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CLASSES OF SPIRALLIKE FUNCTIONS DEFINED BY THE DZIOK-SRIVASTAVA OPERATOR

TAMER M. SEOUDY

Making use of the Dziok-Srivastava operator, in this paper we introduce two classes of analytic functions and investigate convolution properties, the necessary and sufficient condition, coefficient estimates and inclusion properties for these classes.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of $\mathcal{A}$ that consists, respectively, of starlike of order α and convex of order α in $\mathbb{U}$. It is well-known that Let $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(\alpha) = \mathcal{K}(0) \subset \mathcal{K}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition)

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is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\mathbb{U}$, such that $f(z)=g(\omega(z))$, $z\in\mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence, (cf., e.g.,[5], [17] and [18]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

For functions f given by (1) and g given by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k \tag{2}$$

the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k=(g*f)(z). \tag{3}$$

Making use of the principal of subordination between analytic functions, we introduce the subclasses $\mathcal{S}^{\alpha}[A,B]$ and $\mathcal{K}^{\alpha}[A,B]$ of the class $\mathcal{A}$ for $|\alpha|<\frac{\pi}{2}$ and $-1\leq B<A\leq 1$ which are defined by (see [3], [4] and [19])

$$\mathcal{S}^{\alpha}[A,B]=\left\{f\in\mathcal{A}:e^{i\alpha}\frac{zf'(z)}{f(z)}\prec\cos\alpha\left(\frac{1+Az}{1+Bz}\right)+i\sin\alpha\,(z\in\mathbb{U})\right\}, \tag{4}$$

and

$$\mathcal{K}^{\alpha}[A,B]=\left\{f\in\mathcal{A}:e^{i\alpha}\frac{\left(zf'(z)\right)'}{f'(z)}\prec\cos\alpha\left(\frac{1+Az}{1+Bz}\right)+i\sin\alpha\,(z\in\mathbb{U})\right\}. \tag{5}$$

We note that

$$\mathcal{S}^{0}[A,B]=\mathcal{S}[A,B],\ \mathcal{K}^{0}[A;B]=\mathcal{K}[A;B]\ (-1\leq B<A\leq 1),$$

where the classes $\mathcal{S}[A,B]$ and $\mathcal{K}[A;B]$ are introduced and studied by many authors (see [1], [11], [13], [14], and [22]).

For complex parameters $a_1,\dots,a_q;b_1,\dots,b_s$ ($b_j\notin\mathbb{Z}_0^-=\{0,-1,-2,\dots\}$; $j=1,\dots,s$), we define the generalized hypergeometric function

$${}_qF_s(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z)$$

by the following infinite series (see [23]):

$${}_qF_s(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_q)_k}{(b_1)_k\dots(b_s)_k}\frac{z^k}{k!}$$

$$\left(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}; z \in U\right),$$

where $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & (k=0), \\ x(x+1)\dots(x+k-1) & (k \in \mathbb{N}). \end{cases}$$

Dziok and Srivastava [9] considered a linear operator $H(a_1, \dots, a_q; b_1, \dots, b_s) : \mathcal{A} \to \mathcal{A}$ defined by the following Hadamard product:

$$H(a_1, \dots, a_q; b_1, \dots, b_s) f(z) = h(a_1, \dots, a_i, \dots, a_q; b_1, \dots, b_s; z) * f(z), \quad (6)$$

where

$$h(a_1, \dots, a_i, \dots, a_q; b_1, \dots, b_s; z) = z\, {}_qF_s(a_1, \dots, a_i, \dots, a_q; b_1, \dots, b_s; z) \quad (7)$$

$$\left(q \leq s+1; q, s \in \mathbb{N}_0; z \in \mathbb{U}\right).$$

If $f(z) \in \mathcal{A}$ is given by (1), then we have

$$H(a_1, \dots, a_q; b_1, \dots, b_s) f(z) = z + \sum_{k=2}^{\infty} \Gamma_k[a_1; b_1] a_k z^k, \quad (8)$$

where

$$\Gamma_k[a_1; b_1] = \frac{(a_1)_{k-1} \dots (a_q)_{k-1}}{(b_1)_{k-1} \dots (b_s)_{k-1} k!}. \quad (9)$$

If, for convenience, we write

$$H_{q,s}[a_1; b_1] = H(a_1, \dots, a_q; b_1, \dots, b_s),$$

then one can easily verify from the definition (6) or (8) that (see [9]):

$$z\left(H_{q,s}[a_1; b_1] f(z)\right)' = a_1 H_{q,s}[a_1+1; b_1] f(z) - (a_1 - 1) H_{q,s}[a_1; b_1] f(z), \quad (10)$$

and

$$z\left(H_{q,s}[a_1; b_1+1] f(z)\right)' = b_1 H_{q,s}[a_1; b_1] f(z) - (b_1 - 1) H_{q,s}[a_1; b_1+1] f(z). \quad (11)$$

It should be remarked that the linear operator $H_{q,s}[a_1; b_1]$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$, we have

(i) $H_{2,1}(a, b; c) f(z) = \left(I_c^{a,b} f\right)(z) \left(a, b \in \mathbb{C}; c \notin \mathbb{Z}_0^-\right)$, where the linear operator $I_c^{a,b}$ was investigated by Hohlov [12];

(ii) $H_{2,1}(\delta+1,1;1)f(z)=D^{\delta}f(z)(\delta>-1)$, where D^{δ} is the Ruscheweyh derivative of $f(z)$ (see [21]);

(iii) $H_{2,1}(\mu+1,1;\mu+2)f(z)=F_{\mu}(f)(z)=\dfrac{\mu+1}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt$ with $\mu>-1$, where F_{μ} is the Libera integral operator (see [2], [15] and [16]);

(iv) $H_{2,1}(a,1;c)f(z)=L(a,c)f(z)(a\in\mathbb{R};c\in\mathbb{R}\backslash\mathbb{Z}_0^-)$, where $L(a,c)$ is the Carlson-Shaffer operator (see [6]);

(v) $H_{2,1}(\lambda+1,c;a)f(z)=I^{\lambda}(a,c)f(z)(a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-;\lambda>-1)$, where $I^{\lambda}(a,c)$ is the Cho–Kwon–Srivastava operator (see [7]);

(vi) $H_{2,1}(\mu,1;\lambda+1)f(z)=I_{\lambda,\mu}f(z)(\lambda>-1;\mu>0)$, where the operator $I_{\lambda,\mu}$ is the Choi–Saigo–Srivastava operator (see [8]) which is closely related to the Carlson–Shaffer [6] operator $L(\mu,\lambda+1)$;

(vii) $H_{2,1}(1,1;n+1)f(z)=I_nf(z)(n>-1)$, where I_n is Noor operator of $n-th$ order (see [20]).

Next, by using Dziok-Srivastava operator $H_{q,s}[a_1;b_1]$, we introduce the following classes of analytic functions for $s,q\in\mathbb{N}_0,|\alpha|<\frac{\pi}{2}$ and $-1\le B<A\le 1:$

$$\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]=\left\{f\in\mathcal{A}:H_{q,s}[a_1;b_1]f(z)\in\mathcal{S}^{\alpha}[A,B]\right\}, \tag{12}$$

and

$$\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]=\left\{f\in\mathcal{A}:H_{q,s}[a_1;b_1]f(z)\in\mathcal{K}[A,B]\right\}. \tag{13}$$

We also note that

$$f(z)\in\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]\Leftrightarrow zf'(z)\in\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]. \tag{14}$$

In this paper, we investigate convolution properties of $\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]$ and $\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$ associated with the operator $H_{q,s}[a_1;b_1]$. Using convolution properties, we find the necessary and sufficient condition, coefficient estimates and inclusion properties for these classes.

2. Convolution Properties

Unless otherwise mentioned, we assume throughout this paper that $-1\le B<A\le 1,|\alpha|<\frac{\pi}{2},|\zeta|=1$ and $\Gamma_k[a_1;b_1]$ is defined by (9). To prove our convolution properties, we shall need the following lemmas due to Bhoosnurnath and Devadas [3, 4].

Lemma 2.1 ([3])**.** *The function $f(z)$ defined by* (1) *is in the class $\mathcal{S}^{\alpha}[A,B]$ if and only if*

$$\frac{1}{z}\left[f(z)*(1-Mz)\frac{z}{(1-z)^2}\right]\neq 0 \quad (z\in\mathbb{U}), \tag{15}$$

where

$$M=\frac{e^{i\alpha}+(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}. \tag{16}$$

Lemma 2.2 ([4] Lemma 3 with $n=1$)**.** *The function $f(z)$ defined by* (1) *is in the class $\mathcal{K}^{\alpha}[A,B]$ if and only if*

$$\frac{1}{z}\left[f(z)*(1-Nz)\frac{z}{(1-z)^3}\right]\neq 0 \quad (z\in\mathbb{U}), \tag{17}$$

where

$$N=\frac{2e^{i\alpha}+[(A+B)\cos\alpha+i2B\sin\alpha]\zeta}{(A-B)\zeta\cos\alpha}. \tag{18}$$

Theorem 2.3. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{S}^{\alpha}_{q,s}[a_1;A,B]$ is that*

$$1-\sum_{k=2}^{\infty}\frac{(k-1+kB\zeta)e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\Gamma_k[a_1;b_1]a_kz^{k-1}\neq 0 \ \ (z\in\mathbb{U}). \tag{19}$$

Proof. From Lemma 2.1, we find that $f(z)\in\mathcal{S}^{\alpha}_{q,s}[a_1;A,B]$ if and only if

$$\frac{1}{z}\left[H_{q,s}[a_1;b_1]f(z)*(1-Mz)\frac{z}{(1-z)^2}\right]\neq 0 \quad (z\in\mathbb{U}), \tag{20}$$

where M is given by (16). From (8), the left hand side of (20) may be written as

$$\begin{aligned}
&\frac{1}{z}\left[H_{q,s}[a_1;b_1]f(z)*\left(\frac{z}{(1-z)^2}-\frac{Mz^2}{(1-z)^2}\right)\right]\\
=\ &\frac{1}{z}\left[z(H_{q,s}[a_1;b_1]f(z))'-M\left\{z(H_{q,s}[a_1;b_1]f(z))'-H_{q,s}[a_1;b_1]f(z)\right\}\right]\\
=\ &1-\sum_{k=2}^{\infty}\frac{(k-1+kB\zeta)e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\Gamma_k[a_1;b_1]a_kz^{k-1}.
\end{aligned}$$

Thus, the proof of The Theorem 2.3 is completed. □

Theorem 2.4. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$ is that*

$$1-\sum_{k=2}^{\infty} k\frac{(k-1)e^{i\alpha}-[(A-kB)\cos\alpha-i(k-1)B\sin\alpha]\zeta}{(A-B)\zeta\cos\alpha}\Gamma_k[a_1;b_1]a_k z^{k-1}\neq 0\ (z\in\mathbb{U}). \quad (21)$$

Proof. From Theorem 2.3, we find that $f(z)\in\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$ if and only if

$$\frac{1}{z}\left[H_{q,s}[a_1;b_1]f(z)*(1-Nz)\frac{z}{(1-z)^3}\right]\neq 0 \quad (z\in\mathbb{U}), \quad (22)$$

where N is given by (18). From (8), the left hand side of (22) becomes

$$\begin{aligned}
&\frac{1}{z}\left[H_{q,s}[a_1;b_1]f(z)*\left(\frac{z}{(1-z)^3}-\frac{z^2}{(1-z)^3}\right)\right]\\
=\ &\frac{1}{z}\left[\frac{1}{2}z\left(zH_{q,s}[a_1;b_1]f(z)\right)''-\right.\\
&\left.N\left\{\frac{1}{2}z\left(zH_{q,s}[a_1;b_1]f(z)\right)''-z\left(H_{q,s}[a_1;b_1]f(z)\right)'\right\}\right]\\
=\ &1-\sum_{k=2}^{\infty} k\frac{(k-1)e^{i\alpha}-[(A-kB)\cos\alpha-i(k-1)B\sin\alpha]\zeta}{(A-B)\zeta\cos\alpha}\Gamma_k[a_1;b_1]a_k z^{k-1},
\end{aligned}$$

this proves Theorem 2.4. □

3. Coefficient Estimates and Inclusion Properties

Unless otherwise mentioned, we assume throughout this section that $a_i>0$ $(i=1,\ldots,q)$ and $bj>0$ $(j=1,\ldots,s)$.

As an applications of Theorems 2.3 and 2.4, we next determine coefficient estimates and inclusion properties for a function of the form (1) to be in the classes $\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]$ and $\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$.

Theorem 3.1. *If the function $f(z)$ defined by* (1) *belongs to $\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]$, then*

$$\sum_{k=2}^{\infty}\left(k-1+\left|A+iB\sin\alpha-kBe^{i\alpha}\right|\right)\Gamma_k[a_1;b_1]|a_k|\leq (A-B)\cos\alpha. \quad (23)$$

Proof. Since

$$\left|1-\sum_{k=2}^{\infty}\frac{(k-1+kB\zeta)e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\zeta}{(A-B)\zeta\cos\alpha}\Gamma_k[a_1;b_1]a_k z^{k-1}\right|$$

$$\geq 1-\sum_{k=2}^{\infty}\left|\frac{(k-1+kB\zeta)\,e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\,\zeta}{(A-B)\,\zeta\cos\alpha}\right|\Gamma_k\left[a_1;b_1\right]|a_k|,$$

and

$$\begin{aligned}&\left|\frac{(k-1+kB\zeta)\,e^{i\alpha}-(A\cos\alpha+iB\sin\alpha)\,\zeta}{(A-B)\,\zeta\cos\alpha}\right|\\&=\frac{\left|(k-1)\,e^{i\alpha}-\left(A\cos\alpha+iB\sin\alpha-kBe^{i\alpha}\right)\right|}{(A-B)\cos\alpha}\\&\leq\frac{(k-1)+\left|A\cos\alpha+iB\sin\alpha-kBe^{i\alpha}\right|}{(A-B)\cos\alpha},\end{aligned}$$

the result follows from Theorem 2.3. □

Similarly, we can prove the following theorem.

Theorem 3.2. *If the function $f(z)$ defined by* (1) *belongs $\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$, then*

$$\sum_{k=2}^{\infty}k\left\{(k-1)+|(A-kB)\cos\alpha-i\,(k-1)\,B\sin\alpha|\right\}\Gamma_k\left[a_1;b_1\right]|a_k|\leq(A-B)\cos\alpha. \quad (24)$$

We will discuss two inclusion relations for the classes $\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]$ and $\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$. To prove these results we shall require the following lemma:

Lemma 3.3 ([10]). *Let h be convex (univalent) in $\mathbb{U}$, with $\Re\{\gamma h(z)+\eta\}>0$ for all $z\in\mathbb{U}$. If p is analytic in $\mathbb{U}$, with $p(0)=h(0)$, then*

$$p(z)+\frac{zp'(z)}{\gamma p(z)+\eta}\prec h(z)\Rightarrow p(z)\prec h(z).$$

Theorem 3.4. *Suppose that*

$$\Re\left\{e^{-i\alpha}\frac{z}{1+Bz}\right\}>-\frac{a_1}{(A-B)\cos\alpha}\quad(z\in\mathbb{U}). \quad (25)$$

If $f\in\mathcal{S}_{q,s}^{\alpha}[a_1+1;A,B]$, with $H_{q,s}[a_1;b_1]f(z)\neq 0$ $(z\in\mathbb{U})$, then $f\in\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]$.

Proof. Suppose that $f\in\mathcal{S}_{q,s}^{\alpha}[a_1+1;A,B]$, and define the function

$$p(z)=e^{i\alpha}\frac{z\left(H_{q,s}[a_1;b_1]f(z)\right)'}{H_{q,s}[a_1;b_1]f(z)}\quad(z\in\mathbb{U}). \quad (26)$$

Then p is analytic in $\mathbb{U}$ with $p(0)=e^{i\alpha}$, and using the relation (10), from (26) we obtain

$$e^{-i\alpha}p(z)+a_1-1=a_1\frac{H_{q,s}[a_1+1;b_1]f(z)}{H_{q,s}[a_1;b_1]f(z)}. \tag{27}$$

Differentiating logarithmically (27) with respect to z and then using (26), we deduce that

$$p(z)+\frac{zp'(z)}{e^{-i\alpha}p(z)+a_1-1}\prec\cos\alpha\left(\frac{1+Az}{1+Bz}\right)+i\sin\alpha=h(z). \tag{28}$$

From (25), we see that $\Re\left\{e^{-i\alpha}h(z)+a_1-1\right\}>0, z\in\mathbb{U}$. Since the function $h(z)$ is convex (univalent) in $\mathbb{U}$ with $h(0)=e^{i\alpha}$, according to Lemma 3.3 the subordination (28) implies $p(z)\prec h(z)$, which proves that $f\in\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]$. □

From the duality formula (14), the above theorem yields the following inclusion:

Theorem 3.5. *Suppose that* (25) *holds and* $H_{q,s}[a_1;b_1]f(z)\neq 0$ *for all* $z\in\mathbb{U}$. *If* $f\in\mathcal{K}_{q,s}^{\alpha}[a_1+1;A,B]$, *then* $f\in\mathcal{K}_{q,s}^{\alpha}[a_1;A,B]$.

Proof. Applying (14) and Theorem 3.4, we observe that

$$\begin{aligned} f\in\mathcal{K}_{q,s}^{\alpha}[a_1+1;A,B] &\Longleftrightarrow zf'\in\mathcal{S}_{q,s}^{\alpha}[a_1+1;A,B]\,(\text{ from (14) })\\ &\Longrightarrow zf'\in\mathcal{S}_{q,s}^{\alpha}[a_1;A,B]\,(\text{ by Theorem 3.4 })\\ &\Longleftrightarrow f\in\mathcal{K}_{q,s}^{\alpha}[a_1;A,B], \end{aligned}$$

which evidently proves Theorem 3.5. □

Remark 3.6. (i) Taking $q=2, s=1, a_1=n+1\,(n>-1)$ and $a_2=b_1=1$ in Theorems 2.3, 3.1 and 3.4, respectively, we obtain the results obtained by Bhoosnurmath and Devadas [4, Theorems 1,3 and 4, respectively];
(ii) Taking $q=2, s=1, a_1=n+1\,(n>-1)$, $a_2=b_1=1, \alpha=0$ and $\bar{\zeta}=-\zeta$ in Theorems 2.3, 3.1 and 3.4, respectively, we obtain the results obtained by Ahuja [1, Theorems 1, 3 and 5, respectively];
(iii) For special choices for $a_i\,(i=1,\ldots,q)$ and $b_j\,(j=1,\ldots,s)$, where $q,s\in\mathbb{N}_0$, we can obtain corresponding results for different linear operators which are defined in the introduction.

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TAMER M. SEOUDY
Department of Mathematics
Faculty of Science
Fayoum University
Fayoum 63514, Egypt
e-mail: tms00@fayoum.edu.eg