LE MATEMATICHE
Vol. LXVIII (2013) – Fasc. II, pp. 123–130
doi: 10.4418/2013.68.2.9

AN ALTERNATIVE METHOD FOR SOLVING GENERALIZED FRACTIONAL KINETIC EQUATIONS INVOLVING THE GENERALIZED FUNCTIONS FOR THE FRACTIONAL CALCULUS

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The paper is devoted to present an alternative method for deriving the solution of the generalized fractional kinetic equations in terms of K_4-function and generalized M-series. The applied method depends on the fractional differintegral operator technique and the method is different from Laplace transform. The obtained results believed to be new.

1. Introduction

Fractional kinetic equations have gained importance during the last decade due to their occurrence in certain problems in science and engineering. A spherically symmetric non-rotating, self-gravitating model of star like the Sun is assumed to be in thermal equilibrium and hydrostatic equilibrium. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of above characters and some additional information related to the equation of state, nuclear energy generation rate and the opacity.

Entrato in redazione: 19 ottobre 2012

AMS 2010 Subject Classification: 33E12, 33E99.
Keywords: Generalized fractional kinetic equation, K_4-function and generalized M-series.

The generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions is studied by Saxena, Mathai and Haubold [10], which extended the work of Haubold and Mathai [3]. In an another paper Saxena, Mathai and Haubold [11] developed the solutions for fractional kinetic equations associated with the generalized Mittag-Leffler function and R-function.

The fractional kinetic equations are also studied by many authors notably Saichev and Zaslavsky [1], Sharma[5], Saxena et al.[10,11,12], Zaslavsky[2], Saxena and Kalla[14], Chaurasia and Pandey[17,18], Chaurasia and Kumar[16] etc. for their importance in the solution of certain physical problems. Recently, Saxena et al. [15] investigated the solutions of the fractional reaction equation and the fractional diffusion equation.

Generalized M-series is an extension of both Mittag-Leffler function and generalized hypergeometric function pFq and K_4-function is an extension of generalized M-series and G-function. These functions have important role in fractional calculus, physics, biology, engineering and applied sciences, theory of differentiation of arbitrary order and in the solutions of fractional order differential equations.

2. Mathematical Preliminaries

The K_4-function introduced by Sharma [4], is defined by the power series

$$\begin{aligned} K_4^{(\alpha,\beta,\gamma),(a,c):(p;q)}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) &= K_4^{(\alpha,\beta,\gamma),(a,c):(p;q)}(x) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n(\gamma)_n a^n (x-c)^{(n+\gamma)\alpha-\beta-1}}{(b_1)_n\ldots(b_q)_n n!\,\Gamma((n+\gamma)\alpha-\beta)} \end{aligned} \tag{1}$$

where $\alpha,\beta,\gamma,x \in c$ and $R(\alpha\gamma-\beta)>0$ and $(a_i)_n$ $(i=1,2,\ldots,p)$ and $(b_j)_n$ $(j=1,2,\ldots,q)$ are the Pochhammer symbols and none of the parameters b_js is a negative integer or zero.

The generalized M-series is introduced by Sharma and Jain [8], defined as

$${}_pM_q^{\alpha,\beta} = {}_pM_q^{\alpha,\beta}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n} \frac{x^n}{\Gamma(\alpha n+\beta)} \tag{2}$$

Here $x,\alpha,\beta \in C$, $Re(\alpha)>0$ and $(a_j)_n$, $(b_j)_n$ are known Pochhammer symbols. The series (2) is defined when non of the parameters $(b_j)_S$, $j=1,\cdots,q$ is a negative integer or zero. If any numerator parameter a_j is a negative or zero, then the series terminates to a polynomial in it.

3. Relationship between K_4-function and generalized M-series

Setting $\beta = \alpha - \beta,\ \gamma = 1, a = 1$ and $c = 0$ in (1), we obtain

$$K_4^{(\alpha,\alpha-\beta,1),(1,0):(p;q)}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = x^{\beta-1}\,{}_p\overset{\alpha,\beta}{M}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \tag{3}$$

The Riemann-Liouville operators of fractional calculus are defined as [6,7]

$${}_aD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)}\int_a^t (t-u)^{\nu-1} f(u)du,\ R(\nu) > 0, t > a \tag{4}$$

with ${}_aD_t^0 f(t) = f(t)$, and

$${}_aD_t^{\mu} f(t) = \frac{d^n}{dt^n}({}_aD_t^{\mu-n} f(t)), Re(\mu)\ >\ 0,\ n-\mu > 0. \tag{5}$$

If $f(t) = (t-a)^{\rho}$, we have from [9]

$${}_aD_t^{-\nu}(t-a)^{\rho-1} = \frac{\gamma(\rho)}{\Gamma(\rho+\nu)}(t-a)^{\rho+\nu-1}, \tag{6}$$

where $Re(\nu)\ >\ 0,\ Re(\rho)\ >\ 0,\ t\ >\ a$. Also from [9], we have

$${}_aD_t^{\nu}(t-a)^{\rho-1} = \frac{\gamma(\rho)}{\Gamma(\rho-\nu)}(t-a)^{\rho-\nu-1},\ \text{where}\ Re(\nu) > 0, Re(\rho) > 0;\ t > a. \tag{7}$$

when $\rho = 1$ (7) reduces to

$${}_aD_t^{\nu} 1 = \frac{1}{\Gamma(1-\nu)}(t-a)^{-\nu}, t > a; \nu \neq 1, 2\ldots \tag{8}$$

Fractional derivative of 1 is not zero in the Riemann-Liouville sense.

4. Solution of standard kinetic equation in terms of generalized M-series and Solution of generalized fractional kinetic equations

This section deals with the solution of standard fractional kinetic equation in terms of generalized M-series and derive the solution of generalized fractional kinetic equations in terms of generalized M-series and K_4-function.
On integrating the standard kinetic equation

$$\frac{d}{dt}N_i(t) = -c_i N_i(t),\ c_i > 0, \tag{9}$$

Haubold and Mathai [3] derived that

$$N_i(t) - N_i(0) = -c_i\, {}_0D_t^{-1} N_i(t), \tag{10}$$

where ${}_0D_t^{-1}$ is the standard Riemann integral operator, $N_i = N_i(t)$ is the number density of species i, which is a function of time t and $N_i(0) = N_0$ is the number density of that species at time $t = 0$. By dropping the index i and replacing the Riemann integral ${}_0D_t^{-1}$ operator by the fractional Riemann-Liouville operator ${}_0D_t^{-\nu}$ the kinetic equation (10) reduces to

$$N(t) - N_0 = -c^\nu\, {}_0D_t^{-\nu} N(t) \tag{11}$$

Multiplying both side of equation (11) by the operator $(-c^\nu)^m\, {}_0D_t^{-m\nu}$ and taking the sum over m from 0 to ∞, yields

$$\sum_{m=0}^{\infty} (-c^\nu)^m {}_0D_t^{-m\nu} N(t) - \sum_{m=0}^{\infty} (-c^\nu)^{m+1} {}_0D_t^{-(m+1)\nu} N(t) = N_0 \sum_{m=0}^{\infty} (-c^\nu)^m {}_0D_t^{-m\nu} 1$$

replacing m by $(m-1)$ in the second sum of above equation and then cancelling the equal terms on the left hand side and applying the relationship (8) on the right hand side of above equation, we obtain

$$N(t) = N_0 \sum_{m=0}^{\infty} (-c^\nu)^m \frac{t^{m\nu}}{\Gamma(m\nu+1)}$$

$$N(t) = N_0 \sum_{m=0}^{\infty} \frac{(-c^\nu t^\nu)^m}{\Gamma(m\nu+1)}$$

$$N(t) = N_0\ {}_0\overset{\nu,1}{M}_0(-;-;-c^\nu t^\nu)$$

Theorem 4.1. *If $\nu > 0$, $\mu > 0$, then for the solution of generalized fractional kinetic equation*

$$N(t) - N_0 t^{\mu-1} = -c^\nu\, {}_0D_t^{-\nu} N(t) \tag{12}$$

there holds the formula

$$N(t) = N_0 \Gamma \mu t^{\mu-1}\, {}_0\overset{\nu,\mu}{M}_0(-;-;-c^\nu t^\nu) \tag{13}$$

Theorem 4.2. *If $a > 0, \alpha > 0, \beta > 0, \nu > 0$, then the solution of the equation*

$$N(t) - N_0 t^{\beta-1}\, {}_1\overset{\alpha,\beta}{M}_1(\gamma;1;-a^\alpha t^\alpha) = -c^\nu\, {}_0D_t^{-\nu} N(t) \tag{14}$$

is given by

$$N(t) = N_0 \sum_{m=0}^{\infty} (-c^\nu)^m K_4^{(\alpha,\alpha\gamma-\beta-m\nu,\gamma),(-a^\alpha,0):(0;0)}(t) \tag{15}$$

Proof. On solving as above, it is observed that

$$N(t) = N_0 \sum_{m=0}^{\infty} (-c^{\nu})^m {}_0\mathrm{D}_t^{-\nu m}\{t^{\beta-1}\ {}_1\overset{\alpha,\beta}{M}_1(\gamma;1;-a^{\alpha}t^{\alpha})\}$$

Now,

$$\begin{aligned}
{}_0D_t^{-\nu m}\{t^{\beta-1}\ {}_1\overset{\alpha,\beta}{M}_1(\gamma;1;-a^{\alpha}t^{\alpha})\} &= {}_0\mathrm{D}_t^{-\nu m}\ \{t^{\beta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n(-a^{\alpha}t^{\alpha})^n}{(1)_n\Gamma(\alpha n+\beta)}\} \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_n(-a^{\alpha})^n}{(1)_n\Gamma(\alpha n+\beta)}{}_0\mathrm{D}_t^{-\nu m}t^{\alpha n+\beta-1} \\
&= t^{\beta+m\nu-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n(-a^{\alpha})^n}{n!\Gamma(\alpha n+\beta+m\nu)}\ t^{\alpha n} \\
&= t^{\alpha\gamma-(\alpha\gamma-\beta-m\nu)-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n(-a^{\alpha})^n}{n!\Gamma(\alpha(n+\gamma)-(\alpha\gamma-\beta-m\nu)}\ t^{\alpha n}
\end{aligned}$$

$${}_0D_t^{-\nu m}\{t^{\beta-1}\ {}_1\overset{\alpha,\beta}{M}_1(\gamma;1;-a^{\alpha}t^{\alpha})\} = K_4^{(\alpha,\alpha\gamma-\beta-m\nu,\gamma),(-a^{\alpha},0):(0;0)}(t)$$

$$N(t) = N_0\sum_{m=0}^{\infty}(-c^{\nu})^m\ K_4^{(\alpha,\alpha\gamma-\beta-m\nu,\gamma),(-a^{\alpha},0):(0;0)}(t)$$

□

Theorem 4.3. *If $a>0,\ \alpha>0,\ \beta>0,\ \nu>0$, then the solution of the equation*

$$N(t) - N_0 t^{\beta-1}\ {}_0\overset{\alpha,\beta}{M}_0(-;-;-a^{\alpha}t^{\alpha}) = -c^{\nu}\,{}_0D_t^{-\nu}N(t) \tag{16}$$

is given by

$$N(t) = N_0\sum_{m=0}^{\infty}(-c^{\nu})^m\ \mathrm{t}^{\beta+m\nu-1}\ {}_0\overset{\alpha,\beta+m\nu}{M}_0\ (-;-;-a^{\alpha}t^{\alpha}) \tag{17}$$

Theorem 4.4. *If $a>0,\ b\geq 0,\ \gamma>0,\ \alpha>0,\ \beta>0,\ \nu>0$ and $R(\gamma\alpha-\beta)>0$, then the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 K_4^{(\alpha,\beta,\gamma),(-a^{\alpha},b):(p;q)}(t) = -c^{\nu}\,{}_0D_t^{-\nu}N(t) \tag{18}$$

is given by

$$N(t) = N_0\sum_{m=0}^{\infty}(-c^{\nu})^m K_4^{(\alpha,\beta-\nu m,\gamma),(-a^{\alpha},b):(p,q)}(t) \tag{19}$$

Proof. Multiplying both side of equation (18) by the operator $(-c^{v})^{m}\ {}_0D_t^{-mv}$ and taking the sum over m from 0 to ∞, yields

$$\sum_{m=0}^{\infty}(-c^{\nu})^{m}{}_0D_t^{-m\nu}N(t)-\sum_{m=0}^{\infty}(-c^{\nu})^{m+1}{}_0D_t^{-(m+1)\nu}N(t)$$

$$=N_0\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-c^{\nu})^{m}{}_0D_t^{-m\nu}\frac{(a_1)_{n}...(a_p)_n(\gamma)_n a^n (t-b)^{(n+\gamma)\alpha-\beta-1}}{(b_1)_{n}...(b_q)_n n!\Gamma((n+\gamma)\alpha-\beta)}$$

$$=N_0\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-c^{\nu})^{m}\frac{(a_1)_{n}...(a_p)_n(\gamma)_n a^n}{(b_1)_{n}...(b_q)_n n!\Gamma((n+\gamma)\alpha-\beta)}{}_0D_t^{-m\nu}(t-b)^{(n+\gamma)\alpha-\beta-1}$$

By virtue of the relationship (8) on the right hand side of above equation and replacing m by $(m-1)$ in the second sum of above equation and then cancelling the equal terms on the left hand side, we obtain

$$N(t)=N_0\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-c^{\nu})^{m}\frac{(a_1)_{n}\dots(a_p)_n(\gamma)_n a^n (t-b)^{(n+\gamma)\alpha-\beta+\nu m-1}}{(b_1)_{n}\dots(b_q)_n n!\Gamma((n+\gamma)\alpha-\beta+\nu m)}$$

or

$$N(t)=N_0\sum_{m=0}^{\infty}(-c^{\nu})^{m}K_4^{(\alpha,\beta-\nu m,\gamma),(-a^{\alpha},b):(p,q)}(t)$$

$\square$

Corollary 4.5. *If $v>0,\mu=v$, then the solution of integral equation*

$$N(t)-N_0t^{\nu-1}=-c^{\nu}{}_0D_t^{-\nu}N(t) \tag{20}$$

is given by

$$N(t)=N_0\Gamma\nu t^{\nu-1}\,{}_0\overset{\nu,\nu}{M}_0(-;-;-c^{\nu}t^{\nu}) \tag{21}$$

Corollary 4.6. *If $a>0,b=0,\gamma>0,\alpha>0,\beta>0,v>0$ and $R(\gamma\alpha-\beta)>0$, then the solution of the generalized fractional kinetic equation*

$$N(t)-N_0K_4^{(\alpha,\beta,\gamma),(-a^{\alpha},0):(p;q)}(t)=-c^{\nu}{}_0D_t^{-\nu}N(t) \tag{22}$$

is given by

$$N(t)=N_0\sum_{m=0}^{\infty}(-c^{\nu})^{m}K_4^{(\alpha,\beta-\nu m,\gamma),(-a^{\alpha},0):(p,q)}(t) \tag{23}$$

Acknowledgements

The authors are very thankful to the worthy referee for his valuable suggestions in the improvement of the paper.

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