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APPROXIMATION PROPERTIES OF TWO DIMENSIONAL BERNSTEIN-STANCU-CHLODOWSKY OPERATORS

TUNCER ACAR - ALI ARAL

In this paper, as a generalization of Bernstein-Stancu type operators of two variable, we introduce a new positive linear operator $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ called Bernstein-Stancu-Chlodowsky on a triangular domain, with mobile boundaries, which extends to $[0,\infty)\times[0,\infty)$ as $n\to\infty$. We give some shape properties that are preserved and also obtain weighted approximation properties of these operators.

1. Introduction

The approximation operators on triangles with all immobile straight sides were largely studied due to their applications in many fields inside applied mathematics. Most of them was considered as generalizations of Bernstein polynomials. Because of the fact that Bernstein polynomials have simple structure, these polynomials are used for important applications in the branches of mathematics, physics, computer science and a lot of area. Therefore, many researchers studied in this direction and constructed generalizations of these polynomials. For example, in 1968 Stancu [6] introduced one of these generalizations as

$$B_{n,\alpha,\beta}(f;x)=\sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{1}$$

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for each real α,β such that $0\leq\alpha\leq\beta$ and called Bernstein-Stancu polynomials. Similar to $B_n(f;x)$, $B_{n,\alpha,\beta}(f;x)$ also converges to continuous function $f(x)$ uniformly in $[0,1]$.

As another generalization, Bernstein polynomials were defined on an unbounded interval by I. Chlodowsky [2] in 1937 as

$$C_n(f;x)=\sum_{r=0}^{n}f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r}, \tag{2}$$

where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0,b]\subset[0,\infty)$ and $(b_n)_{n\geq1}$ is a positive increasing sequence with the properties

$$b_n\to\infty \text{ and } \frac{b_n}{n}\to0 \text{ as } n\to\infty. \tag{3}$$

In addition to Bernstein polynomials of function of one variable, two dimensional generalizations of these polynomials have been studied increasingly. Recently, Gadjiev and Ghorbanalizadeh [3] introduced the two dimensional generalization of Bernstein-Stancu type polynomials with shifted knots on the triangle $\Delta=\left\{(x,y):x+y\leq\frac{n+2\alpha}{n+\beta},x,y\geq\frac{\alpha}{n+\beta}\right\}$ as

$$S_n^{\alpha,\alpha_k,\beta,\beta_k}(f;x,y)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}f\left(\frac{k+\alpha_1}{n+\beta_1},\frac{l+\alpha_2}{n+\beta_2}\right)p_{n,\alpha,\beta}^{k,l}(x,y), \tag{4}$$

where

$$\begin{aligned}p_{n,\alpha,\beta}^{k,l}(x,y)=&\left(\frac{n+\beta}{n}\right)^n\binom{n}{k}\binom{n-k}{l}\\&\times\left(x-\frac{\alpha}{n+\beta}\right)^k\left(y-\frac{\alpha}{n+\beta}\right)^l\left(\frac{n+2\alpha}{n+\beta}-x-y\right)^{n-k-l},\end{aligned}$$

and $\alpha,\beta,\alpha_k,\ \beta_k,\ (k=1,2)$ are positive real numbers provided $0\leq\alpha\leq\alpha_1\leq\beta_1\leq\beta$ and $0\leq\alpha\leq\alpha_2\leq\beta_2\leq\beta$.

As approximation properties of bivariate positive linear operators, shape preserving properties of them were studied extensively because of applications in many areas. We refer [1] and [5].

In the present paper, inspired by all the above ideas and (4), we introduce two dimensional Bernstein-Stancu-Chlodowsky operators on triangle with mobile boundaries. As distinct from the other studies about two dimensional Benstein type operators on a triangle domain, we study on a triangular domain, with mobile sides, which extends to $[0,\infty)\times[0,\infty)$ as $n\to\infty$. We study the shape preserving and convergence properties of these operators and we give the theorems in weighted space.

2. Preliminaries

We will use the following notations. Let

$$\Delta_a = \{(x,y) : x+y \leq a,\ x,y \geq 0\}$$

be a triangle with the fixed boundary $x+y=a$ for any $a>0$ and (b_n) be a sequence of positive numbers with the properties (3) and α,β are positive real numbers such that $\alpha<\beta$, we denote by $\Delta_a^{n,\alpha,\beta}$ the triangular domain

$$\Delta_a^{n,\alpha,\beta} = \left\{(x,y) : x+y \leq a,\ x,y \geq \frac{\alpha}{n+\beta}b_n\right\},$$

where a is a positive number such that $\frac{n+2\alpha}{n+\beta}b_n \geq a \geq \frac{\alpha}{n+\beta}b_n$, we will consider also triangle $\hat{\Delta}_n^{\alpha,\beta}$ with the mobile boundaries $x+y=\frac{n+2\alpha}{n+\beta}b_n$ and $x=y=\frac{\alpha}{n+\beta}b_n$,

$$\hat{\Delta}_n^{\alpha,\beta} = \left\{(x,y) : x+y \leq \frac{n+2\alpha}{n+\beta}b_n,\ x,y \geq \frac{\alpha}{n+\beta}b_n\right\}$$

which tends to $\mathbb{R}_+^2 = \{(x,y) : x,y \geq 0\}$ as $n\to\infty$. It is clear that $\Delta_a^{n,\alpha,\beta} \subset \Delta_a$ and $\Delta_a^{n,\alpha,\beta} \subset \hat{\Delta}_n^{\alpha,\beta}$.

Let

$$\rho(x,y) = 1+x^2+y^2$$

and denote by $C_\rho(\mathbb{R}_+^2)$ the space of all functions f, which are continuous and satisfies the inequality

$$|f(x,y)| \leq M_f\rho(x,y),$$

where M_f is a constant depending on function f only.

Denote also by $C_\rho^0(\mathbb{R}_+^2)$ a subspace of functions $f \in C_\rho(\mathbb{R}_+^2)$, for which

$$\lim_{x+y\to\infty} \frac{f(x,y)}{1+x^2+y^2} = 0.$$

These spaces are linear normed spaces with norm

$$\|f(x,y)\|_\rho = \sup_{(x,y)\in\mathbb{R}_+^2} \frac{|f(x,y)|}{1+x^2+y^2}. \tag{5}$$

For $x,y \in \hat{\Delta}_n^{\alpha,\beta}$, we introduce the Bernstein-Stancu-Chlodowsky operators for a function f of two variables as follows:

$$C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} f\left(\alpha_3 x+\beta_3\frac{k+\alpha_1}{n+\beta_1}b_n, \alpha_3 y+\beta_3\frac{l+\alpha_2}{n+\beta_2}b_n\right) p_{n,\alpha,\beta}^{k,l}(x,y) \tag{6}$$

where

$$p_{n,\alpha,\beta}^{k,l}(x,y) = \left(\frac{n+\beta}{n}\right)^n \binom{n}{k}\binom{n-k}{l}\left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right)^k \times \left(\frac{y}{b_n} - \frac{\alpha}{n+\beta}\right)^l \left(\frac{n+2\alpha}{n+\beta} - \frac{x+y}{b_n}\right)^{n-k-l}$$

and $f \in C_\rho\left(\mathbb{R}_+^2\right)$, $\alpha, \beta, \alpha_r, \beta_r$ $(r = 1,2,3)$ are positive real numbers provided $0 \leq \alpha \leq \alpha_1 \leq \beta_1 \leq \beta$, $0 \leq \alpha \leq \alpha_2 \leq \beta_2 \leq \beta$ and $\alpha_3 + \beta_3 = 1$.

It is obviously that the family of operators $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is linear and positive and with the special cases of α_r and β_r, we obtain some well-known two dimensional Bernstein polynomials.

1. If we take $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta = 0$ with $b_n = 1$, we get the usual two dimensional Bernstein polynomials on the triangle Δ_1,
2. If we take $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta = 0$, we get the usual two dimensional Bernstein-Chlodowsky polynomials on the triangle Δ_{b_n} given in [4],
3. If we take $\alpha_3 = 0$ with $b_n = 1$, we get the two dimensional Bernstein-Stancu polynomials on the triangle Δ given in [3].

3. Shape preserving properties

In this section we study convexity property of the operators by showing that $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is convex of order (i,j) if $f(x,y)$ is convex of order (i,j) for $0 < i+j \leq 2$.

We first recall a usual definition of convexity for bivariate functions.

For $f \in C(\Delta_1)$, $(x,y) \in \Delta_1$ and $h \in \mathbb{R}^+$, $\Delta_h^{(i,j)}$ is defined by

$$\Delta_h^{(1,0)} f(x,y) = f(x+h,y) - f(x,y),$$
$$\Delta_h^{(0,1)} f(x,y) = f(x,y+h) - f(x,y),$$

$$\Delta_h^{(1,1)} f(x,y) = f(x+h,y+h) + f(x,y) - f(x+h,y) - f(x,y+h),$$

$$\Delta_h^{(2,0)} f(x,y) = f(x+2h,y) - 2f(x+h,y) + f(x,y),$$
$$\Delta_h^{(2,0)} f(x,y) = f(x,y+2h) - 2f(x,y+h) + f(x,y).$$

We know the following definition:

Definition 3.1. $f(x,y)$ is convex of order (i,j), $i,j\in\mathbb{N}$, $0<i+j\leq 2$, if for $h\in\mathbb{R}^{+}$, $\Delta_h^{(i,j)}f\geq 0$.

Remark 3.2. If $f\in C^{i+j}(\Delta_1)$ and for all $(x,y)\in\Delta_1$

$$\frac{\partial^{i+j}f}{\partial x^i\partial y^j}(x,y)\geq 0,$$

then $f(x,y)$ is convex of order (i,j).

Based on Definition 3.1 and Remark 3.2, we give the following theorem.

Theorem 3.3. *Let $f\in C^{i+j}(\hat{\Delta}_n^{\alpha,\beta})$ such that $i,j\in\mathbb{N}$ and $0<i+j\leq 2$. Then the following statements hold:*

i) If $f(x,y)$ is convex of order $(1,0)$ (resp. $(0,1)$), then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is also convex of order $(1,0)$ (resp.$(0,1)$).

ii) If $f(x,y)$ is convex of order $(2,0)$ (resp. $(0,2)$), then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is also convex of order $(2,0)$ (resp.$(0,2)$).

iii) If $f(x,y)$ is convex of order $(1,1)$, then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is also convex of order $(1,1)$.

Proof. We get expressions of the derivatives of $C_n^{\alpha,\alpha_r,\beta,\beta_r}$. By easy computation, we have the equalities

$$\frac{\partial C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)}{\partial x}=\alpha_3\sum_{k=0}^{n}\sum_{l=0}^{n-k}f_x(\sigma_0,\gamma_0)\,p_{n,\alpha,\beta}^{k,l}(x,y)$$
$$+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left[f(\sigma_1,\gamma_0)-f(\sigma_0,\gamma_0)\right]\rho_{n,\alpha,\beta}^{k,l}(x,y), \tag{7}$$

$$\frac{\partial C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)}{\partial y}=\alpha_3\sum_{k=0}^{n}\sum_{l=0}^{n-k}f_y(\sigma_0,\gamma_0)\,p_{n,\alpha,\beta}^{k,l}(x,y)$$
$$+\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left[f(\sigma_0,\gamma_1)-f(\sigma_0,\gamma_0)\right]\rho_{n,\alpha,\beta}^{k,l}(x,y), \tag{8}$$

$$\frac{\partial^2 C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)}{\partial x^2}=\alpha_3^2\sum_{k=0}^{n}\sum_{l=0}^{n-k}f_{xx}(\sigma_0,\gamma_0)\,p_{n,\alpha,\beta}^{k,l}(x,y)$$
$$+2\alpha_3\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left[f_x(\sigma_1,\gamma_0)-f_x(\sigma_0,\gamma_0)\right]\rho_{n,\alpha,\beta}^{k,l}(x,y)$$
$$+\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}\left[f(\sigma_2,\gamma_0)-2f(\sigma_1,\gamma_0)+f(\sigma_0,\gamma_0)\right]r_{n,\alpha,\beta}^{k,l}(x,y), \tag{9}$$

$$\frac{\partial^2 C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)}{\partial y^2} = \alpha_3^2 \sum_{k=0}^{n}\sum_{l=0}^{n-k} f_{yy}\left(\sigma_0,\gamma_0\right) p_{n,\alpha,\beta}^{k,l}\left(x,y\right)$$
$$+2\alpha_3 \sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left[f_y\left(\sigma_0,\gamma_1\right)-f_y\left(\sigma_0,\gamma_0\right)\right]\rho_{n,\alpha,\beta}^{k,l}\left(x,y\right)$$
$$+\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}\left[f\left(\sigma_0,\gamma_2\right)-2f\left(\sigma_0,\gamma_1\right)+f\left(\sigma_0,\gamma_0\right)\right] r_{n,\alpha,\beta}^{k,l}\left(x,y\right), \tag{10}$$

$$\frac{\partial^2 C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)}{\partial x \partial y} = \alpha_3^2 \sum_{k=0}^{n}\sum_{l=0}^{n-k} f_{xy}\left(\sigma_0,\gamma_0\right) p_{n,\alpha,\beta}^{k,l}\left(x,y\right)$$
$$+\alpha_3 \sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left[f_x\left(\sigma_0,\gamma_1\right)-f_x\left(\sigma_0,\gamma_0\right)\right]\rho_{n,\alpha,\beta}^{k,l}\left(x,y\right)$$
$$+\alpha_3 \sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left[f_y\left(\sigma_1,\gamma_0\right)-f_y\left(\sigma_0,\gamma_0\right)\right]\rho_{n,\alpha,\beta}^{k,l}\left(x,y\right)$$
$$+\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}\left[f\left(\sigma_1,\gamma_1\right)-f\left(\sigma_0,\gamma_1\right)-f\left(\sigma_1,\gamma_0\right)+f\left(\sigma_0,\gamma_0\right)\right] r_{n,\alpha,\beta}^{k,l}\left(x,y\right), \tag{11}$$

where

$$\sigma_0 = \alpha_3 x + \beta_3 \frac{k+\alpha_1}{n+\beta_1} b_n \quad , \qquad \gamma_0 = \alpha_3 y + \beta_3 \frac{l+\alpha_2}{n+\beta_2} b_n$$
$$\sigma_1 = \alpha_3 x + \beta_3 \frac{k+1+\alpha_1}{n+\beta_1} b_n \quad , \qquad \gamma_1 = \alpha_3 y + \beta_3 \frac{l+1+\alpha_2}{n+\beta_2} b_n$$
$$\sigma_2 = \alpha_3 x + \beta_3 \frac{k+2+\alpha_1}{n+\beta_1} b_n \quad \text{and} \qquad \gamma_2 = \alpha_3 y + \beta_3 \frac{l+2+\alpha_2}{n+\beta_2} b_n.$$

and

$$\rho_{n,\alpha,\beta}^{k,l}\left(x,y\right) = \frac{n}{b_n}\left(\frac{n+\beta}{n}\right)^n \binom{n-1}{k}\binom{n-k-1}{l}$$
$$\times \left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^k \left(\frac{y}{b_n}-\frac{\alpha}{n+\beta}\right)^l \left(\frac{n+2\alpha}{n+\beta}-\frac{x+y}{b_n}\right)^{n-k-l-1},$$

$$r_{n,\alpha,\beta}^{k,l}\left(x,y\right) = \frac{n(n-1)}{b_n^2}\left(\frac{n+\beta}{n}\right)^n \binom{n-2}{k}\binom{n-k-2}{l}$$
$$\times \left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^k \left(\frac{y}{b_n}-\frac{\alpha}{n+\beta}\right)^l \left(\frac{n+2\alpha}{n+\beta}-\frac{x+y}{b_n}\right)^{n-k-l-2}.$$

Note that

$$\sigma_2-\sigma_1=\sigma_1-\sigma_0=\frac{\beta_3 b_n}{n+\beta_1}=h\in\mathbb{R}^+,\ \gamma_2-\gamma_1=\gamma_1-\gamma_0=\frac{\beta_3 b_n}{n+\beta_2}=h^*\in\mathbb{R}^+,$$

$$\sigma_2-\sigma_0=\frac{2\beta_3 b_n}{n+\beta_1}=h_1\in\mathbb{R}^+ \text{ and } \gamma_2-\gamma_0=\frac{2\beta_3 b_n}{n+\beta_2}=h_1^*\in\mathbb{R}^+.$$

For the proof of *i*), we consider the equality (7) (resp. (8)). Since $f(x,y)$ is convex of order $(1,0)$ (resp. $(0,1)$),

$$f(\sigma_1,\gamma_0)-f(\sigma_0,\gamma_0)\geq 0 \text{ (resp. } f(\sigma_0,\gamma_1)-f(\sigma_0,\gamma_0)\geq 0)$$

by Definition 3.1. Since $f\in C^{(1,0)}(\hat{\Delta}_n^{\alpha,\beta})$ (resp. $f\in C^{(0,1)}(\hat{\Delta}_n^{\alpha,\beta})$), $f_x(\sigma_0,\gamma_0)\geq 0$ (resp. $f_y(\sigma_0,\gamma_0)\geq 0$). Hence, we have $\frac{\partial C_n^{\alpha,\alpha_r,\beta,\beta_r} f}{\partial x}\geq 0$ (resp. $\frac{\partial C_n^{\alpha,\alpha_r,\beta,\beta_r} f}{\partial y}\geq 0$) which implies the convexity of order $(1,0)$ (resp. $(0,1)$) of $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ by Remark 3.2.

For the proof of *ii*), we consider the equality (9) (resp. (10)). Since $f(x,y)$ is convex of order $(2,0)$ (resp. $(0,2)$),

$$f(\sigma_2,\gamma_0)-2f(\sigma_1,\gamma_0)+f(\sigma_0,\gamma_0)\geq 0$$
$$\text{(resp.} f(\sigma_0,\gamma_2)-2f(\sigma_0,\gamma_1)+f(\sigma_0,\gamma_0)\geq 0)$$

by Definition 3.1. Since $f\in C^{(2,0)}(\hat{\Delta}_n^{\alpha,\beta})$ (resp. $f\in C^{(0,2)}(\hat{\Delta}_n^{\alpha,\beta})$), $f_{xx}(\sigma_0,\gamma_0)\geq 0$ (resp. $f_{yy}(\sigma_0,\gamma_0)\geq 0$) and $f_x(\sigma_1,\gamma_0)-f_x(\sigma_0,\gamma_0)\geq 0$ (resp. $f_y(\sigma_0,\gamma_1)-f_y(\sigma_0,\gamma_0)\geq 0$). Hence, we have $\frac{\partial^2 C_n^{\alpha,\alpha_r,\beta,\beta_r} f}{\partial x^2}\geq 0$ (resp. $\frac{\partial^2 C_n^{\alpha,\alpha_r,\beta,\beta_r} f}{\partial y^2}\geq 0$) which implies the convexity of order $(2,0)$ (resp. $(0,2)$) of $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ by Remark 3.2.

For the proof of *iii*), we consider the equality (11). Since $f(x,y)$ is convex of order $(1,1)$,

$$f(\sigma_1,\gamma_1)-f(\sigma_0,\gamma_1)-f(\sigma_1,\gamma_0)+f(\sigma_0,\gamma_0)\geq 0$$

by Definition 3.1. And since $f\in C^{(1,1)}(\hat{\Delta}_n^{\alpha,\beta})$, $f_{xy}(\sigma_0,\gamma_0)\geq 0$ and $f_x(\sigma_0,\gamma_1)-f_x(\sigma_0,\gamma_0)\geq 0$ and $f_y(\sigma_1,\gamma_0)-f_y(\sigma_0,\gamma_0)\geq 0$. Hence, we have $\frac{\partial^2 C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)}{\partial x\partial y}\geq 0$ which implies the convexity of order $(1,1)$ of $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ by Remark 3.2. □

Furthermore, as a corollary of Theorem 3.3, if we choose the $\alpha_3=0$, so that condition of differentiability of f is not needed, we can easily prove the following result.

Corollary 3.4. *Let $f \in C(\hat{\Delta}_n^{\alpha,\beta})$ such that $i, j \in \mathbb{N}$ and $0 < i + j \leq 2$.*

1. *If $f(x,y)$ is convex of order $(1,0)$ (resp.$(0,1)$), then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is also convex of order $(1,0)$ (resp.$(0,1)$).*
2. *If $f(x,y)$ is convex of order $(2,0)$ (resp.$(0,2)$), then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is also convex of order $(2,0)$ (resp.$(0,2)$).*
3. *If $f(x,y)$ is convex of order $(1,1)$, then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ is also convex of order $(1,1)$.*

4. Convergence properties

Lemma 4.1. *Let $f = f(t,\tau) \in C_\rho(\mathbb{R}_+^2)$. For all $n \in \mathbb{N}$ and $(x,y) \in \hat{\Delta}_n^{\alpha,\beta}$, the operator $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ has the following properties*

$$C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y) = 1, \tag{12}$$

$$C_n^{\alpha,\alpha_r,\beta,\beta_r}(t;x,y) = \alpha_3 x + \frac{(n+\beta)}{n+\beta_1}\beta_3 x + \frac{(\alpha_1-\alpha)}{n+\beta_1}\beta_3 b_n, \tag{13}$$

$$C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau;x,y) = \alpha_3 y + \frac{(n+\beta)}{n+\beta_2}\beta_3 y + \frac{(\alpha_2-\alpha)}{n+\beta_2}\beta_3 b_n, \tag{14}$$

$$\begin{aligned} C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2;x,y) &= x^2\left[\alpha_3^2 + 2\beta_3\alpha_3\frac{(n+\beta)}{n+\beta_1} + \beta_3^2\left(\frac{n+\beta}{n+\beta_1}\right)^2\frac{n-1}{n}\right] \\ &+ x\left[2\beta_3\alpha_3\tfrac{b_n(\alpha_1-\alpha)}{n+\beta_1} - \beta_3^2 2\alpha\tfrac{(n+\beta)(n-1)b_n}{(n+\beta_1)^2 n} + \beta_3^2\tfrac{(n+\beta)}{(n+\beta_1)^2}b_n(1+2\alpha_1)\right] \\ &+ \left(\frac{b_n}{n+\beta_1}\right)^2\beta_3^2\left[(\alpha-\alpha_1)^2 - \alpha\left(\frac{\alpha}{n}+1\right)\right], \end{aligned} \tag{15}$$

$$\begin{aligned} C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau^2;x,y) &= y^2\left[\alpha_3^2 + 2\alpha_3\beta_3\frac{(n+\beta)}{n+\beta_2} + \beta_3^2\left(\frac{n+\beta}{n+\beta_2}\right)^2\frac{n-1}{n}\right] \\ &+ y\left[2\alpha_3\beta_3\tfrac{b_n(\alpha_2-\alpha)}{n+\beta_2} - \beta_3^2 2\alpha\tfrac{(n+\beta)(n-1)b_n}{(n+\beta_2)^2 n} + \beta_3^2\tfrac{(n+\beta)}{(n+\beta_2)^2}b_n(1+2\alpha_2)\right] \\ &+ \left(\frac{b_n}{n+\beta_2}\right)^2\beta_3^2\left[(\alpha-\alpha_2)^2 - \alpha(\frac{\alpha}{n}+1)\right]. \end{aligned} \tag{16}$$

Also this sequence of linear positive operators acts from $C_\rho(\mathbb{R}_+^2)$ to $C_\rho(\mathbb{R}_+^2)$.

Proof. We only give the proof for the cases $C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)$, $C_n^{\alpha,\alpha_r,\beta,\beta_r}(t;x,y)$ and $C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2;x,y)$. Others are similar.

For the sake of convenient notation in the proof, we will use the following definition:

$$C_n^{\alpha,\beta}(f(t,\tau);x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} f\left(\frac{k}{n},\frac{l}{n}\right) p_{n,\alpha,\beta}^{k,l}(x,y). \tag{17}$$

By the binomial expansion and (17) it is obvious that

$$C_n^{\alpha,\beta}(1;x,y) = C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} p_{n,\alpha,\beta}^{k,l}(x,y) = 1. \tag{18}$$

We can also obtain

$$\begin{aligned}
C_n^{\alpha,\beta}(t;x,y) &= \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{k}{n} p_{n,\alpha,\beta}^{k,l}(x,y) \\
&= \left(\frac{n+\beta}{n}\right)^n \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right)^k \\
&\quad \times \sum_{l=0}^{n-k} \binom{n-k}{l} \left(\frac{y}{b_n} - \frac{\alpha}{n+\beta}\right)^l \left(\frac{n+2\alpha}{n+\beta} - \frac{x+y}{b_n}\right)^{n-k-l} \\
&= \left(\frac{n+\beta}{n}\right)^n \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right) \\
&\sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right)^{k-1} \left(\frac{n+\alpha}{n+\beta} - \frac{x}{b_n}\right)^{n-k}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{n+\beta}{n}\right)^n \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right) \\
&\sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right)^k \left(\frac{n+\alpha}{n+\beta} - \frac{x}{b_n}\right)^{n-k-1} \\
&= \left(\frac{n+\beta}{n}\right)^n \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right) \left(\frac{n}{n+\beta}\right)^{n-1} \\
&= \left(\frac{n+\beta}{n}\right) \left(\frac{x}{b_n} - \frac{\alpha}{n+\beta}\right)
\end{aligned} \tag{19}$$

and

$$
\begin{aligned}
C_n^{\alpha,\beta}(t^2;x,y) &= \sum_{k=1}^{n}\sum_{l=0}^{n-k}\frac{k^2}{n^2}p_{n,\alpha,\beta}^{k,l}(x,y)\\
&= \left(\frac{n+\beta}{n}\right)^n\sum_{k=1}^{n}\binom{n}{k}\frac{k^2}{n^2}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^k\\
&\sum_{l=0}^{n-k}\binom{n-k}{l}\left(\frac{y}{b_n}-\frac{\alpha}{n+\beta}\right)^l\left(\frac{n+2\alpha}{n+\beta}-\frac{x+y}{b_n}\right)^{n-k-l}\\
&= \left(\frac{n+\beta}{n}\right)^n\frac{n-1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^2\\
&\sum_{k=2}^{n}\binom{n-2}{k-2}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^{k-2}\left(\frac{n+\alpha}{n+\beta}-\frac{x}{b_n}\right)^{n-k}\\
&+\left(\frac{n+\beta}{n}\right)^n\frac{1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)\\
&\sum_{k=1}^{n}\binom{n-1}{k-1}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^{k-1}\left(\frac{n+\alpha}{n+\beta}-\frac{x}{b_n}\right)^{n-k}\\
&= \left(\frac{n+\beta}{n}\right)^n\frac{n-1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^2\left(\frac{n}{n+\beta}\right)^{n-2}\\
&+\left(\frac{n+\beta}{n}\right)^n\frac{1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)\left(\frac{n}{n+\beta}\right)^{n-1}\\
&= \frac{n-1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^2\left(\frac{n+\beta}{n}\right)^2\\
&+\frac{1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)\left(\frac{n+\beta}{n}\right). \qquad (20)
\end{aligned}
$$

From (6) and (17)-(19) we get

$$
\begin{aligned}
C_n^{\alpha,\alpha_r,\beta,\beta_r}(t;x,y) &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\left(\alpha_3x+\beta_3\frac{k+\alpha_1}{n+\beta_1}b_n\right)p_{n,\alpha,\beta}^{k,l}(x,y)\\
&= \alpha_3xC_n^{\alpha,\beta}(1;x,y)+\frac{n\beta_3}{n+\beta_1}b_nC_n^{\alpha,\beta}(t;x,y)+\frac{\alpha_1\beta_3}{n+\beta_1}b_nC_n^{\alpha,\beta}(1;x,y)\\
&= \alpha_3x+\frac{(n+\beta)}{n+\beta_1}\beta_3x+\frac{(\alpha_1-\alpha)}{n+\beta_1}\beta_3b_n.
\end{aligned}
$$

Again using (6) and (17)-(20), we obtain

$$C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2;x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\left(\alpha_3 x+\beta_3\frac{k+\alpha_1}{n+\beta_1}b_n\right)^2 p_{n,\alpha,\beta}^{k,l}(x,y)$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\alpha_3^2 x^2 p_{n,\alpha,\beta}^{k,l}(x,y)+\sum_{k=0}^{n}\sum_{l=0}^{n-k}2\beta_3\alpha_3 x\frac{k+\alpha_1}{n+\beta_1}b_n p_{n,\alpha,\beta}^{k,l}(x,y)$$

$$+\sum_{k=0}^{n}\sum_{l=0}^{n-k}(\beta_3\frac{k+\alpha_1}{n+\beta_1}b_n)^2 p_{n,\alpha,\beta}^{k,l}(x,y)$$

$$= \alpha_3^2 C_n^{\alpha,\beta}(1;x,y)+2\beta_3\alpha_3 x\frac{n}{n+\beta_1}b_n\left[C_n^{\alpha,\beta}(t;x,y)+\frac{\alpha_1}{n}C_n^{\alpha,\beta}(1;x,y)\right]$$

$$+(\beta_3\frac{n}{n+\beta_1}b_n)^2\left[C_n^{\alpha,\beta}(t^2;x,y)+\frac{2\alpha_1}{n}C_n^{\alpha,\beta}(t;x,y)+\frac{\alpha_1^2}{n^2}C_n^{\alpha,\beta}(1;x,y)\right]$$

$$= \alpha_3^2x^2+2\beta_3\alpha_3 x\frac{n}{n+\beta_1}b_n\left[\left(\frac{n+\beta}{n}\right)\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)+\frac{\alpha_1}{n}\right]$$

$$+(\beta_3\frac{n}{n+\beta_1}b_n)^2\left[\frac{n-1}{n}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)^2\left(\frac{n+\beta}{n}\right)^2\right.$$

$$\left.+\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)\frac{n+\beta}{n^2}+\frac{2\alpha_1(n+\beta)}{n^2}\left(\frac{x}{b_n}-\frac{\alpha}{n+\beta}\right)+\frac{\alpha_1^2}{n^2}\right]$$

$$= x^2\left[\alpha_3^2+2\beta_3\alpha_3\frac{n+\beta}{n+\beta_1}+\beta_3^2\left(\frac{n+\beta}{n+\beta_1}\right)^2\frac{n-1}{n}\right]$$

$$+x\left[2\beta_3\alpha_3\frac{b_n(\alpha_1-\alpha)}{n+\beta_1}-\beta_3^2 2\alpha\frac{(n+\beta)(n-1)}{(n+\beta_1)^2}\frac{b_n}{n}+\beta_3^2\frac{(n+\beta)(1+2\alpha_1)}{(n+\beta_1)^2}b_n\right]$$

$$+\left(\frac{b_n}{n+\beta_1}\right)^2\beta_3^2\left[(\alpha-\alpha_1)^2-\alpha\left(\frac{\alpha}{n}+1\right)\right].$$

Let us show that $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)$ acts from $C_\rho(\mathbb{R}_+^2)$ to $C_\rho(\mathbb{R}_+^2)$. If $f\in C_\rho(\mathbb{R}_+^2)$ then we can write from (5)

$$\left|f\left(\alpha_3 x+\beta_3\frac{k+\alpha_1}{n+\beta_1}b_n,\alpha_3 y+\beta_3\frac{l+\alpha_2}{n+\beta_2}b_n\right)\right|$$

$$\leq \|f\|_\rho\left[1+\left(\alpha_3 x+\beta_3\frac{k+\alpha_1}{n+\beta_1}b_n\right)^2+\left(\alpha_3 y+\beta_3\frac{l+\alpha_2}{n+\beta_2}b_n\right)^2\right].$$

Applying the operator $C_n^{\alpha,\alpha_r,\beta,\beta_r}$ to both side of this inequality and using the properties of linearity and monotonicity we get

$$\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)\right| \leq \|f\|_\rho \left(C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)+C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2;x,y)+C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau^2;x,y)\right).$$

By (12), (13) and (14)

$$\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)\right| \leq M_f^* \|f\|_\rho \left(1+x^2+y^2\right)$$

since $\frac{b_n}{n}$ is bounded by (3). Here M_f^* is a constant depending on function f only. □

Now we can prove the following result.

Lemma 4.2. *For any fixed $a>0$, the relation*

$$\lim_{n\to\infty} \max_{(x,y)\in\Delta_a^{n,\alpha,\beta}} \left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|=0$$

holds for all functions $f\in C(\mathbb{R}_+^2)$.

Proof. From (12), (13) and (14), we see that

$$C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)-1=0,$$

$$\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}} \left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(t;x,y)-x\right| = \left(a-\frac{\alpha}{n+\beta}b_n\right)\left(\alpha_3+\frac{n+\beta}{n+\beta_1}\beta_3-1\right) + \frac{b_n}{n+\beta_1}\beta_3(\alpha_1-\alpha),$$

$$\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}} \left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau;x,y)-y\right| = \left(a-\frac{\alpha}{n+\beta}b_n\right)\left(\alpha_3+\frac{n+\beta}{n+\beta_2}\beta_3-1\right) + \frac{b_n}{n+\beta_2}\beta_3(\alpha_2-\alpha).$$

From (15) and (16) we get

$$\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2+\tau^2;x,y)-(t^2+\tau^2)\right|$$
$$\le a^2\left[\left(\frac{n+\beta}{n+\beta_1}\right)^2+\frac{(n+\beta)^2}{(n+\beta_2)^2}-2\right]$$
$$+a\left[2\frac{b_n\alpha_1}{n+\beta_1}+\frac{n+\beta}{(n+\beta_1)^2}b_n(1+2\alpha_1)+2\frac{b_n\alpha_2}{n+\beta_2}+\frac{n+\beta}{(n+\beta_2)^2}b_n(1+2\alpha_2)\right]$$
$$+\left(\frac{b_n}{n+\beta_1}\right)^2\alpha_1^2+\left(\frac{b_n}{n+\beta_2}\right)^2\alpha_2^2.$$

Therefore, using these equalities and (3) we obtain

$$\lim_{n\to\infty}\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)-1\right|=0,$$
$$\lim_{n\to\infty}\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(t;x,y)-x\right|=0,$$
$$\lim_{n\to\infty}\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau;x,y)-y\right|=0. \tag{21}$$

Also, if we consider the equality $\alpha_3+\beta_3=1$ then we have

$$\lim_{n\to\infty}\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2+\tau^2;x,y)-(x^2+y^2)\right|=0. \tag{22}$$

Consider the sequence of operators

$$C_n(f;x,y)=\begin{cases} C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y) & \text{if} \quad (x,y)\in\Delta_a^{n,\alpha,\beta} \\ f(x,y) & \text{if} \quad (x,y)\in\Delta_a\backslash\Delta_a^{n,\alpha,\beta}\end{cases}.$$

Then obviously

$$\|C_nf-f\|_{\Delta_a}=\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right| \tag{23}$$

and using (21) and (22) we obtain

$$\lim_{n\to\infty}\left\|C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)-1\right\|_{\Delta_a}=0,$$
$$\lim_{n\to\infty}\left\|C_n^{\alpha,\alpha_r,\beta,\beta_r}(t;x,y)-x\right\|_{\Delta_a}=0,$$
$$\lim_{n\to\infty}\left\|C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau;x,y)-y\right\|_{\Delta_a}=0,$$

$$\lim_{n\to\infty}\left\|C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2+\tau^2;x,y)-(x^2+y^2)\right\|_{\Delta_a}=0.$$

Since all conditions of two dimensional Korovkin's type theorem (see [7]) is satisfied, we obtain

$$\lim_{n\to\infty}\|C_nf-f\|_{\Delta_a}=0$$

for every continuous function f.

Consequently, (23) gives

$$\lim_{n\to\infty}\max_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|=0$$

and proof is completed. □

Theorem 4.3. *Let $f\in C_\rho(\mathbb{R}_+^2)$. Then the relation*

$$\lim_{n\to\infty}\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}}\frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|}{(1+x^2+y^2)^{1+\alpha}}=0$$

holds for any $\alpha>0$.

Proof. For any function $f\in C_\rho(\mathbb{R}_+^2)$ and any $\alpha>0$, we can write

$$\begin{aligned}\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}}&\frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|}{(1+x^2+y^2)^{1+\alpha}}\\&\le\sup_{(x,y)\in\Delta_a^{n,\alpha,\beta}}\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|\\&\quad+\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}}\frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|}{(1+x^2+y^2)^{1+\alpha}}=I_n^{'}+I_n^{''}.\end{aligned}$$

The first term $I_n^{'}\to0$ as $n\to\infty$ by Lemma 4.2 .

Consider the term $I_n^{''}$.

First of all, given $\varepsilon>0$ there exists a positive number $a>0$ such that

$$\frac{1}{(1+x^2+y^2)^{\alpha}}<\varepsilon, \tag{24}$$

if $x+y>a$.

For $I_n^{''}$, we can have the following inequality

$$I_n^{''}\le\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}}\frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)\right|}{(1+x^2+y^2)^{1+\alpha}}+\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}}\frac{|f(x,y)|}{(1+x^2+y^2)^{1+\alpha}}.$$

From Lemma 4.1, we know that if $f \in C_\rho(\mathbb{R}_+^2)$, then $C_n^{\alpha,\alpha_r,\beta,\beta_r}(f) \in C_\rho(\mathbb{R}_+^2)$. And also using the inequality (24) we obtain

$$\begin{aligned} I_n'' &\leq \left(M_f + M_f^{**}\right) \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{1}{(1+x^2+y^2)^\alpha} \\ &\leq \left(M_f + M_f^{**}\right)\varepsilon. \end{aligned}$$

Here M_f^{**} is a constant depending on f only. This completes the proof. □

Theorem 4.4. *Let $f \in C_\rho^0(\mathbb{R}_+^2)$. Then we have*

$$\lim_{n\to\infty} \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}} \frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y) - f(x,y)\right|}{1+x^2+y^2} = 0.$$

Proof. From the definition of $C_\rho^0(\mathbb{R}_+^2)$, we have

$$\lim_{x+y\to\infty} \frac{f(x,y)}{1+x^2+y^2} = 0 \tag{25}$$

and

$$\lim_{n\to\infty} \frac{f\left(\alpha_3 x + \beta_3 \frac{k+\alpha_1}{n+\beta_1} b_n, \alpha_3 y + \beta_3 \frac{l+\alpha_2}{n+\beta_2} b_n\right)}{1 + \left(\alpha_3 x + \beta_3 \frac{k+\alpha_1}{n+\beta_1} b_n\right)^2 + \left(\alpha_3 y + \beta_3 \frac{l+\alpha_2}{n+\beta_2} b_n\right)^2} = 0. \tag{26}$$

Therefore, by (25), for $\varepsilon > 0$ we can choose a large $a > 0$ such that

$$|f(x,y)| < \varepsilon(1+x^2+y^2), \tag{27}$$

if $x+y > a$. And by (26), for a given $\varepsilon > 0$, there exists a number n_0 such that

$$\begin{aligned} &\left| f\left(\alpha_3 x + \beta_3 \frac{k+\alpha_1}{n+\beta_1} b_n, \alpha_3 y + \beta_3 \frac{l+\alpha_2}{n+\beta_2} b_n\right)\right| \\ &< \varepsilon \left[1 + \left(\alpha_3 x + \beta_3 \frac{k+\alpha_1}{n+\beta_1} b_n\right)^2 + \left(\alpha_3 y + \beta_3 \frac{l+\alpha_2}{n+\beta_2} b_n\right)^2\right], \end{aligned} \tag{28}$$

if $n > n_0$.

Since

$$\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}} \frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|}{1+x^2+y^2}$$
$$\leq \sup_{(x,y)\in\Delta_a^{n,\alpha,\beta}} \frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|}{1+x^2+y^2}$$
$$+ \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)-f(x,y)\right|}{1+x^2+y^2} = I_n^{'}+I_n^{''}$$

and by Lemma 4.2 it sufficient to show that $I_n^{''} \to 0$ as $n\to\infty$.

By (27) and (28)

$$I_n^{''} \leq \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{|f(x,y)|}{1+x^2+y^2} + \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)\right|}{1+x^2+y^2}$$
$$\leq \varepsilon + \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{\left|C_n^{\alpha,\alpha_r,\beta,\beta_r}(f;x,y)\right|}{1+x^2+y^2}$$
$$\leq \varepsilon+\varepsilon \sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)+C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2;x,y)+C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau^2;x,y)}{1+x^2+y^2}$$
$$= \varepsilon\left(1+\sup_{(x,y)\in\hat{\Delta}_n^{\alpha,\beta}\setminus\Delta_a^{n,\alpha,\beta}} \frac{C_n^{\alpha,\alpha_r,\beta,\beta_r}(1;x,y)+C_n^{\alpha,\alpha_r,\beta,\beta_r}(t^2;x,y)+C_n^{\alpha,\alpha_r,\beta,\beta_r}(\tau^2;x,y)}{1+x^2+y^2}\right) \leq C\varepsilon,$$

where C independent on n so $I_n^{''} < C\varepsilon$ which completes the proof. □

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TUNCER ACAR
Department of Mathematics
Kırıkkale University
e-mail: tunceracar@ymail.com

ALI ARAL
Department of Mathematics
Kırıkkale University
e-mail: aliaral73@yahoo.com