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FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS OF COMPLEX ORDER

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In this paper, we obtain Fekete-Szegő inequalities for a certain class of meromorphic functions $f(z)$ for which $1 + \frac{1}{b} \left[\frac{z(f * g)'(z)}{(\lambda - 1)(f * g)(z) + \lambda z(f * g)'(z)} - 1 \right] \prec \varphi(z)$ ($b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \leq \lambda < 1$). Sharp bounds for the Fekete-Szegő functional $|a_1 - \mu a_0^2|$ are obtained.

1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. Let $g(z) \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} g_k z^k, \quad (1.2)$$

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then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k g_k z^k = (g * f)(z). \quad (1.3)$$

A function $f \in \Sigma$ is meromorphic starlike of order α , denoted by $\Sigma^*(\alpha)$, if

$$-\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U^*). \quad (1.4)$$

The class $\Sigma^*(\alpha)$ was introduced and studied by Pommerenke [22], Miller [18], Mogra et al. [19], Cho [8], Cho et al. [9] and Aouf ([4] and [5]).

Let $\varphi(z)$ be an analytic function with positive real part on U satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\varphi)$ be the class of functions $f \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in U^*). \quad (1.5)$$

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [26]. The class $\Sigma^*(\alpha)$ is the special case of $\Sigma^*(\varphi)$ when $\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$ ($0 \leq \alpha < 1$).

For $0 \leq \lambda < 1$ and $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we let $\Sigma_{\lambda,b}^*(g, \varphi)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1), the functions $g(z)$ of the form (1.2) with $g_k > 0$ and satisfying the analytic criterion:

$$1 + \frac{1}{b} \left[\frac{z(f * g)'(z)}{(\lambda - 1)(f * g)(z) + \lambda z(f * g)'(z)} - 1 \right] \prec \varphi(z). \quad (1.6)$$

We note that for suitable choices of $g(z), \lambda, b$ and $\varphi(z)$, we obtain the following subclasses:

- (1) $\Sigma_{0,1}^*(g, \varphi) = M_g^s(\varphi)$ (see Shanmugam and Jeyaraman [25]);
- (2) $\Sigma_{0,1}^* \left(\frac{1}{z(1-z)}, \varphi(z) \right) = \Sigma^*(\varphi)$ (see Silverman et al. [26] and Ali and Ravichandran [1, with $\alpha = 0$]);
- (3) $\Sigma_{0,b}^* \left(\frac{1}{z(1-z)}, \frac{1+z}{1-z} \right) = F^*(b)$ (see Aouf [2]);
- (4) $\Sigma_{0,b}^* \left(\frac{1}{z(1-z)}, \varphi(z) \right) = M_b^*(\varphi)$ (see Mohammed and Darus [20] and Reddy and Sharma [24, with $\gamma = 1$]);
- (5) $\Sigma_{0,1}^* \left(\frac{1}{z(1-z)}, \frac{1 + (1 - 2\alpha)z}{1 - z} \right) = \Sigma^*(\alpha)$ ($0 \leq \alpha < 1$) (see Pommerenke [22]);

(6) $\Sigma_{0,1}^* \left(\frac{1}{z(1-z)}, \frac{1+\beta(1-2\alpha\gamma)z}{1+\beta(1-2\gamma)z} \right) = \Sigma(\alpha, \beta, \gamma)$ ($0 \leq \alpha < 1, 0 < \beta \leq 1, \frac{1}{2} \leq \gamma \leq 1$) (see Kulkarni and Joshi [15]);

(7) $\Sigma_{0,1}^* \left(\frac{1}{z(1-z)}, \frac{1+Az}{1+Bz} \right) = K_1(A, B)$ ($0 \leq B < 1, -B < A < B$) (see Karunakaran [14]);

(8) $\Sigma_{0,(1-\rho)e^{-i\alpha} \cos \alpha}^* \left(\frac{1}{z(1-z)}, \frac{1+z}{1-z} \right) = \Sigma^\alpha(\rho)$ ($|\alpha| < \frac{\pi}{2}; 0 \leq \rho < 1$) (see Kaczmariski [13], Aouf [3] and Ravichandran et al. [23]).

Also, we note that:

(1) $\Sigma_{0,b}^*(g, \varphi) = \Sigma_b^*(g, \varphi)$

$$= \left\{ f(z) \in \Sigma : 1 - \frac{1}{b} \left[\frac{z(f * g)'(z)}{(f * g)(z)} + 1 \right] \prec \varphi(z) \ (z \in U^*) \right\};$$

(2) $\Sigma_{\lambda,b}^* \left(\frac{1}{z(1-z)}, \varphi(z) \right) = \Sigma_{\lambda,b}^*(\varphi)$

$$= \left\{ f(z) \in \Sigma : 1 + \frac{1}{b} \left[\frac{zf'(z)}{(\lambda-1)f(z) + \lambda zf'(z)} - 1 \right] \prec \varphi(z) \ (z \in U^*) \right\};$$

(3) $\Sigma_{\lambda,(1-\rho)e^{-i\alpha} \cos \alpha}^* \left(\frac{1}{z(1-z)}, \varphi(z) \right) = \Sigma_{\lambda,\rho}^*\alpha(\varphi)$

$$= \left\{ f(z) \in \Sigma : \frac{e^{i\alpha} \left[\frac{zf'(z)}{(\lambda-1)f(z) + \lambda zf'(z)} \right]^{-\rho \cos \alpha - i \sin \alpha}}{(1-\rho) \cos \alpha} \prec \varphi(z) \right. \\ \left. (|\alpha| < \frac{\pi}{2}; 0 \leq \rho < 1; z \in U^*) \right\};$$

(4) $\Sigma_{0,(1-\rho)e^{-i\alpha} \cos \alpha}^* \left(\frac{1}{z(1-z)}, \varphi(z) \right) = \Sigma_\rho^*\alpha(\varphi)$

$$= \left\{ f(z) \in \Sigma : \frac{e^{i\alpha} \left[-\frac{zf'(z)}{f(z)} \right]^{-\rho \cos \alpha - i \sin \alpha}}{(1-\rho) \cos \alpha} \prec \varphi(z) \right. \\ \left. (|\alpha| < \frac{\pi}{2}; 0 \leq \rho < 1; z \in U^*) \right\};$$

(5) $\Sigma_{\lambda,b}^* \left(\frac{1}{z} + \sum_{k=0}^{\infty} \left[\frac{\ell + \gamma(k+1)}{\ell} \right]^m z^k, \varphi(z) \right) = \Sigma_{\lambda,b,\gamma,\ell}^{*m}(\varphi)$

$$= \left\{ f(z) \in \Sigma : 1 + \frac{1}{b} \left[\frac{z(I^m(\gamma,\ell)f(z))'}{(\lambda-1)(I^m(\gamma,\ell)f(z)) + \lambda z(I^m(\gamma,\ell)f(z))'} - 1 \right] \prec \varphi(z) \right. \\ \left. (\gamma \geq 0; \ell > 0; m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}) \right\};$$

where the operator

$$I^m(\gamma, \ell)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left[\frac{\ell + \gamma(k+1)}{\ell} \right]^m z^k, \tag{1.7}$$

was introduced and studied by El-Ashwah [10, with $p = 1$] (see also Bulboaca et al. [7, with $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1, 2, \dots\}$], El-Ashwah [11, with $p = 1$ and $m \in \mathbb{N}_0$] and El-Ashwah et al. [12, with $p = 1$ and $m \in \mathbb{N}_0$]);

$$\begin{aligned} (6) \Sigma_{\lambda, b}^* \left(\frac{1}{z} + \sum_{k=0}^{\infty} \Gamma_{k+1}(\alpha_1) z^k, \varphi(z) \right) &= \Sigma_{\alpha, b, q, s}^* \alpha_1(\varphi) \\ &= \left\{ f(z) \in \Sigma : 1 + \frac{1}{b} \left[\frac{z(H_{q,s}(\alpha_1)f(z))'}{(\lambda-1)(H_{q,s}(\alpha_1)f(z)) + \lambda z(H_{q,s}(\alpha_1)f(z))'} - 1 \right] \prec \varphi(z) \right. \\ &\quad \left. (m \in \mathbb{N}_0; q \leq s + 1; q, s \in \mathbb{N}_0) \right\}; \end{aligned}$$

where the operator

$$H_{q,s}(\alpha_1)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \Gamma_{k+1}(\alpha_1) z^k, \tag{1.8}$$

$$\Gamma_{k+1}(\alpha_1) = \frac{(\alpha_1)_{k+1} \dots (\alpha_q)_{k+1}}{(\beta_1)_{k+1} \dots (\beta_s)_{k+1}} \frac{1}{(k+1)!}, \tag{1.9}$$

for $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_s$ are real parameters and $\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$, $j = 1, 2, \dots, s$ was introduced and investigated by Liu and Srivastava [16, with $p = 1$] and Aouf [6, with $p = 1$].

In this paper, we obtain the Fekete-Szegő inequalities for meromorphic functions in the class $\Sigma_{\lambda, b}^*(g, \varphi)$.

2. Fekete-Szegő problem

To prove our results, we need the following lemmas.

Lemma 2.1 ([17]). *If $p(z) = 1 + c_1z + c_2z^2 + \dots$ is a function with positive real part in U and μ is a complex number, then*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Lemma 2.2 ([17]). *If $p_1(z) = 1 + c_1z + c_2z^2 + \dots$ is a function with positive real part in U , then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0, \\ 2 & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if } \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right) \frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1),$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if

$$\frac{1}{p_1(z)} = \left(\frac{1}{2} + \frac{1}{2}\gamma\right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right) \frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1),$$

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \quad (0 < \nu \leq \frac{1}{2}),$$

and

$$|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \quad (\frac{1}{2} < \nu < 1).$$

Unless otherwise mentioned, we assume throughout this paper that

$$0 \leq \lambda < 1, b \in \mathbb{C}^* \text{ and the function } g(z) \text{ is given by (1.2) with } g_k > 0 \text{ (} k \geq 0 \text{)}.$$

Theorem 2.3. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_{\lambda,b}^*(g, \varphi)$ and μ is a complex number. Then*

$$(i) \quad |a_1 - \mu a_0^2| \leq \frac{|B_1 b|}{2(1-\lambda)g_1} \max \left\{ 1, \left| \frac{B_2}{B_1} - \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1 b \right| \right\}, \quad B_1 \neq 0, \tag{2.1}$$

$$(ii) \quad |a_1| \leq \frac{|B_2 b|}{2(1-\lambda)g_1}, \quad B_1 = 0. \tag{2.2}$$

The result is sharp.

Proof. If $f(z) \in \Sigma_{\lambda, b}^*(g, \varphi)$, then there is a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b} \left[\frac{z(f * g)'(z)}{(\lambda - 1)(f * g)(z) + \lambda z(f * g)'(z)} - 1 \right] = \varphi(w(z)). \tag{2.3}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots \tag{2.4}$$

Since $w(z)$ is a Schwarz function, we see that $Re\ p_1(z) > 0$ and $p_1(0) = 1$. Define

$$p(z) = 1 + \frac{1}{b} \left[\frac{z(f * g)'(z)}{(\lambda - 1)(f * g)(z) + \lambda z(f * g)'(z)} - 1 \right] = 1 + b_1z + b_2z^2 + \dots \tag{2.5}$$

In view of (2.3), (2.4) and (2.5), we have

$$p(z) = \varphi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right). \tag{2.6}$$

Since

$$\frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[c_1z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1c_2 \right) z^3 + \dots \right].$$

Therefore, we have

$$\varphi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \dots,$$

and from this equation and (2.6), we obtain

$$b_1 = \frac{1}{2} B_1 c_1,$$

and

$$b_2 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2.$$

Then, from (2.5) and (1.1), we see that

$$bb_1 = -(1 - \lambda) a_0 g_0, \tag{2.7}$$

and

$$bb_2 = (1 - \lambda)^2 a_0^2 g_0^2 - 2(1 - \lambda) a_1 g_1, \tag{2.8}$$

or, equivalently, we have

$$a_0 = \frac{-B_1 b c_1}{2(1 - \lambda) g_0},$$

and

$$a_1 = \frac{-B_1 b c_2}{4(1 - \lambda) g_1} + \frac{b c_1^2}{8(1 - \lambda) g_1} [B_1 - B_2 + B_1^2 b].$$

Therefore

$$a_1 - \mu a_0^2 = \frac{-B_1 b}{4(1-\lambda)g_1} \{c_2 - \nu c_1^2\}, \tag{2.9}$$

where

$$\nu = \frac{1}{2} \left[1 - \frac{B_2}{B_1} + B_1 b - 2\mu \frac{B_1 b g_1}{(1-\lambda)g_0^2} \right]. \tag{2.10}$$

Now, the result (2.1) follows by an application of Lemma 2.1. Also, if $B_1 = 0$, then

$$a_0 = 0 \text{ and } a_1 = \frac{-B_2 b c_1^2}{8(1-\lambda)g_1}.$$

Since $p(z)$ has positive real part, $|c_1| \leq 2$ (see Nehari [21]), so that

$$|a_1| \leq \frac{|B_2 b|}{2(1-\lambda)g_1},$$

this proving (2.2). The result is sharp for the functions

$$1 + \frac{1}{b} \left[\frac{z(f*g)'(z)}{(\lambda-1)(f*g)(z) + \lambda z(f*g)'(z)} - 1 \right] = \varphi(z^2), \tag{2.11}$$

and

$$1 + \frac{1}{b} \left[\frac{z(f*g)'(z)}{(\lambda-1)(f*g)(z) + \lambda z(f*g)'(z)} - 1 \right] = \varphi(z). \tag{2.12}$$

This completes the proof of Theorem 2.3. □

Remark 2.4. (1) Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$, $\lambda = 0$ and $b = 1$ in Theorem 2.3, we obtain the result obtained by Silverman et al. [26, Theorem 2.1];

(2) Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \delta(n,k) z^k$, where $\delta(n,k) = \binom{n+k+1}{k+1}$ ($n > -1$), $\lambda = 0$ and $b = 1$ in Theorem 2.3, we obtain the result obtained by Silverman et al. [26, Theorem 3.3];

(3) Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$ and $\lambda = 0$ in Theorem 2.3, we obtain the result obtained by Mohammed and Darus [20, Theorem 1.1];

(4) Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$ and $B_1 \geq 0$ in Theorem 2.3, we obtain the result obtained by Reddy and Sharma [24, Theorem 2.1 with $\gamma = 1$];

(5) Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$, $b = (1-\rho)e^{-i\alpha} \cos \alpha$ ($|\alpha| < \frac{\pi}{2}$, $0 \leq \rho < 1$), $\lambda = 0$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 2.3, we obtain the result obtained by Mohammed and Darus [20, Example 1.1].

Putting $\lambda = 0$ and $b = 1$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.5. *If $f(z)$ given by (1.1) belongs to the class $M_g^s(\varphi)$ and μ is a complex number, then*

$$(i) |a_1 - \mu a_0^2| \leq \frac{|B_1|}{2g_1} \max \left\{ 1, \left| \frac{B_2}{B_1} - \left[1 - \frac{2\mu g_1}{g_0^2} \right] B_1 \right| \right\}, B_1 \neq 0,$$

$$(ii) |a_1| \leq \frac{|B_2|}{2g_1}, B_1 = 0.$$

Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$, $\lambda = 0$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.6. *If $f(z)$ given by (1.1) belongs to the class $F^*(b)$ and μ is a complex number, then*

$$|a_1 - \mu a_0^2| \leq |b| \max \{1, |1 - 2(1 - 2\mu)b|\}.$$

The result is sharp.

Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$, $\lambda = 0$, $b = 1$ and $\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}$ ($0 \leq \alpha < 1$) in Theorem 2.3, we obtain the following corollary.

Corollary 2.7. *If $f(z)$ given by (1.1) belongs to the class $\Sigma^*(\alpha)$ and μ is a complex number, then*

$$|a_1 - \mu a_0^2| \leq (1 - \alpha) \max \{1, |1 - 2(1 - \alpha)(1 - 2\mu)|\}.$$

The result is sharp.

Putting $\lambda = 0$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.8. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_b^*(g, \varphi)$ and μ is a complex number, then*

$$(i) |a_1 - \mu a_0^2| \leq \frac{|B_1 b|}{2g_1} \max \left\{ 1, \left| \frac{B_2}{B_1} - \left[1 - 2\mu \frac{g_1}{g_0^2} \right] B_1 b \right| \right\}, B_1 \neq 0,$$

$$(ii) |a_1| \leq \frac{|B_2 b|}{2g_1}, B_1 = 0.$$

The result is sharp.

Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.9. Let $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_{\lambda,b}^*(\varphi)$ and μ is a complex number, then

$$(i) \quad |a_1 - \mu a_0^2| \leq \frac{|B_1b|}{2(1-\lambda)} \max \left\{ 1, \left| \frac{B_2}{B_1} - \left[1 - \frac{2\mu}{(1-\lambda)} \right] B_1b \right| \right\}, \quad B_1 \neq 0,$$

$$(ii) \quad |a_1| \leq \frac{|B_2b|}{2(1-\lambda)}, \quad B_1 = 0.$$

The result is sharp.

Remark 2.10. (1) Putting $\lambda = 0$ and $b = (1 - \rho)e^{-i\alpha} \cos \alpha$ ($|\alpha| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Corollary 2.9, we obtain a new result for the class $\Sigma_\rho^{*\alpha}(\varphi)$;

(2) Putting $b = (1 - \rho)e^{-i\alpha} \cos \alpha$ ($|\alpha| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Corollary 2.9, we obtain a new result for the class $\Sigma_{\lambda,\rho}^{*\alpha}(\varphi)$.

By using Lemma 2.2, we can obtain the following theorem.

Theorem 2.11. Let $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$ ($B_i > 0$, $i \in \mathbb{N}$, $b > 0$). If $f(z)$ given by (1.1) belongs to the class $\Sigma_{\lambda,b}^*(g, \varphi)$ and μ is a real number, then

$$|a_1 - \mu a_0^2| \leq \begin{cases} \frac{b}{2(1-\lambda)g_1} \left\{ -B_2 + \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1^2b \right\} & \text{if } \mu \leq \sigma_1, \\ \frac{B_1b}{2(1-\lambda)g_1} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{b}{2(1-\lambda)g_1} \left\{ B_2 - \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1^2b \right\} & \text{if } \mu \geq \sigma_2, \end{cases} \quad (2.13)$$

where

$$\sigma_1 = \frac{\{-(B_2 + B_1) + B_1^2b\}(1-\lambda)g_0^2}{2g_1B_1^2b}$$

and

$$\sigma_2 = \frac{\{-(B_2 - B_1) + B_1^2b\}(1-\lambda)g_0^2}{2g_1B_1^2b}$$

The result is sharp.

Proof. First, let $\mu \leq \sigma_1$. Then

$$\begin{aligned} |a_1 - \mu a_0^2| &\leq \frac{B_1b}{2(1-\lambda)g_1} \left\{ -\frac{B_2}{B_1} + \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1b \right\} \\ &\leq \frac{b}{2(1-\lambda)g_1} \left\{ -B_2 + \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1^2b \right\}. \end{aligned}$$

Let, now $\sigma_1 \leq \mu \leq \sigma_2$. Then, using the above calculations, we obtain

$$|a_1 - \mu a_0^2| \leq \frac{B_1 b}{2(1-\lambda)g_1}.$$

Finally, if $\mu \geq \sigma_2$, then

$$\begin{aligned} |a_1 - \mu a_0^2| &\leq \frac{B_1 b}{2(1-\lambda)g_1} \left\{ \frac{B_2}{B_1} - \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1 b \right\} \\ &\leq \frac{b}{2(1-\lambda)g_1} \left\{ B_2 - \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1^2 b \right\}. \end{aligned}$$

To show that the bounds are sharp, we define the functions $K_{\varphi n} (n \geq 2)$ by

$$1 + \frac{1}{b} \left[\frac{z(K_{\varphi n} * g)'(z)}{(\lambda-1)(K_{\varphi n} * g)(z) + \lambda z(K_{\varphi n} * g)'(z)} - 1 \right] = \varphi(z^{n-1}), K_{\varphi n}(0) = 0 = K'_{\varphi n}(0) - 1,$$

and the functions F_γ and G_γ ($0 \leq \gamma \leq 1$) by

$$1 + \frac{1}{b} \left[\frac{z(F_\gamma * g)'(z)}{(\lambda-1)(F_\gamma * g)(z) + \lambda z(F_\gamma * g)'(z)} - 1 \right] = \varphi\left(\frac{z(z+\gamma)}{1+\gamma z}\right), F_\gamma(0) = 0 = F'_\gamma(0) - 1,$$

and

$$1 + \frac{1}{b} \left[\frac{z(G_\gamma * g)'(z)}{(\lambda-1)(G_\gamma * g)(z) + \lambda z(G_\gamma * g)'(z)} - 1 \right] = \varphi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right), G_\gamma(0) = 0 = G'_\gamma(0) - 1.$$

Clearly the functions $K_{\varphi n}, F_\gamma$ and $G_\gamma \in \Sigma_{\lambda, b}^*(g, \varphi)$. Also we write $K_\varphi = K_{\varphi 2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_γ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_γ or one of its rotations. This completes the proof of Theorem 2.11. □

Remark 2.12. (1) Putting $g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k$, $\lambda = 0$ and $b = 1$ in Theorem 2.11, we obtain the result obtained by Ali and Ravichandran [1, Theorem 5.1]; (2) For different choices of $g(z)$, λ , b and $\varphi(z)$ in Theorem 2.11, we will obtain new results for different classes mentioned in the introduction.

Using arguments similar to those in the proof of Theorem 2.11, we obtain the following theorem.

Theorem 2.13. Let $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$ ($B_i > 0$, $i \in \mathbb{N}$, $b > 0$) and

$$\sigma_3 = \frac{(-B_2 + B_1^2b)(1-\lambda)g_0^2}{2g_1B_1^2b}. \quad (2.14)$$

If $f(z)$ given by (1.1) belongs to the class $\Sigma_{\lambda,b}^*(g, \varphi)$ and μ is a real number, then we have

(i) If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_1 - \mu a_0^2| + \frac{(1-\lambda)g_0^2}{2g_1B_1^2b} \left\{ (B_1 + B_2) + \left[2\mu \frac{g_1}{(1-\lambda)g_0^2} - 1 \right] B_1^2b \right\} |a_0|^2 \leq \frac{B_1b}{2(1-\lambda)g_1}. \quad (2.15)$$

(ii) If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_1 - \mu a_0^2| + \frac{(1-\lambda)g_0^2}{2g_1B_1^2b} \left\{ (B_1 - B_2) + \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2} \right] B_1^2b \right\} |a_0|^2 \leq \frac{B_1b}{2(1-\lambda)g_1}, \quad (2.16)$$

where σ_1 and σ_2 are given in Theorem 2.11.

Remark 2.14. By specializing the function $g(z)$ in Theorems 2.3, 2.11 and 2.13, we will obtain new results for different classes mentioned in the introduction.

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