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DIFFERENTIAL SANDWICH THEOREMS FOR HIGHER-ORDER DERIVATIVES OF p -VALENT FUNCTIONS INVOLVING A GENERALIZED DIFFERENTIAL OPERATOR

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In the present article, we obtain some applications of first order differential subordination, superordination and sandwich results for higher-order derivatives of p -valent functions involving a generalized differential operator. Some of our results improve and generalize previously known results.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} \dots \quad (a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, \dots\}).$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \quad (1)$$

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which are p -valent in U . We write $\mathcal{A}(1) = \mathcal{A}$.

If $f, g \in H(U)$, we say that f is subordinate to g or g is superordinate to f , written $f(z) \prec g(z)$ if there exists a Schwarz function w , which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U , then we have the following equivalence, (cf., e.g., [12], [21] and [22]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi: \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ and h be univalent function in U . If β is analytic function in U and satisfies the first order differential subordination:

$$\phi(\beta(z), z\beta'(z); z) \prec h(z), \quad (2)$$

then β is a solution of the differential subordination (2). The univalent function q is called a dominant of the solutions of the differential subordination (2) if $\beta(z) \prec q(z)$ for all β satisfying (2). A univalent dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If β and ϕ are univalent functions in U and if satisfies first order differential superordination:

$$h(z) \prec \phi(\beta(z), z\beta'(z); z), \quad (3)$$

then β is a solution of the differential superordination (3). An analytic function q is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec \beta(z)$ for all β satisfying (3). A univalent subordinant \tilde{q} that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants of (3) is called the best subordinant.

Using the results of Miller and Mocanu [22], Bulboaca [11] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [12]. Ali et al. [1], have used the results of Bulboaca [11] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [30] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [28] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions $f \in \mathcal{A}(p)$ given by (1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{4}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{5}$$

Upon differentiating both sides of (5) j -times with respect to z , we have

$$(f * g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k; j) a_k b_k z^{k-j}, \tag{6}$$

where

$$\delta(p; j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{7}$$

For functions $f, g \in \mathcal{A}(p)$, Aouf et al. [6] (see also [7]) define the linear operator $D_{\lambda,p}^n (f * g)^{(j)} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ by

$$\begin{aligned} D_{\lambda,p}^n (f * g)^{(j)}(z) &= \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left(\frac{p-j+\lambda(k-p)}{p-j} \right)^n \delta(k; j) a_k b_k z^{k-j} \\ &\quad (\lambda \geq 0; p > j; p \in \mathbb{N}; j, n \in \mathbb{N}_0; z \in U). \end{aligned} \tag{8}$$

From (8), we can easily deduce that

$$\begin{aligned} \frac{\lambda z}{p-j} \left(D_{\lambda,p}^n (f * g)^{(j)}(z) \right)' &= D_{\lambda,p}^{n+1} (f * g)^{(j)}(z) - (1-\lambda) D_{\lambda,p}^n (f * g)^{(j)}(z) \\ &\quad (\lambda > 0; p > j; p \in \mathbb{N}; n, j \in \mathbb{N}_0; z \in U). \end{aligned} \tag{9}$$

We observe that the linear operator $D_{\lambda,p}^n (f * g)^{(j)}(z)$ reduces to several interesting many other linear operators considered earlier for different choices of j, n, λ and the function g :

(i) For $j = 0$, $D_{\lambda,p}^n (f * g)^{(j)} = D_{\lambda,p}^n (f * g)$, where the operator $D_{\lambda,p}^n (f * g)$ ($\lambda \geq 0, p \in \mathbb{N}, n \in \mathbb{N}_0$) was introduced and studied by Selvaraj et al. [26] (see also [10]) and $D_{\lambda,1}^n (f * g)(z) = D_{\lambda}^n (f * g)(z)$, where the operator $D_{\lambda}^n (f * g)$ was introduced by Aouf and Mostafa [9];

(ii) For

$$g(z) = \frac{z^p}{1-z} \quad (p \in \mathbb{N}; z \in U) \tag{10}$$

we have $D_{\lambda,p}^n (f * g)^{(j)}(z) = D_{\lambda,p}^n f^{(j)}(z)$, $D_{\lambda,p}^n f^{(0)}(z) = D_{\lambda,p}^n f(z)$, where the operator $D_{\lambda,p}^n$ is the p -valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [17], $D_{1,p}^n f^{(j)}(z) = D_p^n f^{(j)}(z)$, where the operator $D_p^n f^{(j)}$ ($p > j, p \in \mathbb{N}, n, j \in \mathbb{N}_0$) was introduced and studied by Aouf [3,4] and $D_{1,p}^n f^{(0)} = D_p^n f$, where the operator D_p^n is the p -valent Sălăgean operator which was introduced and studied by Kamali and Orhan [18] (see also [8]);

(iii) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}} \frac{z^k}{(1)_{k-p}} \quad (z \in U), \tag{11}$$

(for complex parameters $\alpha_1, \dots, \alpha_q$ and β_1, \dots, β_s ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$, $j = 1, \dots, s$); $q \leq s + 1; p \in \mathbb{N}; q, s \in \mathbb{N}_0$) where $(v)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ , by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & (k = 0), \\ v(v+1)(v+2) \cdots (v+k-1) & (k \in \mathbb{N}). \end{cases}$$

we have $D_{\lambda,p}^n (f * g)^{(j)}(z) = D_{\lambda,p}^n (H_{p,q,s}(\alpha_1)f)^{(j)}(z)$ and $D_{\lambda,p}^0 (f * g)^{(0)}(z) = H_{p,q,s}(\alpha_1)f(z)$, where the operator $H_{p,q,s}(\alpha_1)$ is the Dziok-Srivastava operator which was introduced and studied by Dziok and Srivastava [15,16] and which contains in turn many interesting operators such as, $H_{1,2,1}(a, 1; c) = L(a, c)$, where the operator $L(a, c)$ was introduced by Carlson and Shaffer [13];

(iv) For

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+l+\alpha(k-p)}{p+l} \right)^m z^k \tag{12}$$

$$(\alpha \geq 0; l \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0; z \in U),$$

we have $D_{\lambda,p}^n (f * g)^{(j)}(z) = D_{\lambda,p}^n (I_p(m, \alpha, l)f)^{(j)}(z)$ and $D_{\lambda,p}^0 (f * g)^{(0)}(z) = I_p(m, \alpha, l)f(z)$, where the operator $I_p(m, \alpha, l)$ was introduced and studied by Cătas [14] which contains in turn many interesting operators such as, $I_p(m, 1, l) = I_p(m, l)$, where the operator $I_p(m, l)$ was investigated by Kumar et al. [19];

(v) For

$$g(z) = z^p + \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \beta)}{\Gamma(k + \alpha + \beta)} z^k \tag{13}$$

$$(\alpha \geq 0; p \in \mathbb{N}; \beta > -1; z \in U)$$

we have $D_{\lambda,p}^n (f * g)^{(j)} = D_{\lambda,p}^n \left(Q_{\beta,p}^\alpha f \right)^{(j)}$ and $D_{\lambda,p}^0 (f * g)^{(0)} = Q_{\beta,p}^\alpha f$, where the operator $Q_{\beta,p}^\alpha$ was introduced and studied by Liu and Owa [20];

(vi) For $j = 0$ and g of the form (11) with $p = 1$, we have $D_{\lambda,1}^n (f * g)(z) = D_{\lambda}^n (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)(z)$, where the operator $D_{\lambda}^n (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)$ was introduced and studied by Selvaraj and Karthikeyan [25];

(vii) For $j = 0, p = 1$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{\Gamma(k + 1)\Gamma(2 - m)}{\Gamma(k + 1 - m)} \right]^n z^k$$

$$(n \in \mathbb{N}_0; 0 \leq m < 1; z \in U)$$

we have $D_{\lambda,1}^n (f * g)(z) = D_{\lambda}^{n,m} f(z)$, where the operator $D_{\lambda}^{n,m}$ was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator $D_{\lambda,p}^n (f * g)^{(j)}$.

2. Definitions and preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 2.1 ([22]). Denote by Q , the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 2.2 ([22]). Let q be univalent in U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \tag{14}$$

Suppose that

(i) $\psi(z)$ is starlike univalent in U ,

$$(ii) \Re \left\{ \frac{zh'(z)}{\psi(z)} \right\} > 0 \text{ for } z \in U.$$

If β is analytic with $\beta(0) = q(0)$, $\beta(U) \subset D$ and

$$\theta(\beta(z)) + z\beta'(z)\phi(\beta(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{15}$$

then $\beta(z) \prec q(z)$ and q is the best dominant.

Lemma 2.3 ([11]). *Let q be convex univalent in U and θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that (i) $\Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0$ for $z \in U$, (ii) $\Psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U . If $\beta(z) \in H[q(0), 1] \cap Q$, with $\beta(U) \subseteq D$, and $\theta(\beta(z)) + z\beta'(z)\phi(\beta(z))$ is univalent in U and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\beta(z)) + z\beta'(z)\phi(\beta(z)), \tag{16}$$

then $q(z) \prec \beta(z)$ and q is the best subordinant.

Lemma 2.4 ([24]). *The function $q(z) = (1 - z)^{-2ab}$ ($a, b \in \mathbb{C}^* (\mathbb{C} \setminus \{0\})$) is univalent in U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.*

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that $\eta, \gamma_i \in \mathbb{C}$ ($i = 1, 2, 3$), $\gamma_4, \mu \in \mathbb{C}^*$, $\lambda > 0$, $\delta(p; j)$ is given by (7), $p > j$, $p \in \mathbb{N}$, $n, j \in \mathbb{N}_0$ and the powers are understood as the principle values.

Theorem 3.1. *Let $q(z)$ be univalent in U with $q(0) = 1$, $q(z) \neq 1$ and let $\frac{zq'(z)}{q(z)}$ be starlike in U . Let $f \in \mathcal{A}(p)$ and assume that f and q satisfy the following conditions:*

$$\left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \neq 0, \tag{17}$$

and

$$\Re \left\{ 1 + \frac{\gamma_2}{\gamma_4} q(z) + \frac{2\gamma_3}{\gamma_4} [q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right\} > 0 \quad (z \in U). \tag{18}$$

If

$$\begin{aligned}
 & \gamma_1 + \gamma_2 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \\
 & + \gamma_3 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^{2\mu} \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^{2\eta} \\
 & + \gamma_4 \mu \left(\frac{p-j}{\lambda} \right) \left[\frac{D_{\lambda,p}^{n+1} (f * g_1)^{(j)}(z)}{D_{\lambda,p}^n (f * g_1)^{(j)}(z)} - 1 \right] \\
 & + \gamma_4 \eta \left(\frac{p-j}{\lambda} \right) \left[1 - \frac{D_{\lambda,p}^{n+1} (f * g_2)^{(j)}(z)}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right] \\
 & \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{z q'(z)}{q(z)}, \tag{19}
 \end{aligned}$$

then

$$\left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \prec q(z) \tag{20}$$

and $q(z)$ is the best dominant.

Proof. Define a function ρ by

$$\rho(z) = \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \quad (z \in U). \tag{21}$$

Then the function ρ is analytic in U and $\rho(0) = 1$. Therefore, differentiating (21) logarithmically with respect to z and using the identity (9) in the resulting equation, we have

$$\begin{aligned}
 & \gamma_1 + \gamma_2 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \\
 & + \gamma_3 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^{2\mu} \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^{2\eta} +
 \end{aligned}$$

$$\begin{aligned}
 & + \gamma_4 \mu \left(\frac{p-j}{\lambda} \right) \left[\frac{D_{\lambda,p}^{n+1} (f * g_1)^{(j)}(z)}{D_{\lambda,p}^n (f * g_1)^{(j)}(z)} - 1 \right] \\
 & + \gamma_4 \eta \left(\frac{p-j}{\lambda} \right) \left[1 - \frac{D_{\lambda,p}^{n+1} (f * g_2)^{(j)}(z)}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right] \\
 & = \gamma_1 + \gamma_2 \rho(z) + \gamma_3 [\rho(z)]^2 + \gamma_4 \frac{z \rho'(z)}{\rho(z)},
 \end{aligned}$$

that is,

$$\gamma_1 + \gamma_2 \rho(z) + \gamma_3 [\rho(z)]^2 + \gamma_4 \frac{z \rho'(z)}{\rho(z)} < \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{z q'(z)}{q(z)}.$$

By setting

$$\theta(w) = \gamma_1 + \gamma_2 w + \gamma_3 w^2 \text{ and } \varphi(w) = \frac{\gamma_4}{w},$$

it can be easily observed that θ is analytic function in \mathbb{C} , φ is analytic function in \mathbb{C}^* and $\varphi(w) \neq 0$. Also we see that

$$\psi(z) = z q'(z) \varphi(q(z)) = \gamma_4 \frac{z q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + \psi(z) = \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{z q'(z)}{q(z)},$$

it is clear that $\psi(z)$ is starlike univalent in U and

$$\Re \left\{ \frac{z h'(z)}{\psi(z)} \right\} = \Re \left\{ 1 + \frac{\gamma_2}{\gamma_4} q(z) + \frac{2\gamma_3}{\gamma_4} [q(z)]^2 - \frac{z q'(z)}{q(z)} + \frac{z q''(z)}{q'(z)} \right\} > 0$$

Therefore, Theorem 3.1 now follows by applying Lemma 2.2. □

Taking $p = \lambda = 1, n = j = 0, g_1 = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k, g_2 = z + \sum_{k=2}^{\infty} \frac{(a+1)_{k-1}}{(c)_{k-1}} z^k,$ and $c \notin \mathbb{Z}_0^-$ in Theorem 3.1, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.1].

Corollary 3.2. . *Let $q(z)$ be univalent in U with $q(0) = 1, q(z) \neq 1$ and $\frac{z q'(z)}{q(z)}$ is starlike in U . Let $f \in \mathcal{A}(p)$ such that*

$$\left[\frac{L(a,c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c) f(z)} \right]^\eta \neq 0, \tag{22}$$

and suppose q satisfies (18). If

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a,c)f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1,c)f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] \\ & \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{zq'(z)}{q(z)}, \end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \prec q(z)$$

and $q(z)$ is the best dominant.

Putting $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.2].

Corollary 3.3. *Let $-1 \leq B < A \leq 1$ and assume that*

$$\Re \left\{ \frac{\gamma_2}{\gamma_4} \left[\frac{1+Az}{1+Bz} \right] + \frac{2\gamma_3}{\gamma_4} \left[\frac{1+Az}{1+Bz} \right]^2 + \frac{1-ABz^2}{(1+Az)(1+Bz)} \right\} > 0 \quad (z \in U),$$

holds. If $f \in \mathcal{A}$ such that (22) holds and satisfy the following subordination condition:

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a,c)f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1,c)f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] \\ & \prec \gamma_1 + \gamma_2 \left[\frac{1+Az}{1+Bz} \right] + \gamma_3 \left[\frac{1+Az}{1+Bz} \right]^2 + \frac{\gamma_4(A-B)z}{(1+Az)(1+Bz)}, \end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^\vartheta$ ($0 < \vartheta \leq 1$) in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.3].

Corollary 3.4. *Assume that*

$$\Re \left\{ \frac{\gamma_2}{\gamma_4} \left(\frac{1+z}{1-z}\right)^\vartheta + \frac{2\gamma_3}{\gamma_4} \left(\frac{1+z}{1-z}\right)^{2\vartheta} + \frac{1-3z^2}{1-z^2} \right\} > 0 \quad (z \in U),$$

holds. If $f \in \mathcal{A}$ such that (22) holds and satisfy the following subordination condition:

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a,c)f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1,c)f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] \\ & \prec \gamma_1 + \gamma_2 \left(\frac{1+z}{1-z}\right)^\vartheta + \gamma_3 \left(\frac{1+z}{1-z}\right)^{2\vartheta} + \frac{2\gamma_4 \vartheta z}{(1-z)^2}, \end{aligned}$$

then

$$\left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \prec \left(\frac{1+z}{1-z}\right)^\vartheta$$

and the function $\left(\frac{1+z}{1-z}\right)^\vartheta$ is the best dominant.

Putting $q(z) = e^{\mu Az}$ ($|\mu A| < \pi$) in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.4].

Corollary 3.5. *Assume that*

$$\Re \left\{ 1 + \frac{\gamma_2}{\gamma_4} e^{\mu Az} + \frac{2\gamma_3}{\gamma_4} e^{2\mu Az} \right\} > 0 \quad (z \in U),$$

holds. If $f \in \mathcal{A}$ such that (22) holds and satisfy the following subordination condition:

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a,c)f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1,c)f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \right] \\ & \prec \gamma_1 + \gamma_2 e^{\mu Az} + \gamma_3 e^{2\mu Az} + \gamma_4 \mu Az, \end{aligned}$$

then

$$\left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a + 1, c) f(z)} \right]^\eta \prec e^{\mu A z}$$

and the function $e^{\mu A z}$ is the best dominant.

Taking $\gamma_1 = p = \lambda = 1, \gamma_2 = \gamma_3 = n = j = \eta = 0, g_1 = z + \sum_{k=2}^\infty z^k, q(z) = \frac{1}{(1-z)^{2ab}}$ ($a, b \in \mathbb{C}^*$), $\mu = a$ and $\gamma_4 = \frac{1}{ab}$ in Theorem 3.1, then combining this to gather with Lemma 2.4 we obtain the following corollary obtained by Obradović et al. [23, Theorem 1].

Corollary 3.6. *Let $a, b \in \mathbb{C}^*$ such that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. Let $f \in \mathcal{A}$ and suppose that $\frac{f(z)}{z} \neq 0$ ($z \in U$). If*

$$1 + \frac{1}{b} \left(\frac{z f'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z} \right)^a \prec \frac{1}{(1-z)^{2ab}}$$

and the function $\frac{1}{(1-z)^{2ab}}$ is the best dominant.

Remark 3.7. For $a = 1$, Corollary 3.6 reduces to the recent result obtained by Srivastava and Lashin [29, Theorem 3].

Taking $\gamma_1 = p = \lambda = 1, \gamma_2 = \gamma_3 = n = j = \eta = 0, g_1 = z + \sum_{k=2}^\infty z^k, q(z) = (1-z)^{-2ab \cos \tau e^{-i\tau}}$ ($a, b \in \mathbb{C}^*, |\tau| < \frac{\pi}{2}$), $\mu = a$ and $\gamma_4 = \frac{e^{i\tau}}{ab \cos \tau}$ in Theorem 3.1, then combining this to gather with Lemma 2.4 we obtain the following corollary obtained by Aouf et al. [5, Theorem 1].

Corollary 3.8. *Let $a, b \in \mathbb{C}^*, |\tau| < \frac{\pi}{2}$ and suppose that $|2ab \cos \tau e^{-i\tau} - 1| \leq 1$ or $|2ab \cos \tau e^{-i\tau} + 1| \leq 1$. Let $f \in \mathcal{A}$ and suppose that $\frac{f(z)}{z} \neq 0$ ($z \in U$). If*

$$1 + \frac{e^{i\tau}}{b \cos \tau} \left(\frac{z f'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z} \right)^a \prec (1-z)^{-2ab \cos \tau e^{-i\tau}}$$

and the function $(1-z)^{-2ab \cos \tau e^{-i\tau}}$ is the best dominant.

Theorem 3.9. Let q be convex univalent in U with $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U . Further assume that

$$\Re \left(\left(\gamma_2 + 2\gamma_3 q(z) \right) \frac{q(z)q'(z)}{\gamma_4} \right) > 0. \tag{23}$$

Let $f \in \mathcal{A}(p)$ such that

$$0 \neq \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \in H[q(0), 1] \cap \mathcal{Q}. \tag{24}$$

If

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^{2\mu} \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^{2\eta} \\ & + \gamma_4 \mu \left(\frac{p-j}{\lambda} \right) \left[\frac{D_{\lambda,p}^{n+1} (f * g_1)^{(j)}(z)}{D_{\lambda,p}^n (f * g_1)^{(j)}(z)} - 1 \right] \\ & + \gamma_4 \eta \left(\frac{p-j}{\lambda} \right) \left[1 - \frac{D_{\lambda,p}^{n+1} (f * g_2)^{(j)}(z)}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right] \end{aligned} \tag{25}$$

is univalent in U and satisfies the following superordination condition

$$\begin{aligned} & \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{zq'(z)}{q(z)} \\ & \prec \gamma_1 + \gamma_2 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^{2\mu} \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^{2\eta} \\ & + \gamma_4 \mu \left(\frac{p-j}{\lambda} \right) \left[\frac{D_{\lambda,p}^{n+1} (f * g_1)^{(j)}(z)}{D_{\lambda,p}^n (f * g_1)^{(j)}(z)} - 1 \right] \end{aligned}$$

$$+\gamma_4\eta \left(\frac{p-j}{\lambda}\right) \left[1 - \frac{D_{\lambda,p}^{n+1}(f * g_2)^{(j)}(z)}{D_{\lambda,p}^n(f * g_2)^{(j)}(z)}\right], \tag{26}$$

holds, then

$$q(z) \prec \left[\frac{D_{\lambda,p}^n(f * g_1)^{(j)}(z)}{\delta(p; j)z^{p-j}}\right]^\mu \left[\frac{\delta(p; j)z^{p-j}}{D_{\lambda,p}^n(f * g_2)^{(j)}(z)}\right]^\eta \tag{27}$$

and q is the best subordinated.

Proof. By setting

$$\theta(w) = \gamma_1 + \gamma_2w + \gamma_3w^2 \text{ and } \varphi(w) = \frac{\gamma_4}{w},$$

it can be easily observed that θ is analytic function in \mathbb{C} , φ is analytic function in \mathbb{C}^* and $\varphi(w) \neq 0$. From the assumption of Theorem 3.9, we see that

$$\Re \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = \Re \left((\gamma_2 + 2\gamma_3q(z)) \frac{q(z)q'(z)}{\gamma_4} \right) > 0 \text{ for } z \in U,$$

Therefore, Theorem 3.9 now follows by applying Lemma 2.3. □

Taking $p = \lambda = 1, n = j = 0, g_1 = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}}z^k, g_2 = z + \sum_{k=2}^{\infty} \frac{(a+1)_{k-1}}{(c)_{k-1}}z^k,$ and $c \notin \mathbb{Z}_0^-$ in Theorem 3.9, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.11].

Corollary 3.10. *Let q be convex univalent in U with $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ is starlike in U . Further assume that (23) holds. Let $f \in \mathcal{A}(p)$ such that*

$$0 \neq \left[\frac{L(a,c)f(z)}{z}\right]^\mu \left[\frac{z}{L(a+1,c)f(z)}\right]^\eta \in H[q(0), 1] \cap \mathcal{Q}.$$

If

$$\begin{aligned} &\gamma_1 + \gamma_2 \left[\frac{L(a,c)f(z)}{z}\right]^\mu \left[\frac{z}{L(a+1,c)f(z)}\right]^\eta \\ &+ \gamma_3 \left[\frac{L(a,c)f(z)}{z}\right]^{2\mu} \left[\frac{z}{L(a+1,c)f(z)}\right]^{2\eta} \\ &+ \gamma_4\mu a \left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right] + \gamma_4\eta(a+1) \left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right] \end{aligned}$$

is univalent in U and satisfies the following superordination condition

$$\begin{aligned} & \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{zq'(z)}{q(z)} \\ & \prec \gamma_1 + \gamma_2 \left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a,c)f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1,c)f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \right], \end{aligned}$$

holds, then

$$q(z) \prec \left[\frac{L(a,c)f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1,c)f(z)} \right]^\eta$$

and q is the best subordinant.

Combining Theorems 3.1 and 3.9, we get the following sandwich theorem for the linear operator $D_{\lambda,p}^n (f * g)^{(j)}$.

Theorem 3.11. Let q_i be convex univalent in U with $q_i(0) = 1$ and let $\frac{zq_i'(z)}{q_i(z)}$ ($i = 1, 2$) be starlike in U . Suppose that q_1 satisfies (23) and q_2 satisfies (18). Let $f \in \mathcal{A}(p)$ such that

$$0 \neq \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \in H[q(0), 1] \cap \mathcal{Q}. \quad (28)$$

If

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^{2\mu} \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^{2\eta} \\ & + \gamma_4 \mu \left(\frac{p-j}{\lambda} \right) \left[\frac{D_{\lambda,p}^{n+1} (f * g_1)^{(j)}(z)}{D_{\lambda,p}^n (f * g_1)^{(j)}(z)} - 1 \right] \\ & + \gamma_4 \eta \left(\frac{p-j}{\lambda} \right) \left[1 - \frac{D_{\lambda,p}^{n+1} (f * g_2)^{(j)}(z)}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right] \end{aligned} \quad (29)$$

is univalent in U and

$$\begin{aligned}
 & \gamma_1 + \gamma_2 q_1(z) + \gamma_3 [q_1(z)]^2 + \gamma_4 \frac{z q_1'(z)}{q_1(z)} \\
 & \prec \gamma_1 + \gamma_2 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \\
 & + \gamma_3 \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^{2\mu} \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^{2\eta} \\
 & + \gamma_4 \mu \left(\frac{p-j}{\lambda} \right) \left[\frac{D_{\lambda,p}^{n+1} (f * g_1)^{(j)}(z)}{D_{\lambda,p}^n (f * g_1)^{(j)}(z)} - 1 \right] \\
 & + \gamma_4 \eta \left(\frac{p-j}{\lambda} \right) \left[1 - \frac{D_{\lambda,p}^{n+1} (f * g_2)^{(j)}(z)}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right] \\
 & \prec \gamma_1 + \gamma_2 q_2(z) + \gamma_3 [q_2(z)]^2 + \gamma_4 \frac{z q_2'(z)}{q_2(z)} \tag{30}
 \end{aligned}$$

holds, then

$$q_1(z) \prec \left[\frac{D_{\lambda,p}^n (f * g_1)^{(j)}(z)}{\delta(p; j) z^{p-j}} \right]^\mu \left[\frac{\delta(p; j) z^{p-j}}{D_{\lambda,p}^n (f * g_2)^{(j)}(z)} \right]^\eta \prec q_2(z) \tag{31}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Taking $p = \lambda = 1, n = j = 0, g_1 = z + \sum_{k=2}^\infty \frac{(a)_{k-1}}{(c)_{k-1}} z^k, g_2 = z + \sum_{k=2}^\infty \frac{(a+1)_{k-1}}{(c)_{k-1}} z^k,$ and $c \notin \mathbb{Z}_0^-$ in Theorem 3.11, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.12].

Corollary 3.12. *Let q_i be convex univalent in U with $q_i(0) = 1$ and $\frac{z q_i'(z)}{q_i(z)}$ be starlike in U for $i = 1, 2$. Suppose that q_1 satisfies (23) and q_2 satisfies (18). Let $f \in \mathcal{A}(p)$ such that*

$$0 \neq \left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \in H[q(0), 1] \cap \mathcal{Q}.$$

If

$$\begin{aligned} & \gamma_1 + \gamma_2 \left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a, c) f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1, c) f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1, c) f(z)}{L(a, c) f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2, c) f(z)}{L(a+1, c) f(z)} \right] \end{aligned}$$

is univalent in U and

$$\begin{aligned} & \gamma_1 + \gamma_2 q_1(z) + \gamma_3 [q_1(z)]^2 + \gamma_4 \frac{z q_1'(z)}{q_1(z)} \\ & \prec \gamma_1 + \gamma_2 \left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \\ & + \gamma_3 \left[\frac{L(a, c) f(z)}{z} \right]^{2\mu} \left[\frac{z}{L(a+1, c) f(z)} \right]^{2\eta} \\ & + \gamma_4 \mu a \left[\frac{L(a+1, c) f(z)}{L(a, c) f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[1 - \frac{L(a+2, c) f(z)}{L(a+1, c) f(z)} \right] \\ & \prec \gamma_1 + \gamma_2 q_2(z) + \gamma_3 [q_2(z)]^2 + \gamma_4 \frac{z q_2'(z)}{q_2(z)}, \end{aligned}$$

holds, then

$$q_1(z) \prec \left[\frac{L(a, c) f(z)}{z} \right]^\mu \left[\frac{z}{L(a+1, c) f(z)} \right]^\eta \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

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