big[\prod_{i\neq p}^{n}f_i^{\gamma_i}f_p^{\beta+\delta}(t)\Big]J^{\omega}\Big[g^{\delta}(t)\prod_{i=1}^{n}f_i^{\gamma_i}(t)\Big] +J^{\omega}\Big[\prod_{i\neq p}^{n}f_i^{\gamma_i}f_p^{\beta+\delta}(t)\Big]J^{\alpha}\Big[g^{\delta}(t)\prod_{i=1}^{n}f_i^{\gamma_i}(t)\Big]\Bigg\}\Bigg/$$
$$\Bigg\{J^{\alpha}\Big[g^{\delta}(t)\prod_{i\neq p}^{n}f_i^{\gamma_i}f_p^{\beta}(t)\Big]J^{\omega}\Big[f_p^{\delta}\prod_{i=1}^{n}f_i^{\gamma_i}(t)\Big] +J^{\omega}\Big[g^{\delta}(t)\prod_{i\neq p}^{n}f_i^{\gamma_i}f_p^{\beta}(t)\Big]J^{\alpha}\Big[f_p^{\delta}\prod_{i=1}^{n}f_i^{\gamma_i}(t)\Big]\Bigg\}\geq 1 \quad (23)$$

is valid.

Remark 3.13. (i) Applying Theorem 3.12 for $\alpha=\omega$, we obtain Theorem 3.10.
(ii) Applying Theorem 3.12, for $\alpha=\omega=1,t=b$, we obtain Theorem 5 of [8].

The third class of fractional integral inequalities is given by the following theorems:

Theorem 3.14. *Let f,g and $(h_i)_{i=1,\dots,n}$ be positive and continuous functions on $[a,b]$, such that*

$$\Big(g(\tau)-g(\rho)\Big)\Big(\frac{f(\rho)}{h_p(\rho)}-\frac{f(\tau)}{h_p(\tau)}\Big)\geq 0; p\in\{1,2,\dots,n\},\tau,\rho\in[a,t],a<t\leq b. \quad (24)$$

Then we have

$$\frac{J^{\alpha}\Big[f(t)\prod_{i\neq p}^{n}h_i(t)\Big]}{J^{\alpha}\Big[\prod_{i=1}^{n}h_i(t)\Big]}\geq\frac{J^{\alpha}\Big[gf\prod_{i\neq p}^{n}h_i(t)\Big]}{J^{\alpha}\Big[g\prod_{i=1}^{n}h_i(t)\Big]}, \quad (25)$$

for any $\alpha>0,a<t\leq b$.

Proof. Suppose that f,g and $(h_i)_{i=1,\dots,n}$ are positive and continuous functions on $[a,b]$. Using (24), we can write

$$g(\tau)\frac{f(\rho)}{h_p(\rho)}+g(\rho)\frac{f(\tau)}{h_p(\tau)}-g(\rho)\frac{f(\rho)}{h_p(\rho)}-g(\tau)\frac{f(\tau)}{h_p(\tau)}\geq 0, \quad (26)$$

for all $\tau,\rho\in[a,t], a<t\le b$ and for any fixed integer $p\in\{1,2,\dots,n\}$.
Therefore,

$$g(\tau)f(\rho)\prod_{i\neq p}^{n}h_i(\rho)\prod_{i=1}^{n}h_i(\tau)+g(\rho)f(\tau)\prod_{i\neq p}^{n}h_i(\tau)\prod_{i=1}^{n}h_i(\rho)$$
$$\ge g(\rho)f(\rho)\prod_{i\neq p}^{n}h_i(\rho)\prod_{i=1}^{n}h_i(\tau)+g(\tau)f(\tau)\prod_{i\neq p}^{n}h_i(\tau)\prod_{i=1}^{n}h_i(\rho), \quad (27)$$

for all $\tau,\rho\in[a,t], 0<t\le b$ and for any fixed integer $p\in\{1,2,\dots,n\}$.
This implies that

$$f(\rho)\prod_{i\neq p}^{n}h_i(\rho)J^{\alpha}\Big[g\prod_{i=1}^{n}h_i(t)\Big]+g(\rho)\prod_{i=1}^{n}h_i(\rho)J^{\alpha}\Big[f(t)\prod_{i\neq p}^{n}h_i(t)\Big]$$
$$\ge g(\rho)f(\rho)\prod_{i\neq p}^{n}h_i(\rho)J^{\alpha}\Big[\prod_{i=1}^{n}h_i(t)\Big]+\prod_{i=1}^{n}h_i(\rho)J^{\alpha}\Big[gf(t)\prod_{i\neq p}^{n}h_i(t)\Big]. \quad (28)$$

Multiplying both sides of (28) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$, then integrating with respect to ρ over $(a,t), 0<t\le b$, we obtain (25). □

Remark 3.15. It is clear that Theorem 7 of [8] would follow as a special case of Theorem 3.14 when $\alpha=1, t=b, n=1$.

Using two fractional parameters, we give the following generalization of Theorem 3.14.

Theorem 3.16. *Let f, g and $(h_i)_{i=1,\dots,n}$ be positive and continuous functions on $[a,b]$, such that*

$$\Big(g(\tau)-g(\rho)\Big)\Big(\frac{f(\rho)}{h_p(\rho)}-\frac{f(\tau)}{h_p(\tau)}\Big)\ge 0; p\in\{1,2,\dots,n\}, \tau,\rho\in[a,t], a<t\le b. \quad (29)$$

Then the inequality

$$\frac{J^{\alpha}\Big[f(t)\prod_{i\neq p}^{n}h_i(t)\Big]J^{\omega}\Big[g\prod_{i=1}^{n}h_i(t)\Big]+J^{\omega}\Big[f(t)\prod_{i\neq p}^{n}h_i(t)\Big]J^{\alpha}\Big[g\prod_{i=1}^{n}h_i(t)\Big]}{J^{\alpha}\Big[\prod_{i=1}^{n}h_i(t)\Big]J^{\omega}\Big[gf\prod_{i\neq p}^{n}h_i(t)\Big]+J^{\omega}\Big[\prod_{i=1}^{n}h_i(t)\Big]J^{\alpha}\Big[gf\prod_{i\neq p}^{n}h_i(t)\Big]}\ge 1 \quad (30)$$

holds for any $\alpha>0, \omega>0, a<t\le b$.

Proof. We use the same arguments as in the proofs of Theorem 3.8 and Theorem 3.14. □

At the end, we give the following corollaries:

Corollary 3.17. *Let f, g and $(h_i)_{i=1,\dots,n}$ be positive functions on $[a,b]$.*
$(1*)$ *Suppose that g is continuous and increasing, f and $(h_i)_{i=1,\dots,n}$ are differentiable and there exist $(M_i)_{i=1,\dots,n}$, such that $M_i := sup_{x\in[a,b]}\left(\frac{f}{h_i}\right)'(x)$. Then the inequality*

$$\frac{J^\alpha\left[f(t)\prod_{i\neq p}^n h_i(t)\right]J^\alpha\left[g\prod_{i=1}^n h_i(t)\right]+M_pJ^\alpha\left[\prod_{i=1}^n h_i(t)\right]J^\alpha\left[tg(t)\prod_{i=1}^n h_i(t)\right]}{J^\alpha\left[\prod_{i=1}^n h_i(t)\right]J^\alpha\left[gf\prod_{i\neq p}^n h_i(t)\right]+M_pJ^\alpha\left[g(t)\prod_{i=1}^n h_i(t)\right]J^\alpha\left[t\prod_{i=1}^n h_i(t)\right]}\geq 1 \tag{31}$$

holds for any $\alpha>0, a<t\leq b$ and for any fixed $p\in\{1,2,\dots,n\}$.
$(2*)$ *Suppose that g is continuous and decreasing, f and $(h_i)_{i=1,\dots,n}$ are differentiable and there exist $(m_i)_{i=1,\dots,n}$, such that $m_i := inf_{x\in[a,b]}\left(\frac{f}{h_i}\right)'(x), i=1,\dots,n$. Then the inequality*

$$\frac{J^\alpha\left[f(t)\prod_{i\neq p}^n h_i(t)\right]J^\alpha\left[g\prod_{i=1}^n h_i(t)\right]+m_pJ^\alpha\left[\prod_{i=1}^n h_i(t)\right]J^\alpha\left[tg(t)\prod_{i=1}^n h_i(t)\right]}{J^\alpha\left[\prod_{i=1}^n h_i(t)\right]J^\alpha\left[gf\prod_{i\neq p}^n h_i(t)\right]+J^\alpha m_p\left[g(t)\prod_{i=1}^n h_i(t)\right]J^\alpha\left[t\prod_{i=1}^n h_i(t)\right]}\geq 1 \tag{32}$$

holds for any $\alpha>0, a<t\leq b$ and for any fixed $p\in\{1,2,\dots,n\}$.

Proof. $(1*)$ In Theorem 3.14, we replace $\frac{f}{h_p}(x)$ by $V_p(x) := \frac{f}{h_p}(x) - M_px, x\in[a,b]$. It is clear that V_p is decreasing, and since g is increasing, hence V_p and g are monotonic in opposite sense; the condition (24) is satisfied. Then, by a simple calculation we obtain (31).

To prove $(2*)$, in Theorem 3.14 we replace $\frac{f}{h_p}(x)$ by $W_p(x) := \frac{f}{h_p}(x) - m_px, x\in[a,b]$. □

Corollary 3.18. *Let f, g and $(h_i)_{i=1,\dots,n}$ be positive functions on $[a,b]$.*
$(1*)$ *Suppose that g is continuous and increasing, f and $(h_i)_{i=1,\dots,n}$ are differentiable and there exist $(M_i)_{i=1,\dots,n}$, such that $M_i := sup_{x\in[a,b]}\left(\frac{f}{h_i}\right)'(x)$. Then for any $p\in\{1,2,\dots,n\}, \alpha>0, \omega>0, 0<t\leq b$, we have*

$$\frac{R_{\alpha,\omega,M_p}(t)}{S_{\alpha,\omega,M_p}(t)}\geq 1, \tag{33}$$

where

$$R_{\alpha,\omega,M_p}(t) := J^\alpha\left[f(t)\prod_{i\neq p}^n h_i(t)\right]J^\omega\left[g\prod_{i=1}^n h_i(t)\right] + J^\omega\left[f(t)\prod_{i\neq p}^n h_i(t)\right]J^\alpha\left[g\prod_{i=1}^n h_i(t)\right] + M_pJ^\alpha\left[\prod_{i=1}^n h_i(t)\right]J^\omega\left[tg(t)\prod_{i=1}^n h_i(t)\right] + M_pJ^\omega\left[\prod_{i=1}^n h_i(t)\right]J^\alpha\left[tg(t)\prod_{i=1}^n h_i(t)\right] \tag{34}$$

and

$$S_{\alpha,\omega,M_p}(t) := J^{\alpha}\Big[\prod_{i=1}^{n} h_i(t)\Big] J^{\omega}\Big[gf\prod_{i\neq p}^{n} h_i(t)\Big] + J^{\omega}\Big[\prod_{i=1}^{n} h_i(t)\Big] J^{\alpha}\Big[gf\prod_{i\neq p}^{n} h_i(t)\Big] + M_p J^{\alpha}\Big[g(t)\prod_{i=1}^{n} h_i(t)\Big] J^{\omega}\Big[t\prod_{i=1}^{n} h_i(t)\Big] + M_p J^{\omega}\Big[g(t)\prod_{i=1}^{n} h_i(t)\Big] J^{\alpha}\Big[t\prod_{i=1}^{n} h_i(t)\Big]. \quad (35)$$

$(2*)$ *Suppose that* g *is continuous and decreasing,* f *and* $(h_i)_{i=1,\dots,n}$ *are differentiable and there exist* $(m_i)_{i=1,\dots,n}$, *such that* $m_i := inf_{x\in[a,b]}\left(\frac{f}{h_i}\right)'(x), i = 1,..n.$ *Then the inequality*

$$\frac{R_{\alpha,\omega,m_p}(t)}{R_{\alpha,\omega,m_p}(t)} \geq 1 \quad (36)$$

holds for any $\alpha > 0, \omega > 0, a < t \leq b$ *and for any fixed* $p \in \{1,2,\dots,n\}$.

Proof. We apply Theorem 3.16. □

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ZOUBIR DAHMANI
Laboratory LPAM, Faculty SEI
UMAB University
e-mail: zzdahmani@yahoo.fr