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A GENERALIZED VARIATIONAL PRINCIPLE IN b-METRIC SPACES

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In this paper we establish and prove a generalized variational principle for b-metric spaces. As a consequence, we obtain a weak Zhong-type variational principle in b-metric spaces. We show the applicability of the mentioned generalized variational principle by presenting a Caristi-type fixed point theorem and an extension of the main result for bifunctions - both of them stated in b-metric spaces.

1. Introduction

One of the most important results of the mathematical analysis is the well-known Ekeland's variational principle, which has appeared in 1974 in [13]. This principle is considered the basis of modern calculus of variations, because since its discovery there have also appeared many generalizations and equivalent formulations of it (see [17–19], etc.). Ekeland's variational principle has been widely used in nonlinear analysis, since it entails the existence of approximate solutions of minimization problems for lower semicontinuous functions on complete metric spaces (see, for instance [5]). Since minimization problems are particular cases of equilibrium problems, many authors deal to derive

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existence results for solutions of equilibrium problems using Ekeland-type variational principles. M. Bianchi, G. Kassay, R. Pini extended in [8] a version of Ekeland's variational principle for bifunctions.

Recently, there have appeared many generalizations and extensions of these results to quasimetric spaces. In the literature, the notion of quasimetric space is used in two different ways: the first concept means asymmetry of the metric (see, for example, [1]), while in the second concept a quasimetric satisfies a relaxed triangle inequality, rather than the usual triangle inequality; see [6, 11, 15]. In this paper we will focus our attention to the latter quasimetric which is also called b-metric. The concept of this space was introduced by Bakthin in [6] and Czerwik in [11]. Since the publication of these works, several papers have appeared which were concerned to these spaces obtaining important results in many fields of mathematics: fixed point theory (study of fixed point theorems for single-valued and multivalued operators), geometry, calculus of variations. One of the advantages of using a b-metric is highlighted by the results of fixed point theory extended to these type of spaces, offering informations about the existence of the fixed point, the convergence of sequences of successive approximations, data dependence of the fixed point set and the study of well-posedness of the fixed point problem. Recently, B. Monica, A. Molnár and Cs. Varga in [9] formulated and proved a version of Ekeland's variational principle in b-metric spaces and using this theorem the authors gave a very simple proof of Caristi's fixed point theorem, stated also in b-metric spaces.

The aim of this paper is to present a Zhong-type theorem in different spaces. Before to state our main result, let us introduce some notations and symbols used in the paper.

Let (X,d) be a complete b-metric space such that the b-metric d is continuous, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous and bounded from below function. Let us also consider the continuous non-increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ and the non-negative number sequence $\delta_n \subset \mathbb{R}_+$ such that $\delta_0 > 0$. We also assume the following:

(A) The function $\rho : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies:

- (i) for each $x \in X$, we have $\rho(x,x) = 0$;
- (ii) for each $(y_n,z_n) \in X \times X$ as $n \to \infty$ such that $\rho(y_n,z_n) \to 0$, we have $d(y_n,z_n) \to 0$ as $n \to \infty$;
- (iii) for each $z \in X$, the function $y \mapsto \rho(y,z)$ is lower semi-continuous.

At the end, we introduce the notation below:

$$\mathcal{A}[x;m] = f(x) + h(d(x_0,x)) \sum_{n=0}^{m} \delta_n \rho(x,x_n), m \in \mathbb{N} \cup \{\infty\}.$$

Now, we can formulate our main result.

Theorem 1.1. *Let $\rho : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ be a function which satisfies the assumption (A). Then for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{1}$$

there exists of a sequence $\{x_n\} \subset X$ which converges to some x_ε ($x_n \to x_\varepsilon$ as $n \to \infty$) such that

$$h(d(x_0, x_n))\rho(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n \delta_0}, \text{ for all } n \in \mathbb{N}. \tag{2}$$

If $\delta_n > 0$ for infinitely many $n \in \mathbb{N}$, then we have

$$\mathcal{A}[x_\varepsilon; +\infty] \leq f(x_0), \tag{3}$$

and for $x \neq x_\varepsilon$ we have

$$\mathcal{A}[x; +\infty] > \mathcal{A}[x_\varepsilon; +\infty]. \tag{4}$$

If $\delta_k > 0$ for some $k \in \mathbb{N}^*$ and $\delta_j = 0$ for every $j > k$, then for each $x \neq x_\varepsilon$ there exists $m \in \mathbb{N}$, $m \geq k$ such that

$$\mathcal{A}[x; k-1] + h(d(x_0, x))\delta_k \rho(x, x_m) > \mathcal{A}[x_\varepsilon; k-1] + h(d(x_0, x_\varepsilon))\delta_k \rho(x_\varepsilon, x_m). \tag{5}$$

The paper is structured as follows. In Section 2, we recall the definition of b-metric space given by Czerwick in [11], then we present some important properties and examples concerning these spaces. We also state Cantor's intersection theorem in b-metric spaces, which will be crucial on the proof of our main result. In Section 3, we prove our paper's chief result. Section 4 is dedicated to present some applications of this result, for example a Caristi-type fixed point theorem. Section 5 is concerned to fomulate a version of the main variational principle for functions with two variables.

2. Preliminaries

For the convenience of the reader, let us first recall some notations and preliminary results from theory of b-metric spaces.

Definition 2.1 (Bakhtin [6], Czerwik [11]). Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a *b-metric* if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;

2. $d(x,y) = d(y,x)$;
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a *b-metric space*.

Remark 2.2. Obviously, the class of b-metric spaces is larger than the class of metric spaces, since a b-metric space is a metric space, when $s = 1$ in the third assumption of the above definition. An example, which shows that a b-metric on X need not to be a metric on X at the same time, can be found in [20].

Definition 2.3. Let (X,d) be a b-metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) Cauchy sequence if and only if $d(x_n,x_m) \to 0$ as $m,n \to \infty$.

Remark 2.4. Let (X,d) be a b-metric space.

(a) Every convergent sequence in (X,d) is a Cauchy sequence ($d(x_n,x_m) \leq s(d(x_n,x) + d(x,x_m))$).

(b) If every Cauchy sequence in (X,d) is convergent, then we say that d is a complete b-metric on X.

(c) In general, a b-metric is not continuous.

We also notice that almost all concepts and results obtained for metric spaces can be extended to the case of b-metric spaces. For a large amount of results concerning b-metric spaces see for instance [11] and [15], respectively.

Example 2.5 (Berinde [7])**.** The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}, \text{ for each } x,y \in L^p[0,1],$$

is a b-metric space.

By an elementary calculation, we obtain that $d(x,z) \leq 2^{\frac{1}{p}}[d(x,y) + d(y,z)]$. Therefore, $s = 2^{1/p} > 0$.

Example 2.6 (Xia [23])**.** Let us consider $(\mathbb{R}^n, \|\cdot\|)$. Then, for every $\beta > 1, \lambda \geq 0, \mu > 0$, $J(x,y) = \lambda\|x-y\| + \mu\|x-y\|^{\beta}$ is not a metric on $\mathbb{R}^n$. Nevertheless, $(\mathbb{R}^n, J)$ is a b-metric space with $s = 2^{\beta-1}$.

For some other examples, the reader is invited to consult the papers [7], [9] and [23]. Finally, we mention Cantor's intersection theorem for b-metric spaces. Notice that, if (X,d) is a b-metric space and Y is a nonempty subset of X, then, as in metric spaces, $\mathrm{diam}(Y)$ denotes the diameter of the set Y, i.e.,

$$\mathrm{diam}(Y) := \sup\{d(a,b) \mid a,b \in Y\}.$$

Lemma 2.7 (M. Bota et al. [9])**.** *Let (X,d) be a b-metric space. We suppose that (X,d) is complete. Then, for every non-increasing sequence $\{F_n\}_{n\geq 1}$ of nonempty closed subsets of X, that is,*

$$F_1 \supset F_2 \supset F_3 \supset \cdots \supset F_n \supset \dots \tag{6}$$

such that

$$\mathrm{diam}(\mathrm{F_n}) \to 0 \textit{ as } \mathrm{n} \to \infty, \tag{7}$$

we have that the intersection $\bigcap\limits_{n=1}^{\infty} F_n$ *contains one and only one point.*

3. Proof of Theorem 1.1

Now, we are in the position to prove our main result.

Proof. In the first case (for infinitely many $n \in \mathbb{N}$), without loss of generality, we can assume that $\delta_n > 0$, for every $n \in \mathbb{N}$. Let us define the following set:

$$\mathcal{W}(x_0) = \{x \in X | \mathcal{A}[x;0] \leq f(x_0)\}. \tag{8}$$

By the assumption (i), we have $\rho(x_0,x_0) = 0$, so $x_0 \in \mathcal{W}(x_0)$. Therefore, the set $\mathcal{W}(x_0)$ is nonempty. From the lower semi-continuity of the functions f and $\rho(\cdot,x_0)$ and the continuity of the function h, we deduce that $\mathcal{W}(x_0)$ is a closed subset of X. We can choose the element $x_1 \in \mathcal{W}(x_0)$ such that

$$\mathcal{A}[x_1;0] \leq \inf_{x\in\mathcal{W}(x_0)} \mathcal{A}[x;0] + \frac{\varepsilon\cdot\delta_1}{2\delta_0}$$

and consider:

$$\mathcal{W}(x_1) = \{x \in \mathcal{W}(x_0) | \mathcal{A}[x;1] \leq \mathcal{A}[x_1;0]\}.$$

Similarly as above, we obtain that $\mathcal{W}(x_1) \neq \emptyset$ (since $x_1 \in \mathcal{W}(x_1)$), and $\mathcal{W}(x_1)$ is a nonempty closed subset of $\mathcal{W}(x_0)$, which it means that $\mathcal{W}(x_1)$ is a nonempty closed subset of X as well.

Using the method of mathematical induction, we can define a sequence $x_{n-1} \in \mathcal{W}(x_{n-2})$ and a set $\mathcal{W}(x_{n-1})$ such that:

$$\mathcal{W}(x_{n-1}) = \{x \in \mathcal{W}(x_{n-2}) | \mathcal{A}[x; n-1] \leq \mathcal{A}[x_{n-1}; n-2]\}.$$

It is easy to see that $\mathcal{W}(x_{n-1}) \neq \emptyset$, and $\mathcal{W}(x_{n-1})$ is a closed subset of X. We can choose $x_n \in \mathcal{W}(x_{n-1})$ such that

$$\mathcal{A}[x_n; n-1] \leq \inf_{x \in \mathcal{W}(x_{n-1})} \mathcal{A}[x; n-1] + \frac{\delta_n \cdot \varepsilon}{2^n \delta_0},$$

and consider the set

$$\mathcal{W}(x_n) = \{x \in \mathcal{W}(x_{n-1}) | \mathcal{A}[x; n] \leq \mathcal{A}[x_n; n-1]\},$$

which is also a closed subset of X.

Let z be an arbitrary element of $\mathcal{W}(x_n)$. Then, from the definition of $\mathcal{W}(x_n)$, we have the following inequality

$$\mathcal{A}[z; n] \leq \mathcal{A}[x_n; n-1],$$

which means that,

$$f(z) + h(d(x_0, z))\delta_n \rho(z, x_n) + h(d(x_0, z)) \sum_{i=0}^{n-1} \delta_i \rho(z, x_i) \leq f(x_n) + h(d(x_0, x_n)) \sum_{i=0}^{n-1} \delta_i \rho(x_n, x_i).$$

Hence, we obtain that

$$\begin{aligned} h(d(x_0, z))\delta_n \rho(z, x_n) &\leq \left[f(x_n) + h(d(x_0, x_n)) \sum_{i=0}^{n-1} \delta_i \rho(x_n, x_i) \right] \\ &\quad - \left[f(z) + h(d(x_0, z)) \sum_{i=0}^{n-1} \delta_i \rho(z, x_i) \right] \\ &\leq \mathcal{A}[x_n; n-1] - \inf_{x \in \mathcal{W}(x_{n-1})} \mathcal{A}[x; n-1] \leq \frac{\delta_n \varepsilon}{2^n \delta_0}, \end{aligned}$$

therefore

$$h(d(x_0, z))\rho(z, x_n) \leq \frac{\varepsilon}{2^n \delta_0}. \tag{9}$$

So, if $n \to \infty$, then $\rho(z, x_n) \to 0$. Then, from (ii) it follows that $d(z, x_n) \to 0$ as $n \to \infty$. Therefore $\mathrm{diam}(\mathcal{W}(\mathrm{x_n})) \to 0$, whenever $n \to \infty$ and we obtain a descending sequence $\{\mathcal{W}(x_n)\}_{n \geq 0}$ of nonempty closed subsets of X,

$$\mathcal{W}(x_0) \supset \mathcal{W}(x_1) \supset \ldots . \supset \mathcal{W}(x_n) \supset \ldots$$

such that $\mathrm{diam}(\mathcal{W}(\mathrm{x_n})) \to 0$, as $n \to \infty$. Applying the Cantor intersection theorem for the set sequence $\{\mathcal{W}(x_n)\}_{n \in \mathbb{N}}$, we conclude that there exists an $x_\varepsilon \in X$ such that (see Lemma 2.7)

$$\bigcap_{n=0}^{\infty} \mathcal{W}(x_n) = \{x_\varepsilon\}.$$

We can observe that $z = x_\varepsilon$ satisfies the inequality (9), therefore $x_n \to x_\varepsilon$ as $n \to \infty$. If $x \neq x_\varepsilon$, then there exists $m \in \mathbb{N}$ such that

$$\mathcal{A}[x; m] > \mathcal{A}[x_m; m-1]. \tag{10}$$

It is clear that, if $q \geq m$ then

$$\mathcal{A}[x_m; m-1] \geq \mathcal{A}[x_q; q-1] \geq \mathcal{A}[x_\varepsilon; q-1].$$

Combining this relation with inequality (10), we get the following estimation

$$\mathcal{A}[x; m] \geq \mathcal{A}[x_\varepsilon; q-1].$$

Hence, if $q, m \to \infty$, we obtain the required relation (4).

It remains to treat the second case. Let us assume the existence of a number $k \in \mathbb{N}$ such that $\delta_k > 0$ and $\delta_j = 0$, for each $j > k \geq 0$. Without loss of generality, we can assume that $\delta_i > 0$ for every $i \leq k$. If $n \leq k$, then we can take x_n and $\mathcal{W}(x_n)$ similarly as above. If $n > k$, we can choose $x_n \in \mathcal{W}(x_{n-1})$, so that

$$\mathcal{A}[x_n; k-1] \leq \inf_{x \in \mathcal{W}(x_{n-1})} \mathcal{A}[x; k-1] + \frac{\delta_k \varepsilon}{2^n \delta_0},$$

and we define the following set

$$\mathcal{W}(x_n) = \{x \in \mathcal{W}(x_{n-1}) | \mathcal{A}[x; k-1] + h(d(x_0, x)) \delta_k \rho(x, x_n) \leq \mathcal{A}[x_n; k-1]\}.$$

Similarly to the "infinity" case, we can show that the statement of Theorem 1.1 holds. But, if we have $x \neq x_\varepsilon$, then there exists an $m > k$ such that

$$\begin{aligned} \mathcal{A}[x; k-1] + h(d(x_0, x)) \delta_k(x, x_m) &> f(x_m) + h(d(x_0, x)) \sum_{i=0}^{k-1} \delta_i \rho(x_m, x_i) \\ &\geq \mathcal{A}[x_\varepsilon; k-1] + h(d(x_0, x_\varepsilon)) \delta_k \rho(x_\varepsilon, x_m), \end{aligned}$$

which concludes the proof. □

Remark 3.1. If in the previous theorem we choose $h(x) \equiv 1$ and $\delta_n = \frac{1}{s^n}$, we get back to the main result of [9]. In other words, for every $x \neq x_\varepsilon$ we obtain the following inequality:

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x, x_n) > f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x_\varepsilon, x_n). \tag{11}$$

4. Applications

In this section we give some applications of the main result. First, we formulate and prove an extension of Caristi's fixed point theorem.

4.1. Caristi-type fixed point theorem

In the paper [14], Ekeland gave a new proof of Caristi's fixed point theorem (for the original proof, see [10]), applying a consequence of Ekeland's variational principle. Following the main ideas of these proofs, we obtained a new Caristi-type fixed point theorem in b-metric spaces, which shows the applicability of Theorem 1.1 to fixed point theory.

Throughout this section, we denote by ξ the sum of the convergent series $\sum_{n=0}^{\infty} \delta_n$, i.e.

$$\xi := \sum_{n=0}^{\infty} \delta_n.$$

Theorem 4.1. *Let (X,d) be a complete b-metric space such that d is a continuous b-metric and $\rho : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ be a continuous function. Let us consider the operator $\varphi : X \to X$ such that there exists a lower semi-continuous mapping $f : X \to \mathbb{R}_+ \cup \{\infty\}$ satisfying the following assumptions:*

(i) $h(d(x_0, \varphi(x)))\rho(\varphi(x), y) - h(d(x_0, x))\rho(x, y) \leq \rho(x, \varphi(x))$,

(ii) $\xi \rho(u, \varphi(u)) \leq f(u) - f(\varphi(u))$.

Then φ has at least one fixed point.

Proof. We argue by contradiction. We assume that

$$\varphi(x) \neq x, \text{ for all } x \in X. \tag{12}$$

Using Theorem 1.1, we have that for each $\varepsilon > 0$ there exists a sequence δ_j of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \to x_\varepsilon$ as $n \to \infty$, $x_\varepsilon \in X$ such that for every $x \in X$, $x \neq x_\varepsilon$ we have $\mathcal{A}[x; +\infty] > \mathcal{A}[x_\varepsilon; +\infty]$, i.e,

$$f(x) + h(d(x_0, x)) \sum_{n=0}^{\infty} \delta_n \rho(x, x_n) > f(x_\varepsilon) + h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n). \tag{13}$$

By the initial assumption, in (13) we choose $x := \varphi(x_\varepsilon)$. Hence, we get the following inequality:

$$f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < h(d(x_0, \varphi(x_\varepsilon))) \sum_{n=0}^{\infty} \delta_n \rho(\varphi(x_\varepsilon), x_n) - h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n),$$

which is equivalent with

$$f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < \sum_{n=0}^{\infty} \delta_n \left[h(d(x_0, \varphi(x_\varepsilon))) \rho(\varphi(x_\varepsilon), x_n) - h(d(x_0, x_\varepsilon)) \rho(x_\varepsilon, x_n) \right]. \quad (14)$$

From (i), it follows that

$$f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < \sum_{n=0}^{\infty} \delta_n \left[h(d(x_0, \varphi(x_\varepsilon))) \rho(\varphi(x_\varepsilon), x_n) - h(d(x_0, x_\varepsilon)) \rho(x_\varepsilon, x_n) \right]$$
$$\leq \sum_{n=0}^{\infty} \delta_n \left[\rho(x_\varepsilon, \varphi(x_\varepsilon)) \right] = \rho(x_\varepsilon, \varphi(x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n = \xi \rho(x_\varepsilon, \varphi(x_\varepsilon)). \quad (15)$$

We observe, that if in (ii) we choose $u = x_\varepsilon$, then we derive that

$$\xi \rho(x_\varepsilon, \varphi(x_\varepsilon)) \leq f(x_\varepsilon) - f(\varphi(x_\varepsilon)). \quad (16)$$

From the relation (15), we have

$$f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < \xi \rho(x_\varepsilon, \varphi(x_\varepsilon)). \quad (17)$$

If we compare the inequalities (17) and (16), we have that

$$\xi \rho(x_\varepsilon, \varphi(x_\varepsilon)) \leq f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < \xi \rho(x_\varepsilon, \varphi(x_\varepsilon)),$$

which is a contradiction.

Thus, there exists $\tilde{x} \in X$ such that $\tilde{x} \in \varphi(\tilde{x})$. □

4.2. The weak Zhong-type variational principle

Ekeland's variational principle states that if a Gâteaux differentiable function f has a finite lower bound, then for every $\varepsilon > 0$, there exists some point x_ε such that $\|f'(x_\varepsilon)\| \leq \varepsilon$. We want to show that a similar result can be proved for b-metric spaces. First of all, we need to prove the following two lemmas. Let us denote by $B(x_0; r)$ the closed ball of radius r centered at a point x_0 from the metric space (X, d), defined by

$$B(x_0; r) = \{x \in X | d(x, x_0) \leq r\}.$$

Lemma 4.2. *If $g:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous non-decreasing function, and $x\notin B(x_0;d(x_0,x_\varepsilon))$ then*

$$\frac{d(x_0,x)}{1+g(d(x_0,x))}-s\cdot\frac{d(x_0,x_\varepsilon)}{1+g(d(x_\varepsilon,x_0))}\le s\cdot\frac{d(x,x_\varepsilon)}{1+g(d(x_\varepsilon,x_0))}. \tag{18}$$

Proof. We introduce the following notations:

$$\begin{cases} d(x_0,x)=a,\\ d(x_0,x_\varepsilon)=c,\\ d(x,x_\varepsilon)=b.\end{cases}$$

It is easy to see that a,b,c are exactly the three sides of a triangle. Therefore, the inequality (18) can be rewritten in the following form

$$\frac{a}{1+g(a)}\le s\cdot\frac{b+c}{1+g(c)}$$

which is equivalent with

$$a+ag(c)\le s(b+c)+s(b+c)g(a). \tag{19}$$

If $a\ge c$, then we have $g(a)\ge g(c)$ (since the choice of g), so

$$ag(c)\le ag(a)\le s(b+c)g(a).$$

□

Lemma 4.3. *If $g:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous non-decreasing function, and $\frac{g(x)}{x}$ is decreasing on $(0,d(x_0,x_\varepsilon)]$, then*

$$\frac{d(x_0,x)}{1+g(d(x_0,x))}-s\cdot\frac{d(x_0,x_\varepsilon)}{1+g(d(x_\varepsilon,x_0))}\le s\cdot\frac{d(x,x_\varepsilon)}{1+g(d(x_\varepsilon,x_0))}.$$

Proof. The proof works similarly as in the case of Lemma 4.2. We can observe that, if $x\mapsto\frac{g(x)}{x}$ is a non-increasing function, then $\frac{g(c)}{c}\le\frac{g(a)}{a}$, hence

$$a+ag(c)\le a+cg(a)\le s(b+c)+cg(a)\le s(b+c)+s(b+c)g(a).$$

□

Next we show that in a special case of the Theorem 1.1 we get Zhong's variational principle (see for instance [21, 22]). In order to obtain this result let us choose the functions h, ρ, and the sequence δ_n as follows. Let $\delta_0=1$ and

$\delta_n = 0$, for every $n > 0$. Let $\varepsilon, \lambda > 0$ and $h(t) = \dfrac{\varepsilon}{\lambda(1+g(t))}$, where $g : [0,\infty) \to [0,\infty)$ is a continuous non-decreasing function. In this case, we have

$$\sum_{n=0}^{\infty} \delta_n \rho(x,x_n) = \delta_0 \rho(x,x_0) = \rho(x,x_0).$$

If $\rho = d$, then Theorem 1.1 has the following form:

$$f(x) \geq f(x_\varepsilon) + \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x_\varepsilon,x_0) - \frac{\varepsilon}{\lambda(1+g(d(x_0,x)))} d(x,x_0). \tag{20}$$

From Lemma 4.3 and 4.2, we get the following weak Zhong-type variational principle in b-metric space.

Corollary 4.4 (Weak Zhong-type variational principle)**.** *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function. Let (X,d) be a complete b-metric space (the b-metric d is continuous) and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semi-continuous and bounded from below function. Then for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{21}$$

there exists a sequence $\{x_n\} \subset X$ which converges to some x_ε (where $x_n \to x_\varepsilon$ as $x \to \infty$) such that

$$h(d(x_0,x_n)) d(x_\varepsilon,x_n) \leq \frac{\varepsilon}{2^n}, \ n \in \mathbb{N}. \tag{22}$$

Then we can distinguish two cases:
1. If $x \notin B(x_0, d(x_0,x_\varepsilon))$, then we have

$$f(x) \geq f(x_\varepsilon) - s \cdot \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x,x_\varepsilon). \tag{23}$$

2. If $\dfrac{g(x)}{x}$ is decreasing on $(0, d(x_0,x_\varepsilon)]$, then for all we have $x \neq x_\varepsilon$

$$f(x) \geq f(x_\varepsilon) - s \cdot \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x,x_\varepsilon).$$

Remark 4.5. Let X be a b-Banach spaces. Let us recall first that a b-norm on a vector space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a map $X \to [0,\infty)$ with the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \|x\|$ if $\alpha \in \mathbb{K}$, $x \in X$;

(iii) There is a constant $s \geq 1$ such that if $x_1, x_2 \in X$ we have that

$$\|x_1 + x_2\| \leq s(\|x_1\| + \|x_2\|).$$

X is called a b-Banach space if X is complete for this norm. The best-known examples of b-Banach spaces are the spaces ℓ_p and $L_p(0,1)$, when $0 < p < 1$. It is readily seen that $\ell_p^* = \ell_\infty$ but $L_p(0,1)^* = \{0\}$. This latter result is due to Day [12] in what is arguably the first paper on b-Banach spaces. For more details we invite the reader to look at Day [12] and Kalton [16].

We recall that a functional $f : X \to \mathbb{R} \cup \{\infty\}$ is called Gâteaux differentiable if at every point x with $f(x) < \infty$, there exists a continuous linear functional $f'(x_0)$ such that for every $y \in X$,

$$\lim_{t \to 0} \frac{f(x_0 + ty) - f(x_0)}{t} = \langle f'(x_0); y \rangle.$$

In the following, we assume that $g : [0, +\infty) \to [0, +\infty)$ is a continuous non-decreasing function, the mapping $\frac{g(x)}{x}$ is decreasing, and $f : X \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous, Gâteaux differentiable function and not identically with $+\infty$. In this case we can extend Theorem 2.1 from Zhong [21]. More precisely, if f is bounded from below, then for $\varepsilon > 0$, every $y \in X$ such that

$$f(y) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{24}$$

and every $\lambda > 0$, there exists a $z \in X$ such that

$$f(z) \leq f(y),$$

$$\|f'(z)\| \leq s \cdot \frac{\varepsilon}{\lambda(1 + g(\|z\|))}.$$

From the above, one can extend the notion of weak (PS)-condition form [21], and we can prove that a function, which is bounded below, and satisfies the weak (PS)-condition has a minimal point.

5. Extension of Theorem 1.1 for bifunctions

As above, we introduce some new symbols and notations which we will also use in the rest of the paper.
Let C be a closed subset of the complete b-metric space X (suppose that d is continuous), $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function and $\rho : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ be a function which satisfies the assumption (A).

We assume the following assumption:

$(\mathcal{F})$ For the mapping $f : C\times C \to \mathbb{R}$ the following assertions hold:

- (i) $f(x,\cdot)$ is lower bounded and lower semicontinuous, for every $x\in C$;
- (ii) $f(z,z) = 0$, for every $z\in C$;
- (iii) $f(z,x) \le f(z,y) + f(y,x)$ for every $x,y,z\in C$.

We will also use the following notation:

$$\mathcal{Q}[y,x;m] = f(y,x) + h(d(x_0,x))\sum_{n=0}^{m}\delta_n\rho(x,x_n).$$

Theorem 5.1. *Let us consider the mapping $f : C\times C\to\mathbb{R}$ which satisfies the assumption $(\mathcal{F})$. Then, for every $x_0\in X$ and $\varepsilon > 0$ there exists a sequence $\{x_n\}\subset C$ which converges to some x_ε ($x_n\to x_\varepsilon$ as $n\to\infty$) such that*

$$h(d(x_0,x_\varepsilon))\rho(x_\varepsilon,x_n) \le \frac{\varepsilon}{2^n\delta_0},\ n\in\mathbb{N}, \tag{25}$$

and

$$\mathcal{Q}[x_0,x_\varepsilon;0] \le 0, \tag{26}$$

and

$$\forall x\neq x_\varepsilon,\ \mathcal{Q}[x_\varepsilon,x;+\infty] - h(d(x_0,x_\varepsilon))\sum_{n=0}^{\infty}\delta_n\rho(x_\varepsilon,x_n) > 0. \tag{27}$$

Proof. We use the same argument as in Theorem 1.1. Let us define the following set:

$$\mathcal{W}(x_0) = \{y\in C|\mathcal{Q}[x_0,y;0]\le 0\}.$$

Obviously, $\mathcal{W}(x_0)$ is a nonempty set, and by $(\mathcal{F})(i)$, we obtain that it is also a closed subset of C. Now we can choose an element $x_1\in\mathcal{W}(x_0)$ such that the following inequality holds:

$$h(d(x_0,x_1))\delta_0\rho(x_1,x_0) + f(x_0,x_1) \le \inf_{z\in\mathcal{W}(x_0)}\mathcal{Q}[x_0,y;0] + \frac{\delta_1\cdot\varepsilon}{2\delta_0}. \tag{28}$$

Let us consider the set

$$\mathcal{W}(x_1) = \{y\in\mathcal{W}(x_0)|\mathcal{Q}[x_1,y;1]\le h(d(x_0,x_1))\delta_0\rho(x_1,x_0)\}.$$

Due to $A(a)$, $(\mathcal{F})(i)$ and $(\mathcal{F})(ii)$, it is easy to see that $\mathcal{W}(x_0)\supset\mathcal{W}(x_1)$, and $\mathcal{W}(x_1)$ is an nonempty closed set. In general, let us suppose that we have defined $x_{n-1}\in\mathcal{W}(x_{n-2})$ in the following way

$$\mathcal{W}(x_{n-1}) = \left\{y\in\mathcal{W}(x_{n-2})|\mathcal{Q}[x_{n-1},y;n-1]\le h(d(x_0,x_{n-1}))\sum_{i=0}^{n-2}\delta_{n-1}\rho(x_{n-1},x_i)\right\}.$$

We readily see that $\mathcal{W}(x_{n-1})$ is a nonempty closed subset of $\mathcal{W}(x_{n-2})$. We can take $x_n \in \mathcal{W}(x_{n-1})$ such that

$$\mathcal{Q}[x_{n-1},x_n;n-1] \leq \inf_{z\in\mathcal{W}(x_{n-1})} \{\mathcal{Q}[x_{n-1},z;n-1]\} + \frac{\delta_n \cdot \varepsilon}{2^n \delta_0}. \tag{29}$$

Similarly to the above, we can define $\mathcal{W}(x_n)$

$$\mathcal{W}(x_n) = \left\{ y \in \mathcal{W}(x_{n-1}) | \mathcal{Q}[x_n,y;n] \leq h(d(x_0,x_n)) \sum_{i=0}^{n-1} \delta_n \rho(x_n,x_i) \right\}, \tag{30}$$

which is a nonempty, closed subset of $\mathcal{W}(x_{n-1})$. We can prove that, in the same way as in the proof of Theorem 1.1, the below inequality is true:

$$h(d(x_0,z))\rho(z,x_n) \leq \frac{\varepsilon}{2^n \delta_0} \to 0 \text{ as } n \to \infty. \tag{31}$$

Therefore $\rho(z,x_n) \to 0$ as $n \to \infty$. At the same time, by $A(b)$, we have $d(z,x_n) \to 0$ as $n \to \infty$. Then, if we consider the points $z_1,z_2 \in \mathcal{W}(x_n)$, it follows that $d(z_1,z_2) \leq s(d(z_1,x_n)+d(z_2,x_n))$. This means that $\operatorname{diam}(\mathcal{W}(\mathrm{x_n})) \to 0$ as $n \to \infty$. Applying Cantor's intersection theorem, we deduce that

$$\bigcap_n \mathcal{W}(x_n) = \{x_\varepsilon\}. \tag{32}$$

From the above relation we can conclude that $x_\varepsilon \in \mathcal{W}(x_0)$ which means that

$$f(x_0,x_\varepsilon) + \delta_0 h(d(x_0,x_\varepsilon))\rho(x_\varepsilon,x_0) < 0.$$

So we have that $\mathcal{W}(x_\varepsilon) \subset \mathcal{W}(x_n)$ for all $n \in \mathbb{N}$. By 32, we obtain that

$$\mathcal{W}(x_\varepsilon) = \{x_\varepsilon\},$$

therefore if $x \neq x_\varepsilon$ then it is clear that $x \notin \mathcal{W}(x_\varepsilon)$, i.e.:

$$\mathcal{Q}[x_\varepsilon,x;+\infty] > h(d(x_0,x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon,x_n). \qquad \square$$

One of the particular cases of Theorem 5.1 is the following version ($h \equiv 1$, $\delta_n = \dfrac{1}{s^n}$ and $\rho = d$):

Theorem 5.2. *Let (X,d) be a complete b-metric space with $s > 1$ (d is continuous), $C \subset X$ be a closed set and $f : C \times C \to \mathbb{R}$ be a function which satisfies* *the assertion* $(\mathcal{F})$. *Then for every* $x_0 \in X$ *and* $\varepsilon > 0$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset C$ *and an element* $x_\varepsilon \in C$ *such that* $x_n \to x_\varepsilon$*, as* $n \to \infty$ *and*

$$d(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n}, \quad n \in \mathbb{N}, \tag{33}$$

$$f(x_0, x_\varepsilon) + d(x_\varepsilon, x_0) \leq 0, \tag{34}$$

and for every $x \neq x_\varepsilon$*, we have*

$$f(x_\varepsilon, x) + \sum_{i=0}^{\infty} \frac{1}{s^i} d(x, x_i) - \sum_{i=0}^{\infty} \frac{1}{s^i} d(x_\varepsilon, x_i) > 0. \tag{35}$$

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