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## SOME RESULTS FOR HADAMARD-TYPE INEQUALITIES IN QUANTUM CALCULUS

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In this paper, we establish the  $q$ -analogue of Hermite-Hadamard inequalities for some convex type functions.

### 1. Introduction

A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ ,  $I$  is an interval, is said to be a convex function on  $I$  if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ . If the reversed inequality in (1) holds, then  $f$  is said to be concave. Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and  $a, b \in I$  with  $a < b$ . Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (2)$$

is known in the literature as Hermite-Hadamard inequality for convex mappings [7].

Some basic definitions can be given as follows:

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**Definition 1.1** ([8]). Let  $s \in (0, 1]$ . A function  $f : [0, \infty) \rightarrow [0, \infty)$  is said to be  $s$ -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (3)$$

for all  $x, y \in [0, \infty)$  and  $t \in [0, 1]$ . This class of  $s$ -convex functions is usually denoted by  $K_s^2$ .

**Definition 1.2** ([20]). Let  $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a positive function and  $[0, 1] \subset J$ . We say that  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is  $h$ -convex function, or that  $f$  belong to the class  $SX(h, I)$ , if  $f$  is nonnegative and for all  $x, y \in I$  and  $\alpha \in (0, 1)$  we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y). \quad (4)$$

If inequality (4) is reversed, then  $f$  is said to be  $h$ -concave, i.e.  $f \in SV(h, I)$ . Obviously, if  $h(t) = t$ , then all nonnegative convex functions belong to  $SX(h, I)$  and all nonnegative concave functions belong to  $SV(h, I)$ ; and if  $h(t) = t^s$ , where  $s \in (0, 1)$ , then  $SX(h, I) \supseteq K_s^2$ .

In [19], G. H. Toader defined the concept of  $m$ -convexity as the following:

**Definition 1.3.** The function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be  $m$ -convex, where  $m \in [0, 1]$ , if for every  $x, y \in [0, b]$  and  $t \in [0, 1]$  we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \quad (5)$$

Denote by  $K_m(b)$  the set of the  $m$ -convex functions on  $[0, b]$ .

In [12], V. G. Miheşan introduced the class of  $(s, m)$ -convex functions as the following:

**Definition 1.4.** The function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be  $(s, m)$ -convex, where  $(s, m) \in (0, 1]^2$ , if for every  $x, y \in [0, b]$  and  $t \in [0, 1]$  we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y). \quad (6)$$

Denote by  $K_m^s(b)$  the set of the  $(s, m)$ -convex functions on  $[0, b]$ . If we choose  $(s, m) = (1, m)$ , it can be easily seen that  $(s, m)$ -convexity reduces to  $m$ -convexity and for  $(s, m) = (1, 1)$ , we have ordinary convex functions on  $[0, b]$ . For the recent results based on the above definition see the papers ([1, 12–15, 18]). In [4], S. S. Dragomir and N. M. Ionescu introduced the following class of functions:

**Definition 1.5.** Let  $g : I \rightarrow \mathbb{R}$  be a given convex function on the interval  $I$  from  $\mathbb{R}$ . The real function  $f : I \rightarrow \mathbb{R}$  is called  $g$ -convex dominated on  $I$  if the following condition is satisfied:

$$\begin{aligned} & |tf(x) + (1-t)f(y) - f(tx + (1-t)y)| \\ & \leq tg(x) + (1-t)g(y) - g(tx + (1-t)y) \end{aligned} \quad (7)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

The following theorems are some known results obtained in recent years: In [3], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for  $s$ -convex functions in the second sense:

**Theorem 1.6** ([3]). *Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is an  $s$ -convex function in the second sense, where  $s \in (0, 1]$  and let  $a, b \in [0, \infty)$ ,  $a < b$ . If  $f \in L^1[0, 1]$ , then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1}, \quad (8)$$

the constant  $k = \frac{1}{s+1}$  is the best possible in the second inequality in (8). The above inequalities are sharp.

In [17], Sarikaya, Saglam and Yildirim proved some Hadamard-type inequalities for  $h$ -convex functions:

**Theorem 1.7** ([17]). *Let  $f \in SX(h, I)$ ,  $a, b \in I$ , with  $a < b$  and  $f \in L^1[a, b]$ . Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a)+f(b)] \int_0^1 h(\alpha) d\alpha. \quad (9)$$

In [6], the following inequality of Hermite-Hadamard type for  $m$ -convex functions holds:

**Theorem 1.8** ([6]). *Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $m$ -convex function with  $m \in (0, 1]$ . If  $0 \leq a < b < \infty$  and  $f \in L^1[a, b]$ , then one has the inequality:*

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}. \quad (10)$$

For  $0 < q < 1$ , the  $q$ -Jackson integral from 0 to  $b$  is defined by [10]

$$\int_0^b f(x) d_q x = (1-q)b \sum_{n=0}^{\infty} f(bq^n) q^n \quad (11)$$

provided the sum converge absolutely. The  $q$ -Jackson integral in a generic interval  $[a, b]$  is given by [10]

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \quad (12)$$

In [16], the authors presented a Riemann-type  $q$ -integral by

$$R_q(f; a, b) = (b-a)(1-q) \sum_{k=0}^{\infty} f(a + (b-a)q^k) q^k \quad (13)$$

The aim of this work is to divide the interval in two parts, then we can get another definition from the Riemman-type  $q$ -integral:

$$\begin{aligned} & \frac{2}{b-a} \int_a^b f(x) d_q^R x \\ &= (1-q) \sum_{k=0}^{\infty} \left( f \left( \frac{a+b}{2} + q^k \left( \frac{b-a}{2} \right) \right) + f \left( \frac{a+b}{2} - q^k \left( \frac{b-a}{2} \right) \right) \right) q^k \end{aligned}$$

From the  $q$ -Jackson integral we can write:

$$\begin{aligned} \frac{2}{b-a} \int_a^b f(x) d_q^R x &= \int_{-1}^1 f \left( \frac{1-t}{2} a + \frac{1+t}{2} b \right) d_q t \\ &= \int_{-1}^1 f \left( \frac{1+t}{2} a + \frac{1-t}{2} b \right) d_q t. \end{aligned}$$

In recent years, many authors have studied several inequalities connected to this famous integral inequality (2). For some results which generalize, improve and extend the inequality (2) (see [2], [3], [5], [6], [17], [18]). S.S. Dragomir ([2],[6]) proved several Hermite-Hadamrd type inequalities for  $m$ -convex functions. S.S. Dragomir and S. Fitzpatrick [3] proved a variant of Hadamard's inequality which holds for  $s$ -convex functions in the second sense. S.S Dragomir, C.E.M. Pearce and J.E. Pečarić [5] proved the Hermite-Hadamrd's inequality for convex-dominated functions. M.Z. Sarikaya, A. Saglam and H. Yildirim [17] proved some Hadamard-type inequalities for  $h$ -convex functions. E. Set, M. Sardari, M.E. Ozdemir and J. Roojin [18] proved the Hermite-Hadamard for  $(s, m)$ -convex functions. For more inequalities on convex functions see also the references in the above cited papers.

The main purpose of this paper is to establish the  $q$ -analogue of the Hermite-Hadamard inequality for some inequalities proved in ([2], [3], [5], [6], [17], [18]).

## 2. Main Results

**Theorem 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function. Then one has the inequalities:*

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \frac{f(a) + f(b)}{2}.$$

*Proof.* According to the definition of convex function we have

$$f \left( \frac{x+y}{2} \right) \leq \frac{f(x)}{2} + \frac{f(y)}{2}$$

for all  $x \in [a, b]$ .

Choose  $x = \frac{1-t}{2}a + \frac{1+t}{2}b$  and  $y = \frac{1+t}{2}a + \frac{1-t}{2}b$ ,  $t \in [-1, 1]$ . We obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \frac{1}{2}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)$$

for all  $t \in [-1, 1]$ .

Integrating over  $t \in [-1, 1]$ , we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \int_{-1}^1 d_q t \\ & \leq \frac{1}{2} \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t + \frac{1}{2} \int_{-1}^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_q t \end{aligned}$$

then

$$2f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(t) d_q^R t$$

which proves the first inequality. The proof of the second inequality is given by

$$f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \leq \left(\frac{1-t}{2}\right)f(a) + \left(\frac{1+t}{2}\right)f(b).$$

We integrate  $t$  over  $[-1, 1]$ , we get

$$\frac{2}{b-a} \int_a^b f(t) d_q^R t \leq f(a) \int_{-1}^1 \frac{1-t}{2} d_q t + f(b) \int_{-1}^1 \frac{1+t}{2} d_q t.$$

Then

$$\frac{2}{b-a} \int_a^b f(x) d_q^R x \leq f(a) + f(b).$$

The proof of Theorem 2.1 is completed.  $\square$

**Theorem 2.2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a  $s$ -convex function. Then one has the following:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \frac{(2+q)^s + q^s}{2^{s+1} [2]_q^s} (f(a) + f(b)),$$

where  $[2]_q = 1 + q$ .

*Proof.* According to the definition of  $s$ -convex function with  $x = \frac{1-t}{2}a + \frac{1+t}{2}b$  and  $y = \frac{1+t}{2}a + \frac{1-t}{2}b$ ,  $t \in [-1, 1]$ , we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)}{2}\right) \\ &\leq \frac{1}{2^s} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \frac{1}{2^s} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \end{aligned}$$

Integrating with respect  $t$  over  $[-1, 1]$ , we obtain

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^s} \frac{4}{b-a} \int_a^b f(t) d_q^R t$$

the first inequality follows.

Secondly, we have

$$f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \leq \left(\frac{1-t}{2}\right)^s f(a) + \left(\frac{1+t}{2}\right)^s f(b)$$

then, integrating this inequality over  $t \in [-1, 1]$ , we obtain

$$\frac{2}{b-a} \int_a^b f(t) d_q^R t \leq \frac{f(a)}{2^s} \int_{-1}^1 (1-t)^s d_q t + \frac{f(b)}{2^s} \int_{-1}^1 (1+t)^s d_q t.$$

Moreover,

$$\int_{-1}^1 (1+t)^s d_q t = \int_{-1}^0 (1+t)^s d_q t + \int_0^1 (1+t)^s d_q t$$

using the  $q$ -Hölder's inequality, we get

$$\frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \frac{(2+q)^s + q^s}{2^{s+1} [2]_q^s} (f(a) + f(b))$$

The result is thus proved. □

**Remark 2.3.** Applying Theorem 2.2 for  $s = 1$ , we obtain Theorem 2.1.

**Theorem 2.4.** Let  $f \in SX(h, I)$ ,  $a, b \in I$ , with  $a < b$  and  $f \in L^1[a, b]$ , then

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(t) d_q^R t \leq (f(a) + f(b)) \int_0^1 h(t) d_q^R t.$$

*Proof.* Since  $f \in SX(h, I)$  with  $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ ,  $y = \frac{1+t}{2}a + \frac{1-t}{2}b$ , and  $\alpha = \frac{1}{2}$ , we obtain

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right) \left[ f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right].$$

Integrating over  $t \in [-1, 1]$ , we obtain

$$\begin{aligned} & 2f\left(\frac{a+b}{2}\right) \\ & \leq h\left(\frac{1}{2}\right) \left[ \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t + \int_{-1}^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_q t \right]. \end{aligned}$$

Then

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(t) d_q^R t$$

and the first inequality is proved. Now the proof of second inequality is given by

$$f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \leq h\left(\frac{1-t}{2}\right) f(a) + h\left(\frac{1+t}{2}\right) f(b).$$

We integrate  $t$  on  $[-1, 1]$ , we obtain

$$\begin{aligned} & \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ & \leq f(a) \int_{-1}^1 h\left(\frac{1-t}{2}\right) d_q t + f(b) \int_{-1}^1 h\left(\frac{1+t}{2}\right) d_q t. \end{aligned}$$

However,

$$\int_{-1}^1 h\left(\frac{1-t}{2}\right) d_q t = \int_{-1}^1 h\left(\frac{1+t}{2}\right) d_q t = \int_0^1 h(t) d_q^R t$$

then

$$\frac{2}{b-a} \int_a^b f(t) d_q^R t \leq (f(a) + f(b)) \int_0^1 h(t) d_q^R t. \tag{14}$$

Theorem 2.4 is thus proved. □

**Remark 2.5.** Applying Theorem 2.4 for  $h(t) = t^s$ , we obtain Theorem 2.2.

**Theorem 2.6.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $m$ -convex function with  $m \in [0, 1]$ . If  $0 \leq a < b < \infty$  and  $f \in L^1[a, b]$ , then one has the inequality:

$$\frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \min \left\{ \frac{f(a) + m f\left(\frac{b}{m}\right)}{2}, \frac{f(b) + m f\left(\frac{a}{m}\right)}{2} \right\}.$$

*Proof.* Since  $f$  is  $m$ -convex, we have

$$f\left(\frac{1-t}{2}x + m\frac{1+t}{2}y\right) \leq \left(\frac{1-t}{2}\right)f(x) + m\left(\frac{1+t}{2}\right)f(y),$$

for all  $x, y \geq 0$  and  $t \in [-1, 1]$ , which gives

$$f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \leq \left(\frac{1-t}{2}\right)f(a) + m\left(\frac{1+t}{2}\right)f\left(\frac{b}{m}\right)$$

and

$$f\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right) \leq \left(\frac{1-t}{2}\right)f(b) + m\left(\frac{1+t}{2}\right)f\left(\frac{a}{m}\right)$$

then, by integrating both sides with respect  $t$  over  $[-1, 1]$ , we obtain

$$\frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}$$

and

$$\frac{1}{b-a} \int_a^b f(t) d_q^R t \leq \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}.$$

So the proof is completed.  $\square$

**Theorem 2.7.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $m$ -convex function with  $m \in (0, 1]$  and  $0 \leq a < b$ . If  $f \in L^1[a, b]$ , then one has the inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b \frac{f(t) + mf\left(\frac{t}{m}\right)}{2} d_q^R t \\ &\leq \frac{1}{4} \left[ f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right) \right] \end{aligned}$$

*Proof.* By the  $m$ -convexity of  $f$  we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} \left[ f(x) + mf\left(\frac{y}{m}\right) \right]$$

for all  $x, y \in [0, \infty)$ .

Choose  $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ ,  $y = \frac{1+t}{2}a + \frac{1-t}{2}b$ , we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + mf\left(\left(\frac{1+t}{2}\right)\left(\frac{a}{m}\right) + \left(\frac{1-t}{2}\right)\left(\frac{b}{m}\right)\right) \right]$$



for all  $t \in [-1, 1]$ .

Integrating on  $[-1, 1]$

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ + \frac{m}{2} \int_{-1}^1 f\left(\left(\frac{1+t}{2}\right)\left(\frac{a}{m}\right) + \left(\frac{1-t}{2}\right)\left(\frac{b}{m}\right)\right) d_q t$$

then

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ \frac{2}{b-a} \int_a^b f(t) d_q^R t + \frac{2m}{b-a} \int_a^b f\left(\frac{t}{m}\right) d_q^R t \right]$$

and we get the proof of the first inequality. Now, by using the  $m$ -convexity of  $f$ , we get

$$\frac{1}{2} \left[ f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + mf\left(\left(\frac{1+t}{2}\right)\left(\frac{a}{m}\right) + \left(\frac{1-t}{2}\right)\left(\frac{b}{m}\right)\right) \right] \\ \leq \frac{1}{2} \left[ \left(\frac{1-t}{2}\right)f(a) + m\left(\frac{1+t}{2}\right)f\left(\frac{b}{m}\right) \right. \\ \left. + m\left(\frac{1+t}{2}\right)f\left(\frac{a}{m}\right) + m^2\left(\frac{1-t}{2}\right)f\left(\frac{b}{m^2}\right) \right]$$

for all  $t \in [-1, 1]$ . We integrate over  $t$  on  $[-1, 1]$ , we obtain

$$\frac{1}{b-a} \left( \int_a^b f(t) d_q^R t + m \int_a^b f\left(\frac{t}{m}\right) d_q^R t \right) \\ \leq \frac{1}{2} \left[ f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right) \right]. \quad \square$$

**Theorem 2.8.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a  $m$ -convex function with  $m \in (0, 1]$ . If  $f \in L^1[am, b]$  where  $0 \leq a < b$ , then one has the inequality:

$$\frac{1}{mb-a} \int_a^{mb} f(t) d_q^R t + \frac{1}{b-ma} \int_{ma}^b f(t) d_q^R t \leq (m+1) \frac{f(a)+f(b)}{2}$$

*Proof.* By the  $m$ -convexity of  $f$  we get

$$f\left(\frac{1-t}{2}a + m\left(\frac{1+t}{2}\right)b\right) + f\left(\frac{1+t}{2}a + m\left(\frac{1-t}{2}\right)b\right) \\ + f\left(\frac{1-t}{2}b + m\left(\frac{1+t}{2}\right)a\right) + f\left(\frac{1+t}{2}b + m\left(\frac{1-t}{2}\right)a\right) \\ \leq (m+1)(f(a) + f(b)).$$

Integrating over  $t \in [-1, 1]$ , we obtain

$$\begin{aligned} & \int_{-1}^1 f\left(\frac{1-t}{2}a + m\left(\frac{1+t}{2}\right)b\right) d_q t + \int_{-1}^1 f\left(\frac{1+t}{2}a + m\left(\frac{1-t}{2}\right)b\right) d_q t \\ & + \int_{-1}^1 f\left(\frac{1-t}{2}b + m\left(\frac{1+t}{2}\right)a\right) d_q t + \int_{-1}^1 f\left(\frac{1+t}{2}b + m\left(\frac{1-t}{2}\right)a\right) d_q t \\ & \leq 2(m+1)(f(a) + f(b)). \end{aligned}$$

then

$$\begin{aligned} & \frac{1}{mb-a} \int_a^{mb} f(t) d_q^R t + \frac{1}{mb-a} \int_a^{mb} f(t) d_q^R t + \frac{1}{b-ma} \int_{ma}^b f(t) d_q^R t \\ & + \frac{1}{b-ma} \int_{ma}^b f(t) d_q^R t \leq (m+1)(f(a) + f(b)) \quad \square \end{aligned}$$

and the result is proved.

**Theorem 2.9.** Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be an  $(s, m)$ -convex function with  $(s, m) \in (0, 1)^2$ . If  $0 \leq a < b < \infty$  and  $f \in L^1[a, b]$ , then one has the inequality:

$$\frac{2}{b-a} \int_a^b f(t) d_q^R t \leq C_s \min\{L(a, b), L(b, a)\}$$

$$\text{where } C_s = \frac{q^s + (2+q)^s}{2^s [2]_q^s}, \quad L(a, b) = f(a) - mf\left(\frac{b}{m}\right) + \frac{2m}{C_s} f\left(\frac{b}{m}\right).$$

*Proof.* Since  $f$  is an  $(s, m)$ -convex function on  $[a, b]$ , we know that for any  $t \in [-1, 1]$

$$f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \leq \left(\frac{1+t}{2}\right)^s f(a) + m\left(1 - \left(\frac{1+t}{2}\right)^s\right) f\left(\frac{b}{m}\right) \quad (15)$$

and

$$f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \leq \left(\frac{1+t}{2}\right)^s f(b) + m\left(1 - \left(\frac{1+t}{2}\right)^s\right) f\left(\frac{a}{m}\right). \quad (16)$$

By integrating both side (15) and (16) with respect to  $t$  over  $[-1, 1]$ , we get

$$\begin{aligned} & \int_{-1}^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_q t \\ & \leq f(a) \int_{-1}^1 \left(\frac{1+t}{2}\right)^s d_q t + mf\left(\frac{b}{m}\right) \int_{-1}^1 \left(1 - \left(\frac{1+t}{2}\right)^s\right) d_q t \end{aligned}$$

and

$$\int_{-1}^1 f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) d_q t \leq f(b) \int_{-1}^1 \left(\frac{1+t}{2}\right)^s d_q t + mf\left(\frac{a}{m}\right) \int_{-1}^1 \left(1 - \left(\frac{1+t}{2}\right)^s\right) d_q t.$$

Moreover,

$$\int_{-1}^1 \left(\frac{1+t}{2}\right)^s d_q t = \int_{-1}^0 \left(\frac{1+t}{2}\right)^s d_q t + \int_0^1 \left(\frac{1+t}{2}\right)^s d_q t.$$

Then by the  $q$ -Hölder’s inequality, we have

$$\frac{2}{b-a} \int_a^b f(t) d_q^R t \leq \left(f(a) - mf\left(\frac{b}{m}\right)\right) \left(\frac{q^s + (2+q)^s}{2^s [2]_q^s}\right) + 2mf\left(\frac{b}{m}\right)$$

and

$$\frac{2}{b-a} \int_a^b f(t) d_q^R t \leq \left(f(b) - mf\left(\frac{a}{m}\right)\right) \left(\frac{q^s + (2+q)^s}{2^s [2]_q^s}\right) + 2mf\left(\frac{a}{m}\right).$$

The proof of Theorem 2.9 is completed. □

**Remark 2.10.** Applying Theorem 2.9 for  $s = 1$ , we obtain Theorem 2.6.

**Theorem 2.11.** Let  $g : I \rightarrow \mathbb{R}$  be a convex mapping on  $I$  and  $f : I \rightarrow \mathbb{R}$  a  $g$ -convex-dominated mapping. Then, for all  $a, b \in I$  with  $a < b$ , one has the inequalities:

$$\frac{1}{b-a} \int_a^b (g+f)(t) d_q^R t \leq \frac{(f+g)(a) + (f+g)(b)}{2}$$

and

$$\frac{1}{b-a} \int_a^b (g-f)(t) d_q^R t \leq \frac{(g-f)(a) + (g-f)(b)}{2}.$$

*Proof.* From (7), we can write

$$\begin{aligned} & g\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) - \left(\frac{1-t}{2}\right)g(x) - \left(\frac{1+t}{2}\right)g(y) \\ & \leq \left(\frac{1-t}{2}\right)f(x) + \left(\frac{1+t}{2}\right)f(y) - f\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) \\ & \leq \left(\frac{1-t}{2}\right)g(x) + \left(\frac{1+t}{2}\right)g(y) - g\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) \end{aligned}$$

for all  $x, y \in I$  and  $t \in [-1, 1]$ , or additionally, with

$$\begin{aligned} & f\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) + g\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) \\ & \leq \left(\frac{1-t}{2}\right)(f(x) + g(x)) + \left(\frac{1+t}{2}\right)(f(y) + g(y)) \end{aligned}$$

and

$$\begin{aligned} & g\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) - f\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) \\ & \leq \left(\frac{1-t}{2}\right)(g(x) - f(x)) + \left(\frac{1+t}{2}\right)(g(y) - f(y)) \end{aligned}$$

for all  $x, y \in I$  and  $t \in [-1, 1]$ . Then, by using  $x = a$ ,  $y = b$  and integrating with respect  $t$  over  $[-1, 1]$ , we get

$$\begin{aligned} & \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t + \int_{-1}^1 g\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ & \leq (f(a) + g(a)) \int_{-1}^1 \left(\frac{1-t}{2}\right) d_q t + (f(b) + g(b)) \int_{-1}^1 \left(\frac{1+t}{2}\right) d_q t \end{aligned}$$

and

$$\begin{aligned} & \int_{-1}^1 g\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t - \int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t \\ & \leq (g(a) - f(a)) \int_{-1}^1 \left(\frac{1-t}{2}\right) d_q t + (g(b) - f(b)) \int_{-1}^1 \left(\frac{1+t}{2}\right) d_q t \end{aligned}$$

therefore

$$\frac{1}{b-a} \int_a^b (f+g)(t) d_q^R t \leq \frac{(f+g)(a) + (f+g)(b)}{2}$$

and

$$\frac{1}{b-a} \int_a^b (g-f)(t) d_q^R t \leq \frac{(g-f)(a) + (g-f)(b)}{2}.$$

Then, the proof of Theorem 2.11 is completed.  $\square$

**Theorem 2.12.** Let  $g : I \rightarrow \mathbb{R}$  be a convex mapping on  $I$  and  $f : I \rightarrow \mathbb{R}$  a  $g$ -convex-dominated mapping. Then, for all  $a, b \in I$  with  $a < b$ , one has the inequalities:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) d_q^R t - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{b-a} \int_a^b g(t) d_q^R t - g\left(\frac{a+b}{2}\right) \end{aligned} \quad (17)$$

and

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) d_q^R t \right| \\ & \leq \frac{g(a)+g(b)}{2} - \frac{1}{b-a} \int_a^b g(t) d_q^R t. \end{aligned} \tag{18}$$

*Proof.* Since  $f$  is  $g$ -convex dominated, we have

$$\left| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right| \leq \frac{g(x)+g(y)}{2} - g\left(\frac{x+y}{2}\right)$$

for all  $x, y \in [a, b]$ .

Choose  $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ ,  $y = \frac{1+t}{2}a + \frac{1-t}{2}b$ ,  $t \in [-1, 1]$ . Then we get

$$\begin{aligned} & \left| \frac{f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)}{2} - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{g\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + g\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)}{2} - g\left(\frac{a+b}{2}\right). \end{aligned}$$

Now, we integrate  $t$  over  $[-1, 1]$  we obtain

$$\begin{aligned} & \left| \frac{\int_{-1}^1 f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t + \int_{-1}^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_q t}{2} - 2f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{\int_{-1}^1 g\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_q t + \int_{-1}^1 g\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_q t}{2} - 2g\left(\frac{a+b}{2}\right), \end{aligned}$$

therefore

$$\begin{aligned} & \left| \frac{1}{b-a} \left( \int_a^b f(t) d_q^R t + \int_a^b f(x) d_q^R t \right) - 2f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{b-a} \left( \int_a^b g(t) d_q^R t + \int_a^b g(t) d_q^R t \right) - 2g\left(\frac{a+b}{2}\right) \end{aligned}$$

and the inequality (17) is proved.

Secondly, the proof of the second inequality is given by

$$\begin{aligned} & \left| \left(\frac{1-t}{2}\right) f(x) + \left(\frac{1+t}{2}\right) f(y) - f\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right) \right| \\ & \leq \left(\frac{1-t}{2}\right) g(x) + \left(\frac{1+t}{2}\right) g(y) - g\left(\frac{1-t}{2}x + \frac{1+t}{2}y\right). \end{aligned}$$

Then, by integrating over  $t \in [-1, 1]$  with  $x = a$  and  $y = b$ , we obtain

$$\begin{aligned} & \left| f(a) \int_{-1}^1 \left( \frac{1-t}{2} \right) d_q t + f(b) \int_{-1}^1 \left( \frac{1+t}{2} \right) - \int_{-1}^1 f \left( \frac{1-t}{2} a + \frac{1+t}{2} b \right) d_q t \right| \\ & \leq g(a) \int_{-1}^1 \left( \frac{1-t}{2} \right) d_q t + g(b) \int_{-1}^1 \left( \frac{1+t}{2} \right) d_q t - \int_{-1}^1 g \left( \frac{1-t}{2} a + \frac{1+t}{2} b \right) d_q t. \end{aligned}$$

Therefore,

$$\left| f(a) + f(b) - \frac{2}{b-a} \int_a^b f(t) d_q^R t \right| \leq g(a) + g(b) - \frac{2}{b-a} \int_a^b g(t) d_q^R t$$

then, we deduce the inequalities (18). □

**Remark 2.13.** From Theorem 2.11 and Theorem 2.12, we deduce

$$\begin{aligned} (f+g) \left( \frac{a+b}{2} \right) & \leq \frac{1}{b-a} \int_a^b (f+g)(t) d_q^R t \\ & \leq \frac{(f+g)(a) + (f+g)(b)}{2} \end{aligned}$$

and

$$\begin{aligned} (g-f) \left( \frac{a+b}{2} \right) & \leq \frac{1}{b-a} \int_a^b (g-f)(t) d_q^R t \\ & \leq \frac{(g-f)(a) + (g-f)(b)}{2}. \end{aligned}$$

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